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Simultaneous diophantine approximation of rational numbers

by

T. W. Cusick (Buffalo, N.Y.)

1. Introduction. For any real number x, let $\|x\|$ denote the distance from x to the nearest integer; thus $\frac{1}{2} \geqslant \|x\| \geqslant 0$ for all x.

Let n be any positive integer and let $\sigma = (s_1, \ldots, s_n)$ denote an arbitrary point in the set S^n of n-dimensional points all of whose coordinates are rational noninteger numbers. Define the function $\omega(n)$ by

$$\omega(n) = \inf_{\sigma\in S^n} \sup_{q} \min_{1\leqslant i\leqslant n} \|qs_i\| \tag{1}$$

where the supremum (or maximum) is taken over all integers q (in what follows, $\max_q$ will always be taken over all integers q).

If $z > 1$ is an integer with prime factorization $\prod_{i=1}^{k} p_i^{a_i}$, define $h(z) = k$. Then for each positive integer n define the function $w(n)$ by $w(1) = 1/3$, $w(2) = 1/5$ and

$$w(n) = \max\{z\colon\ h(z)+\tfrac{1}{2}\varphi(z) \leqslant n\} \tag{2}$$

for $n \geqslant 3$ (here φ is Euler's function).

The main purpose of this paper is to propose the conjecture that $\omega(n) = 1/w(n)$ for every positive integer n, and to prove the conjecture for $n \leqslant 7$.

THEOREM 1. *For $n \leqslant 7$, $\omega(n) = 1/w(n)$. Numerically,*

$$\begin{aligned}\omega(1) &= 1/3,\\ \omega(2) &= 1/5,\\ \omega(3) &= 1/8,\\ \omega(4) &= 1/12,\\ \omega(5) &= 1/18,\\ \omega(6) &= 1/24,\\ \omega(7) &= 1/30.\end{aligned}$$

The problem of evaluating $\omega(n)$ originated in two papers of Wills ([6], [7]); he showed $\omega(1) = 1/3$ and $(2n^2)^{-1} \leqslant \omega(n) \leqslant 1/w(n)$ for $n \geqslant 2$. In a later paper, Wills [8] obtained the better lower bound $\omega(n)$

$\geqslant c(n\log n)^{-1}$, and found an asymptotic formula for $w(n)$ (see Lemma 7 below).

Actually, the function considered by Wills was defined by (1) with the infimum taken over all points in the set of n-dimensional points *none of whose coordinates is an integer*. The fact that Wills' function can also be defined by (1) (which simplifies the study of $\omega(n)$) was proved by Wills [7].

The results of this paper were announced in [3].

2. Preliminary results. I begin by giving a proof that $\omega(n)\leqslant 1/w(n)$ for every n; this proof uses the same idea as the one given by Wills ([7], pp. 376–377), but avoids his unnecessary use of certain auxiliary integer sequences in the argument.

LEMMA 1. *For each positive integer n, $\omega(n)\leqslant 1/w(n)$.*

Proof. The result is obvious for $n=1, 2$, so suppose $n\geqslant 3$ and $w(n)$ has prime factorization $\prod_{i=1}^{h} p_i^{a_i}$. Define s_i, $1\leqslant i\leqslant h+\frac{1}{2}\varphi(w(n))$ by

$$s_i=\begin{cases} p_i^{-1} & (1\leqslant i\leqslant h),\\ a_{i-h}/w(n) & (h+1\leqslant i\leqslant h+\frac{1}{2}\varphi(w(n))),\end{cases}$$

where the numbers a_j $(1\leqslant j\leqslant \frac{1}{2}\varphi(w(n)))$ are those positive integers less than $\frac{1}{2}w(n)$ which are relatively prime to $w(n)$, taken in some order. Since $h+\frac{1}{2}\varphi(w(n))\leqslant n$ by the definition of $w(n)$, there are no more than n s_i's. If in fact $h+\frac{1}{2}\varphi(w(n))<n$, define $s_i=p_1^{-1}$ $(h+\frac{1}{2}\varphi(w(n))+1\leqslant i\leqslant n)$, say. In order to prove the lemma, it suffices to show $\max_q \min_{1\leqslant i\leqslant n}\|qs_i\|\leqslant 1/w(n)$.

Clearly we need only consider integers q satisfying $1\leqslant q\leqslant w(n)$. If q is not relatively prime to $w(n)$, then $\min_{1\leqslant i\leqslant h}\|qs_i\|=0$. If q is relatively prime to $w(n)$, then $\min_{h+1\leqslant i\leqslant n}\|qs_i\|\leqslant 1/w(n)$: for each of the $\varphi(w(n))$ congruences $a_jx\equiv\pm 1 \bmod w(n)$ $(1\leqslant j\leqslant\frac{1}{2}\varphi(w(n)))$ has a unique solution $x=q$ with $1\leqslant q\leqslant w(n)$ and q prime to $w(n)$, and no two of these solutions are the same because the a_j are distinct and satisfy $1\leqslant a_j<\frac{1}{2}w(n)$. This proves the lemma.

Thus in order to prove Theorem 1, it is only necessary to show that $\omega(n)\geqslant 1/w(n)$ for $1\leqslant n\leqslant 7$. The following lemma will play an important role:

LEMMA 2. *Let n, k and m be any positive integers, and suppose m has prime factorization $\prod_{i=1}^{h} p_i^{a_i}$. If there exist rational noninteger numbers $s_1,\ldots,s_n$ such that $\max_q\min_{1\leqslant i\leqslant n}\|qs_i\|\leqslant k/m$, then we may assume without loss of generality that $h\leqslant n$, $s_i=b_i/p_i$ $(1\leqslant i\leqslant h)$ where the b_i are integers satisfying*

$0<b_i<p_i$, and $s_i=a_i/m$ $(h+1\leqslant i\leqslant n)$ where the a_i are integers satisfying $0<a_i<m$.

Proof. This is a lemma of Wills ([7], Lemma 3, p. 372) expressed in a different form. I give a simplified proof, as follows:

We may obviously assume that $\max_q\min_{1\leqslant i\leqslant n}\|qs_i\|=k/m$ and $s_i=a_i/m$ $(1\leqslant i\leqslant n)$ where the a_i are integers satisfying $0<a_i<m$. Let p be any prime dividing m and suppose that ps_i is not an integer for $1\leqslant i\leqslant n$. Then if we define $s_i'=ps_i$ $(1\leqslant i\leqslant n)$ the s_i' are rational noninteger numbers and clearly $\max_q\min_{1\leqslant i\leqslant n}\|qs_i'\|\leqslant k/m$. Hence we need only consider the case where for each p dividing m, ps_i is an integer for some subscript $i=i(p)$. The subscripts $i(p)$ must be different for different p, and this proves the lemma.

The arguments used below to establish $\omega(n)\geqslant 1/w(n)$ for $3\leqslant n\leqslant 7$ do not seem to apply for $n=1, 2$, so I give special proofs for these cases.

LEMMA 3. *The equality $\omega(n)=1/w(n)$ holds for $n=1,2$.*

Proof. The case $n=1$ is trivial. For $n=2$, we have $w(2)=5$, so it suffices to show that for any two rational noninteger numbers a/b and c/d which satisfy

$$\max_q\min(\|qa/b\|,\|qc/d\|)\leqslant 1/5, \tag{3}$$

equality must hold in (3).

We assume without loss of generality that a/b and c/d are in lowest terms. There are clearly $\leqslant d(2[b/5]+1)$ integers q in the range $1\leqslant q\leqslant bd$ which satisfy $\|qa/b\|\leqslant 1/5$, with a similar result for the inequality $\|qc/d\|\leqslant 1/5$. It follows that (3) implies

$$2\left(b\left[\frac{d}{5}\right]+d\left[\frac{b}{5}\right]\right)+b+d\geqslant bd. \tag{4}$$

Using the trivial estimate for the greatest integer function in (4), we obtain $b^{-1}+d^{-1}\geqslant 1/5$, so at least one of b, d is $\leqslant 10$. We assume without loss of generality that $2\leqslant b\leqslant 10$, $b\leqslant d$. Then a little arithmetic shows that for $b\neq 5$, the only pairs (b,d) which satisfy (4) are $(2,5)$, $(2,6)$, $(2,10)$, $(6,6)$, $(6,10)$ and $(10,10)$. It is a simple matter to verify that (3) cannot hold for any fractions a/b, c/d in lowest terms if (b,d) is one of these pairs.

Hence $b=5$, and it is easily seen that strict inequality cannot hold in (3) for any choice of a, c and d.

3. The cases $3\leqslant n\leqslant 7$ of Theorem 1. Assume that for some n, there exist positive integers k and m such that

$$\omega(n)\leqslant\frac{k}{m}<\frac{1}{w(n)}. \tag{5}$$

I prove Theorem 1 by showing that for $3 \leqslant n \leqslant 7$, the assumption (5) leads to a contradiction.

By Lemma 2, we can suppose without loss of generality that there exist distinct primes r_i $(1 \leqslant i \leqslant h,\ r_1 < r_2 < \ldots < r_h)$ and rational numbers $s_i = a_i/m$ $(1 \leqslant i \leqslant n)$ such that

(6) $$m = \prod_{i=1}^{h} r_i^{\beta_i}, \qquad h \leqslant n;$$

(7) $$\max_{q} \min_{1\leqslant i\leqslant n} \|qs_i\| \leqslant k/m;$$

and

(8) $$0 < a_i < m \ (1 \leqslant i \leqslant n), \qquad a_i = mb_i/r_i \text{ for integers } b_i,$$
$$0 < b_i < r_i \ (1 \leqslant i \leqslant h).$$

Define

$v_n(m)$ = the number of distinct primes p such that p divides m and $p < w(n)$.

Lemma 4. *If* (5), (6), (7) *and* (8) *hold, then*

$$\frac{m}{\varphi(m)} > \begin{cases} \dfrac{w(n)}{2n-h(m)-v_n(m)}, & m \text{ even}, \\ \dfrac{w(n)}{2(n-v_n(m))}, & m \text{ odd}. \end{cases}$$

Proof. Let (a, b) denote the greatest common divisor of the integers a and b. There are at most $2k$ different values of x, $1 \leqslant x \leqslant m$, which satisfy at least one of the $2k$ congruences

(9) $$a_i x \equiv \pm j \bmod m \qquad (1 \leqslant j \leqslant k).$$

This is clear if $(a_i, m) = 1$, for then each of the congruences in (9) has a unique solution $\bmod m$. If $(a_i, m) > 1$, the congruence $a_i x \equiv j \bmod m$ is solvable if and only if (a_i, m) divides j, in which case there are (a_i, m) solutions x, $1 \leqslant x \leqslant m$. The number of j such that $1 \leqslant |j| \leqslant k$ and (a_i, m) divides j is $2[k/(a_i, m)]$, so the total number of different x, $1 \leqslant x \leqslant m$, which satisfy at least one of the congruences in (9) is $\leqslant 2(a_i, m)[k/(a_i, m)] \leqslant 2k$.

Since $0 < a_i < m$ by (8), any integer q such that $(q, m) = 1$ and $\min_{1\leqslant i\leqslant n} \|qs_i\| \leqslant k/m$ must be a solution of at least one of the congruences in (9) for some i, $v_n(m)+1 \leqslant i \leqslant n$. The range $1 \leqslant i \leqslant v_n(m)$ need not be considered, because for these i we have

$$(a_i, m) = \frac{m}{r_i} > \frac{m}{w(n)} > k$$

by (8), the definition of $v_n(m)$ and (5); thus none of the congruences in (9) has solutions if $1 \leqslant i \leqslant v_n(m)$. There are $\varphi(m)$ values of q such that $(q, m) = 1$ and $1 \leqslant q \leqslant m$, so (7) implies that there are at least $\varphi(m)/2k$ different values of a_i $(v_n(m)+1 \leqslant i \leqslant n)$. Hence, using (5),

(10) $$\varphi(m) \leqslant 2k(n-v_n(m)) < \frac{2m(n-v_n(m))}{w(n)},$$

which gives the lemma if m is odd.

For m even, we shall show that if $v_n(m)+1 \leqslant i \leqslant h(m)$, then there are at most k different values of x such that $(x, m) = 1$, $1 \leqslant x \leqslant m$ and x satisfies at least one of the congruences in (9). Then the argument which led to (10) will give

$$\varphi(m) \leqslant 2k(n-h(m)) + k(h(m)-v_n(m)) < \frac{m(2n-h(m)-v_n(m))}{w(n)},$$

which is the desired result.

The (a_i, m) solutions of any solvable congruence $a_i x \equiv j \bmod m$ are given by

(11) $$x = x_0 + t\frac{m}{(a_i, m)} \qquad (1 \leqslant t \leqslant (a_i, m))$$

where

$$\frac{a_i}{(a_i, m)} x_0 \equiv \frac{j}{(a_i, m)} \bmod \frac{m}{(a_i, m)}.$$

If m is even and $v_n(m)+1 \leqslant i \leqslant h(m)$, then by (8) $(a_i, m) = m/r_i$ is even and $m/(a_i, m) = r_i$ is odd. Hence exactly half of the solutions (11) are even, namely those for which t has the same parity as x_0.

We saw previously that at most $2k$ different values of x, $1 \leqslant x \leqslant m$, satisfy at least one of the congruences in (9). The above remarks show that at most k of these values of x also satisfy $(x, m) = 1$. This completes the proof of the lemma.

Define for each positive integer n

$$P_n = \prod_{i=1}^{n} \frac{p_i}{p_i - 1}$$

where $p_1 = 2, \ldots, p_i$ = the ith prime. Values of P_n for various $n < 20$ occur frequently in the calculations necessary in the proofs of the next two lemmas. For this work a table of P_n (or P_n^{-1}) is very convenient; such tables are given in [1], [4], [5].

Lemma 5. *If* (5), (6), (7) *and* (8) *hold and n satisfies $3 \leqslant n \leqslant 7$, then m is even.*

Proof. If m is not even, it follows from (6) that

$$\frac{m}{\varphi(m)} = \prod_{i=1}^{h} \frac{r_i}{r_i-1} \leqslant \frac{1}{2} P_{n+1}. \tag{12}$$

For each n in the range $3 \leqslant n \leqslant 7$, it is a simple calculation to verify that for any odd m the inequality (12) contradicts the inequality of Lemma 4. For example, if $n = 4$ and both inequalities hold, then $v_4(m) < 4 - \big(w(4)/P_5\big) = 1.50\ldots$ This implies either $v_4(m) = 0$, so that $m/\varphi(m) \leqslant P_9 P_5^{-1} = 1.27\ldots$, or $v_4(m) = 1$, so that $m/\varphi(m) \leqslant 1.5 P_8 P_5^{-1} = 1.82\ldots$; in both cases the inequality of Lemma 4 is contradicted.

The range of n for which Lemma 5 is valid could be greatly extended, by the same type of calculation. It is the next lemma which leads to the restriction $n \leqslant 7$ in Theorem 1.

LEMMA 6. *If n satisfies $3 \leqslant n \leqslant 7$, then there is no even integer m for which* (5), (6), (7) *and* (8) *are valid.*

Proof. We show that for $3 \leqslant n \leqslant 7$, the assumption of the existence of an even m such that (5), (6), (7) and (8) hold contradicts Lemma 4. For each n, the first step is to deduce an upper bound for $h(m) + v_n(m)$ from the inequalities

$$P_n \geqslant \frac{m}{\varphi(m)} > \frac{w(n)}{2n - h(m) - v_n(m)}, \tag{13}$$

which follow from (6) and Lemma 4. Then calculations similar to those in the proof of Lemma 5 show that the conditions imposed on m cannot all be satisfied. I illustrate the calculations with the cases $n = 3$ and $n = 7$; the other cases are much the same. Notice that the trivial inequality $h(m) \geqslant v_n(m)$ is frequently used.

The case $n = 3$. Here (13) implies $h(m) + v_3(m) \leqslant 3$, so the only possibilities are $v_3(m) = 1$, $h(m) = 2$ or 1. If $h(m) = 2$, then $m/\varphi(m) \leqslant 2(11/10) = 2.2$; if $h(m) = 1$, then $m/\varphi(m) \leqslant 2$. In both cases the inequality of Lemma 4 is contradicted.

The case $n = 7$. Here (13) implies $h(m) + v_7(m) \leqslant 8$, so $v_7(m) \leqslant 4$.

If $v_7(m) = 4$, then also $h(m) = 4$, so $m/\varphi(m) \leqslant P_4 = 4.375$. This contradicts the inequality of Lemma 4.

If $v_7(m) = 3$, then $3 \leqslant h(m) \leqslant 5$. If $h(m) = 5$, we have $P_5 = 4.8125 \geqslant m/\varphi(m) > 30/6$ (contradiction). If $h(m) = 4$, we have $P_4 = 4.375 \geqslant m/\varphi(m) > 30/7$ (contradiction). If $h(m) = 3$, $P_3 = 3.75 \geqslant m/\varphi(m) > 30/8$ (contradiction).

If $v_7(m) = 2$, then $2 \leqslant h(m) \leqslant 6$, so $m/\varphi(m) \leqslant 3 P_{14} P_{10}^{-1} = 3.34\ldots$ However, by Lemma 4, $m/\varphi(m) > 30/\big(12 - h(m)\big) \geqslant 3.75$ for $h(m) \geqslant 4$; this eliminates the cases $4 \leqslant h(m) \leqslant 6$. If $h(m) = 3$, we have $3(31/30) = 3.1 \geqslant m/\varphi(m) > 30/9$ (contradiction). If $h(m) = 2$, we have $3 \geqslant m/\varphi(m) > 30/10$ (contradiction).

If $v_7(m) = 1$, then $1 \leqslant h(m) \leqslant 7$, so $m/\varphi(m) \leqslant 2 P_{16} P_{10}^{-1} = 2.32\ldots$ However, by Lemma 4, $m/\varphi(m) > 30/\big(13 - h(m)\big) \geqslant 2.5$ for $h(m) \geqslant 1$, which gives a contradiction for any possible value of $h(m)$.

Lemma 6 completes the proof of Theorem 1. The method used to prove Lemma 6 breaks down for $n \geqslant 8$ by failing to exclude all possible even values of m. For example, in the case $n = 8$ $(w(8) = 36)$, the method excludes all even integers m except m of the form (6) with $h = 3$, $r_1 = 2$, $r_2 = 3$, $r_3 = 5$; that is, the assumption that m exists leads to a contradiction except when $v_8(m) = h(m) = 3$ and 30 divides m. Thus the technique of Lemma 6 is insufficient to evaluate $\omega(8)$, although the calculations will give the estimates $1/36 \geqslant \omega(8) \geqslant 2/75$.

4. A theorem about $w(n)$. The function $w(n)$ is of some interest in its own right. (See table of $w(n)$ below. This table was easily constructed by using Tables I and II of [2]. Table I gives n, the factorization of n, and $\varphi(n)$; Table II gives the values of n for which $\varphi(n)$ takes on a given value.)

Theorem 2 below states one of the more striking properties of $w(n)$. The proof makes use of the fact that $w(n)/n$ tends to infinity, which is an immediate consequence of the following known result:

LEMMA 7. *If γ denotes Euler's constant, then*

$$\lim_{n\to\infty} \frac{w(n)}{n \log\log n} = 2e^{\gamma}.$$

Proof. This was proved by Wills ([8], Lemma 1, p. 167).

THEOREM 2. *Given any prime q, q divides $w(n)$ for all sufficiently large n.*

Proof. We first obtain the weaker result that $h\big(w(n)\big) \to \infty$ as $n \to \infty$. First, we have

$$\varphi\big(w(n)\big) = w(n) \prod_{p} \Big(1 - \frac{1}{p}\Big) \geqslant \frac{w(n)}{h\big(w(n)\big) + 1},$$

where the product is over the $h\big(w(n)\big)$ primes p which divide $w(n)$ and the inequality is trivial. Second, we have $w(n)/\varphi\big(w(n)\big) \to \infty$ as $n \to \infty$: for if $w(n)/\varphi\big(w(n)\big) \leqslant B$ for infinitely many n, then the inequality $n > \frac{1}{2}\varphi\big(w(n)\big)$ implies $w(n)/n \leqslant 2B$ for infinitely many n, in contradiction to Lemma 7. It follows at once that $h\big(w(n)\big) \to \infty$ as $n \to \infty$.

Suppose now that $w(n) = z_0$ and the prime q does not divide z_0; we shall deduce a contradiction if n is sufficiently large. Suppose the prime factorization of z_0 is $\prod_{i=1}^{k} p_i^{a_i}$, $p_1 < p_2 < \ldots < p_k$.

Let $N(r)$ be a function with the property that $x \geqslant N(r)$ implies the existence of a prime p satisfying $x < p < (1+r)x$ $(0 < r \leqslant 1)$. The existence of such a function of course follows from the prime number theorem.

Since $h(w(n)) \to \infty$ as $n \to \infty$, we may suppose that n is so large that $k \geqslant 3$ and p_{k-1}, p_k satisfy

$$\frac{q}{q-1}\left(\frac{p_{k-1}-1}{p_{k-1}}\right)\left(\frac{p_k-1}{p_k}\right) \geqslant 1+\frac{1}{2q}, \tag{14}$$

$$\frac{p_{k-1}p_k}{q} > N\left(\frac{1}{2q}\right) \tag{15}$$

and

$$p_{k-1} > q. \tag{16}$$

Then (15) and (16) imply that there exists a prime p such that

$$p_k < p_{k-1}^{\alpha_{k-1}} p_k^{\alpha_k} q^{-1} < p < \left(1+\frac{1}{2q}\right) p_{k-1}^{\alpha_{k-1}} p_k^{\alpha_k} q^{-1}. \tag{17}$$

Now define $z_1 = pq \prod_{i=1}^{k-2} p_i^{\alpha_i}$; neither p nor q is one of the p_i's because $p_{k-2} < p_k < p$ by (17) and q does not divide z_0 by hypothesis. We have $z_1 > z_0$ by (17), $h(z_1) = h(z_0)$ and $\varphi(z_1) < \varphi(z_0)$ (because

$$\varphi(pq) = (p-1)(q-1) < \varphi(p_{k-1}^{\alpha_{k-1}} p_k^{\alpha_k}) = (p_{k-1}-1)(p_k-1) p_{k-1}^{\alpha_{k-1}-1} p_k^{\alpha_k-1}$$

follows from (14) and the third inequality in (17)). This contradicts the hypothesis $w(n) = z_0$, and so proves the theorem.

Table of $w(n)$, $1 \leqslant n \leqslant 50$

n	$w(n)$	n	$w(n)$	n	$w(n)$	n	$w(n)$	n	$w(n)$
1	3	11	60	21	126	31	210	41	270
2	5	12	60	22	126	32	210	42	270
3	8	13	66	23	150	33	210	43	300
4	12	14	72	24	150	34	210	44	330
5	18	15	90	25	150	35	240	45	330
6	24	16	90	26	150	36	240	46	330
7	30	17	90	27	180	37	240	47	330
8	36	18	96	28	210	38	240	48	330
9	42	19	120	29	210	39	270	49	330
10	48	20	120	30	210	40	270	50	330

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