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On the analytic theory of quadratic forms

by

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Dedicated to the memory of W. Sierpiński

1. The analytic theory of quadratic forms, as developed by Siegel [6], leads to a fundamental formula, now called the 'Siegel formula', which is an identity for $m>4$ between the theta series associated with the genus of a quadratic form in $m>4$ variables and the Eisenstein-Siegel series associated with it. In a beautiful reworking of the theory, Weil [10] has obtained, among others, a proof of the Siegel formula for $m>4$ by an analytic method which lends itself to important generalizations (see the recent paper of J. Igusa, Inventiones Math. 1971).

In this note we present a proof of this formula for $m\geqslant 3$ by using an idea due to Hecke [2]. In the case $m=2$ a similar formula is proved by Hecke [2] for definite forms and by Maass [4] for indefinite forms. However the summation in these cases is over all classes of forms with a given determinant. The result for summation over classes in a given genus is, in general, false. In case $m=1$, this formula is proved by Siegel [8] and Maass [3]. In [5] Raghavan and Rangachari have extended Weil's methods to the case of quadratic forms in 4 variables with index $\leqslant 1$.

An interesting consequence of the analysis is that one proves, analytically, that the Minkowski–Siegel constant (for semi-simple algebraic groups this is called the Tamagawa number) is two. However one has to prove it first in the case $m=2$. This is well-known by the classical results of Dirichlet–Minkowski–Siegel.

Generalizations of this formula can be obtained for quadratic or hermitian forms over arbitrary algebraic number fields and over quaternion algebras. The generalization where one deals with representation of matrices by matrices seems difficult and is related to the analytic continuation of Eisenstein series in the Siegel half space.

2. Let S be a semi-integral non-singular m rowed symmetric matrix so that $2S$ is an integral matrix with even diagonal elements. Put

$$d = |2S|. \tag{1}$$

Let S have signature $n, m-n$ and H a matrix in the majorant space [7] of S. Put

$$f(S, H, z) = \sum_{\underline{x}} e^{2\pi i(\xi S + i\eta H)[\underline{x}]} \tag{2}$$

where $z = \xi + i\eta$, $\eta > 0$ is a parameter in the upper half complex plane and $\underline{x}$ runs through all m-rowed integral columns.

Let $r = \frac{a}{b}$, $(a, b) = 1$, $b > 0$ be a rational number. When $z = r + i\eta$ tends to the point $z = r$ on the ξ axis, $f(S, H, z)$ has the behaviour

$$f(S, H, z) \sim e^{-\frac{\pi i}{4}(m-2n)} |d|^{-1/2} b^{-m} \sum_{\underline{u} (\mathrm{mod}\, b)} e^{2\pi i r S[\underline{u}]} \eta^{-m/2}. \tag{3}$$

For every rational number $r = \frac{a}{b}$, $(a, b) = 1$, $b > 0$, put

$$\gamma(r) = e^{-\frac{\pi i}{4}(2n-m)} |d|^{-1/2} b^{-m} \sum_{\underline{u} (\mathrm{mod}\, b)} e^{2\pi i r S[\underline{u}]}. \tag{4}$$

We then define the Eisenstein–Siegel series associated with S as

$$\varphi(S, z, s) = 1 + \sum_r b^{-s} \gamma(r) (z-r)^{-n/2} (\bar{z}-r)^{-\left(\frac{m-n}{2}\right)} |z-r|^{-s} \tag{5}$$

where the sum runs through all rational numbers and s is a complex variable with

$$\mathrm{Re}\, s > 2 - \frac{m}{2} \tag{6}$$

and the radicals $\left((z-r)^{1/2}\right)^{-n}$, $\left((\bar{z}-r)^{1/2}\right)^{-(m-n)}$ are taken with their principal parts. In the region (6), the series $\varphi(S, z, s)$ converges absolutely and represents for $\eta > 0$, an analytic function of s. Since

$$\gamma\left(\frac{a}{b}\right) = \gamma\left(\frac{a'}{b}\right) \quad \text{for} \quad a \equiv a' \ (\mathrm{mod}\, b), \quad (a, b) = 1,$$

we have, in the region (6)

$$\varphi(S, z, s) = 1 + \sum_{b=1}^{\infty} \sum_{\substack{a (\mathrm{mod}\, b) \\ (a,b)=1}} b^{-s} \gamma\left(\frac{a}{b}\right) \times$$
$$\times \sum_{k=-\infty}^{\infty} \left(z - \frac{a}{b} + k\right)^{-(n+s)/2} \left(\bar{z} - \frac{a}{b} + k\right)^{-(m-n+s)/2}.$$

The Poisson summation formula gives

$$\sum_{k=-\infty}^{\infty} \left(z - \frac{a}{b} + k\right)^{-(n+s)/2} \left(\bar{z} - \frac{a}{b} + k\right)^{-(m-n+s)/2} \tag{7}$$
$$= \sum_{t=-\infty}^{\infty} e^{-\frac{2\pi i a t}{b}} \int_{-\infty}^{\infty} \frac{e^{-2\pi i x t}}{(x+z)^{(n+s)/2} (x+\bar{z})^{(m-n+s)/2}} \, dx.$$

Using (7) we get

$$\varphi(S, z, s) = 1 + e^{\frac{\pi i}{4}(2n-m)} |d|^{-1/2} \sum_{t=-\infty}^{\infty} B(t, s) A(t, s, z) \tag{8}$$

where

$$A(t, s, z) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x t}}{(x+z)^{(n+s)/2} (x+\bar{z})^{(m-n+s)/2}} \, dx \tag{9}$$

and

$$B(t, s) = \sum_{b=1}^{\infty} G(b, t) b^{-s}, \tag{10}$$

$$G(b, t) = b^{-m} \sum_{\underline{u} (\mathrm{mod}\, b)} \sum_{\substack{a (\mathrm{mod}\, b) \\ (a,b)=1}} e^{\frac{2\pi i a}{b}(S[\underline{u}] - t)}.$$

3. We shall now consider (9) and (10) separately. Using the estimate

$$\Big| \sum_{\underline{u} (\mathrm{mod}\, b)} e^{\frac{2\pi i a}{b} S[\underline{u}]} \Big| \leqslant c_1 b^{m/2}$$

where c_1 is a constant, we get

$$|G(b, t)| \leqslant c_1 b^{-m/2} \varphi(b) < c_2 b^{1-m/2}$$

where c_2 is a constant and $\varphi(b)$ is Euler's function. Because of (6) therefore, the Dirichlet series $B(t, s)$ converges absolutely. Furthermore we have in this domain

$$B(t, s) = \prod_p B_p(t, s)$$

where the product runs over all prime numbers and

$$B_p(t, s) = \sum_{k=0}^{\infty} G(p^k, t) p^{-ks}.$$

Let $A_{p^k}(S, t)$ denote the number of $\mathrm{mod}\, p^k$ incongruent solutions of the congruence

$$S[\underline{u}] - t \equiv 0 \ (\mathrm{mod}\, p^k).$$

Then

$$G(p^k, t) = p^{k(1-m)} A_{p^k}(S, t) - p^{(k-1)(1-m)} A_{p^{k-1}}(S, t). \tag{11}$$

By Siegel ([6], Hilfssatz 13), it follows that $G(p^k, t) = 0$ for all sufficiently large k, if $t \neq 0$. This means that $B_p(t, s)$ is a rational function of p^{-s}. If $t = 0$ it also follows that [9] $(1-p^{2-m-2s})B_p(0, s)$ is a polynomial in p^{-s}. More explicitly if $m \geqslant 2$ and $p \nmid 2d$ then for $t \neq 0$

$$B_p(t, s) = \begin{cases} \left(1-\chi(p)p^{-\frac{m}{2}-s}\right)\left(\sum_{h=0}^{l} \chi(p^h)p^{h\left(1-\frac{m}{2}-2\right)}\right), & m \text{ even}, \\ \dfrac{1-p^{-\frac{m}{2}-s}}{1-\chi_t(p)p^{\frac{1-m}{2}-s}}\left(\sum_{k=0}^{l/2} p^{2-m-2s} - \chi_t(p)\sum_{k=0}^{[(l-1)/2]} p^{2-m-2s}\right), & m \text{ odd} \end{cases} \tag{12}$$

and for $t = 0$

$$B_p(0, s) = \begin{cases} \dfrac{1-\chi(p)p^{-\frac{m}{2}+s}}{1-\chi(p)p^{-\frac{m}{2}+1-s}}, & m \text{ even}, \\ \dfrac{1-p^{-2\left(s+\frac{m-1}{2}\right)}}{1-p^{-2\left(s+\frac{m}{2}-1\right)}}, & m \text{ odd} \end{cases} \tag{13}$$

where $\chi(p)$ and $\chi_t(p)$ are given by

$$\chi(p) = \left(\frac{(-1)^{m/2}d}{p}\right), \quad m \text{ even},$$

$$\chi_t(p) = \left(\frac{(-1)^{\frac{m-1}{2}}2dt^*}{p}\right), \quad m \text{ odd}$$

where t^* is the discriminant of the quadratic field $Q(\sqrt{t})$ and p^l is the highest power of p dividing $t \neq 0$.

Let us denote the L-series

$$L_\chi(s) = \sum_{k=1}^{\infty} \frac{\chi(k)}{k^s}, \tag{14}$$

$$L_{\chi_t}(s) = \sum_{k=1}^{\infty} \frac{\chi_t(k)}{k^s}$$

for $\operatorname{Re} s > 1$. Since

$$B(t, s) = \prod_p B_p(t, s) = \prod_{p\mid 2d} B_p(t, s) \prod_{p\nmid 2d} B_p(t, s),$$

we get, for $t \neq 0$,

$$B(t, s) = \begin{cases} \dfrac{\prod_{p\mid 2d} F_p(s)}{L_\chi\left(\frac{m}{2}+s\right)}, & m \text{ even}, \\ \dfrac{L_{\chi_t}\left(s+\frac{m-1}{2}\right)}{\zeta\left(s+\frac{m}{2}\right)} \prod_{p\mid 2d} \left(1-p^{-s-\frac{m}{2}}\right)^{-1} F_p(s), & m \text{ odd} \end{cases} \tag{15}$$

where $F_p(s)$ is a polynomial in p^{-s} with rational coefficients and $\zeta(s)$ is the Riemann zeta function.

If $t = 0$, we have

$$B(0, s) = \begin{cases} \dfrac{L_\chi\left(s+\frac{m}{2}-1\right)}{L_\chi\left(s+\frac{m}{2}\right)} \prod_{p\mid 2d} (1-p^{2-m-2s})^{-1}E_p(s), & m \text{ even}, \\ \dfrac{\zeta(2s+m-2)}{\zeta(2s+m-1)} \prod_{p\mid 2d} (1-p^{1-m-2s})^{-1}E_p(s), & m \text{ odd}. \end{cases} \tag{16}$$

4. From formulae (15) and (16) it is obvious that $B(t, s)$ has an analytic continuation throughout the s-plane as a meromorphic function of s.

We now wish to study the nature of $B(t, s)$ in a complete neighbourhood of $s = 0$. We shall assume that $m \geqslant 2$ and that $\underline{x}'S\underline{x}$ is not a binary, ternary or quaternionic zero form.

Let us first take the case $t \neq 0$.

If $m \geqslant 4$ then from (15) it is obvious that $B(t, s)$ has an analytic continuation as a regular function of s into a full neighbourhood of $s = 0$. Let now $m = 3$. Under the conditions above, $\chi_t(k)$ is a non-principal character and so $L_{\chi_t}(s+1)$ is regular at $s = 0$. Therefore $B(t, s)$ is regular in a complete neighbourhood of $s = 0$. If $m = 2$ then $-d$ is not a square since $\underline{x}'S\underline{x}$ is not a zero form. Therefore $\chi(k)$ is a non-principal character and $L_\chi(1+s) \neq 0$ at $s = 0$.

Let us take the case $t = 0$.

If $m > 4$ then (16) shows that $B(0, s)$ can be continued analytically to $s = 0$ as a regular function of s. Suppose $m = 4$. $L_\chi(s+1)$ has a pole at $s = 0$ if $\chi(k)$ is a principal character. But from our assumption d is not a perfect square and so $\chi(k)$ is a non-principal character. However if d is a perfect square then since $\underline{x}'S\underline{x}$ is not a quaternionic zero form, $B_p(0, s)$ is zero at $s = 0$ for at least one p. This would mean that since $L_\chi(s+1)$ has a simple pole at $s = 0$ when χ is a principal character, $B_p(0, s)$ will be regular in a full neighbourhood of $s = 0$. Let now $m = 3$. Since S is not the matrix of a ternary zero form, $B_p(0, s)$ vanishes at $s = 0$ for at least one p dividing $2d$ so that $B(0, s)$ is regular at $s = 0$. If $m = 2$ and $-d$ is not a square then $\underline{x}'S\underline{x}$ is not a zero form. But if $2d$ has more than one distinct prime factor, then, by (16), $B(0, s)$ has a pole of order > 1 since $L_\chi(s)/L_\chi(s+1)$ is regular at $s = 0$. Even if $L_\chi(s)$ is zero at $s = 0$, it will be a simple zero. If $d > 0$ and 2 is the only prime dividing d then $B(0, s)$ is regular at $s = 0$.

Summing up we have

LEMMA 1. *Let $m \geqslant 2$ and let S be not the matrix of a binary, ternary or quaternionic zero form. Then $B(t, s)$ can be continued analytically to a full neighbourhood of $s = 0$ if $m \geqslant 3$. This is still true if $m = 2$ provided $t \neq 0$. If $m = 2$ and $t = 0$ then, in general, $B(t, s)$ is not regular at $s = 0$.*

From the nature of the series $B_p(t, s)$ it is clear that, under the assumptions of Lemma 1,

$$B(t, 0) = \prod_p \alpha_p(S, t) \tag{17}$$

where

$$\alpha_p(S, t) = \lim_{k\to\infty} p^{-k(m-1)} A_{p^k}(S, t). \tag{18}$$

It is known from Siegel's theory [6], that $\alpha_p(S, t)$ is a rational number for $m \geqslant 1$ and that if $m \leqslant 4$ and the infinite product in (17) is taken in the natural order of the primes, then the product converges. Furthermore, as Siegel has shown [6], $B(t, 0)$ is zero if and only if at least one $B_p(t, 0)$ is zero.

5. We now discuss the function $A(t, s, z)$ introduced in (9).

Let $n(m-n) \neq 0$ so that $m \geqslant 2$ and S is indefinite. It is then known that

$$A(t, s, z) = \frac{e^{\frac{\pi i}{4}(m-2n)} (2\pi)^{\frac{m}{2}+s}}{\Gamma\left(\frac{n+s}{2}\right)\Gamma\left(\frac{m-n+s}{2}\right)} h_t(z, s) \tag{19}$$

where

$$h_t(z, s) = e^{2\pi i t\bar{z}} \int\limits_{x>0,t} x^{\frac{n+s}{2}-1} (x-t)^{\frac{m-n+s}{2}-1} e^{-4\pi\eta x}\, dx.$$

If we define the confluent hypergeometric integral

$$h(a, b, \eta) = \int_0^\infty x^{a-1}(x+1)^{b-1} e^{-x\eta}\, dx$$

where $\mathrm{Re}\, a > 0$, $\mathrm{Re}\, b > 0$, $\eta > 0$, then

$$h_t(z, s) = \begin{cases} t^{\frac{m}{2}+s-1} e^{2\pi i t z} h\left(\frac{m-n+s}{2}, \frac{n+s}{2}, 4\pi t\eta\right), & t > 0, \\ (-t)^{\frac{m}{2}+s-1} e^{2\pi i t\bar{z}} h\left(\frac{n+s}{2}, \frac{m-n+s}{2}, -4\pi t\eta\right), & t < 0, \\ \Gamma\left(\frac{m}{2}+s-1\right)(4\pi\eta)^{1-\frac{m}{2}-s}. & \end{cases}$$

These formulae show that if $m \geqslant 3$ and $n(m-n) \neq 0$, then $A(t, s, z)$ are analytic in a full neighbourhood of $s = 0$. If however $n = (m-n) = 1$, then $m = 2$ and $A(t, s, z)$ is analytic in a complete neighbourhood of $s = 0$ provided $t \neq 0$. If $t = 0$ however $A(t, s, z)$ has a pole at $s = 0$.

Let now $S > 0$ so that $m = n$. Hecke [1] has shown that if $t \neq 0$ then $A(t, s, z)$ represents an entire function of s and

$$A(t, 0, z) = \begin{cases} e^{-\frac{\pi i m}{4}} \dfrac{(2\pi)^{m/2}}{\Gamma(m/2)} t^{\frac{m}{2}-1} e^{2\pi i t z}, & t > 0 \\ 0, & t < 0 \end{cases} \tag{20}$$

If $t = 0$, and $m \geqslant 3$ then $A(0, s, z)$ is regular even in the domain $\mathrm{Re}\, s > -\frac{1}{2}$ and

$$A(0, 0, z) = 0. \tag{21}$$

If now $m = 2$, again Hecke has shown that $A(0, s, z)$ is regular in a complete neighbourhood of $s = 0$ and

$$A(0, 0, z) \neq 0. \tag{22}$$

Having discussed the nature of the functions $B(t, s)$ and $A(t, s, z)$ at $s = 0$ we go back to the series (8). From the results in Section 4 and 5 and the behaviour of the L-series for increasing moduli, we have

$$|B(t, s)| = O(|t|^\lambda), \quad |t| \to \infty \tag{23}$$

for $t \neq 0$ in a small open, relatively compact neighbourhood of $s = 0$. On the other hand, from the integral representation, it follows that

$$|A(t, s, z)| = O(e^{-\delta|t|}), \quad \delta > 0, \ |t| \to \infty. \tag{24}$$

Therefore

$$\sum_{t\neq 0} B(t, s) A(t, s, z) = O\Big(\sum_{t>0} t^\lambda e^{-\delta t}\Big). \tag{25}$$

If $m \geqslant 3$ it follows that $\varphi(S, z, s)$ is regular in a complete neighbourhood of $s = 0$ and

$$\varphi(S, z) = \lim_{s \to 0} \varphi(S, z, s) \tag{26}$$

exists. Furthermore

$$\lim_{s \to 0} B(t, s) = \prod_p \alpha_p(S, t).$$

Therefore we have

LEMMA 2. *If $m \geqslant 3$ and S satisfies conditions of Lemma 1, then*

$$\varphi(S, z) = 1 + e^{\frac{\pi i}{4}(2n-m)} |d|^{1/2} \sum_{t=-\infty}^{\infty} \prod_p \alpha_p(S, t) A(t, 0, z).$$

In particular, if $S > 0$ and $m \geqslant 3$, then

$$\varphi(S, z) = 1 + \frac{(2\pi)^{m/2}}{\Gamma(m/2)} d^{-1/2} \sum_{t=1}^{\infty} \prod_p \alpha_p(S, t) t^{\frac{m}{2}-1} e^{2\pi i t z} \tag{27}$$

and if $n(m-n) \neq 0$

$$\varphi(S, z) = 1 + \frac{(2\pi)^{m/2} |d|^{-1/2}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m-n}{2}\right)} \sum_{t=-\infty}^{\infty} \prod_p \alpha_p(S, t) h_t(z) \tag{28}$$

where $h_t(z) = h(t, z, 0)$.

6. Let us now consider the theta function associated with the quadratic form $\underline{x}' S \underline{x}$.

Let $\mathfrak{H}$ be the majorant space of S and dv the invariant volume measure in $\mathfrak{H}$. Let $\Gamma(S)$ be the unit group of S and F a fundamental region for $\Gamma(S)$ in $\mathfrak{H}$. Let V be the volume of F measured with dv. We then put

$$f(S, z) = \frac{1}{V} \int_F f(S, H, z) dv$$

where $f(S, H, z)$ is defined in (2). If $n(m-n) \neq 0$, Siegel [7] has shown that $(m \geqslant 2)$

$$f(S, z) = 1 + \sum_{t=-\infty}^{\infty} a_t(S) h_t(z) \tag{29}$$

where

$$a_t(S) = \frac{(2\pi)^{m/2}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m-n}{2}\right)} |d|^{-1/2} \frac{M(S, t)}{\mu(S)}, \tag{30}$$

where $M(S, t)$ is the measure of representation of t by S and $\mu(S)$ is the measure of the unit group. From Section 5 we have

$$e^{\frac{\pi i}{4}(2n-m)} \int_{-\infty}^{\infty} \frac{e^{-2\pi i t x}}{(x+z)^{n/2}(x+\bar{z})^{(m-n)/2}} dx = \frac{(2\pi)^{m/2}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m-n}{2}\right)} h_t(z). \tag{31}$$

We can therefore write for $m \geqslant 2$, under the restrictions on S given in Lemma 1,

$$f(S, z) = 1 + \sum_{t=-\infty}^{\infty} |d|^{-1/2} \frac{M(S, t)}{\mu(S)} e^{\frac{\pi i}{4}(2n-m)} A(t, 0, z). \tag{32}$$

In this set up the right side also makes sense for $S > 0$. For, from Siegel [7], we see that for $S > 0$

$$M(S, t) = \varrho_{m-1} \cdot |S|^{-m/2} t^{1-\frac{m}{2}} \frac{A(S, t)}{E(S)}$$

where $A(S, t)$ is the number of representations of t by S and $E(S)$ is the order of the finite group $\Gamma(S)$. Further

$$\varrho_k = \prod_{l=1}^{k} \frac{\pi^{l/2}}{\Gamma(l/2)}, \quad k = 1, 2, \ldots$$

and $\varrho_0 = 1$. Also

$$\mu(S) = \frac{\varrho_m |S|^{-(m+1)/2}}{E(S)}.$$

We therefore obtain

$$|d|^{-1/2} \frac{M(S, t)}{\mu(S)} = (2\pi)^{-m/2} \cdot \Gamma\left(\frac{m}{2}\right) t^{1-\frac{m}{2}} A(S, t).$$

Using (20) we get

$$|d|^{-1/2} \frac{M(S, t)}{\mu(S)} e^{\pi i m/4} A(t, 0, z) = A(S, t) e^{2\pi i t z}. \tag{33}$$

(33) shows that for $S > 0$, (32) reduces to

$$f(S, z) = 1 + \sum_{t > 0} A(S, t) e^{2\pi i t z}$$

which is equal to $\sum_{\underline{x}} e^{2\pi i S[\underline{x}] z}$. (32) thus makes sense for all S definite or indefinite; in case S is indefinite it satisfies conditions of Lemma 1.

Let now $S_1, S_2, \ldots, S_h$ be a complete system of representatives of classes of the genus of S $(= S_1)$. Put

$$M(S) = \sum_{k=1}^{h} \mu(S_k), \tag{34}$$

$M(S)$ is called the measure of the genus of S.

Siegel has shown that [6]

$$\alpha_p(S, S) = \tfrac{1}{2}\lim_{k\to\infty} p^{-km(m-1)/2} A_{p^k}(S, S)$$

exists for every prime p and that further the product

(35) $$\prod_p \alpha_p(S, S)$$

converges when the product is extended over all primes in their natural order. Put

(36) $$\tau(S) = M(S)\prod_p \alpha_p(S, S).$$

Minkowski and Siegel realized the importance of this number for the analytical theory of quadratic forms. (If we put $\varepsilon_m = 1$ for $m = 1$ and $\frac{1}{2}$ if $m > 1$, then $\tau(S)$ is the same as $\varrho(S)/\varepsilon_m$ in the notation of Siegel ([6], I, p. 563).

Let us now write

(37) $$\alpha(S, t) = \frac{\sum_{k=1}^{h} M(S_k, t)}{M(S)}$$

where $M(S_k, t)$ is defined in (30) and $M(S)$ in (34). $\alpha(S, t)$ like $\tau(S)$ depends only on the genus of S. Put

(38) $$F(S, z) = \frac{\sum_{k=1}^{h} \mu(S_k) f(S_k, z)}{M(S)}$$

where $f(S_k, z)$ is given in (32). We call $F(S, z)$ the *theta function associated with the genus of* S. From (32) and (37) we get

(39) $$F(S, z) = 1 + \sum_{t=-\infty}^{\infty} \alpha(S, t)\, |d|^{-1/2} e^{\frac{\pi i}{4}(2n-m)} A(t, 0, z).$$

We now prove the main

THEOREM. *Let $m \geqslant 3$ and let S satisfy conditions of Lemma 1. Then*

$$F(S, z) = \varphi(S, z).$$

In order to prove the theorem we require certain analytical and arithmetical facts.

Consider the modular matrices $M = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ with a_1, b_1, c_1, d_1 integers satisfying $a_1 d_1 - b_1 c_1 = 1$ and

(40) $$c_1 \geqslant 0, \quad c_1 \equiv 0 \pmod{2d}.$$

Siegel [7] has shown that if $M\langle z\rangle = \dfrac{a_1 z + b_1}{c_1 z + d_1}$, then

(41) $$(c_1 z + d_1)^{-n/2} (c_1 \bar{z} + d_1)^{-(m-n)/2} F(S, M\langle z\rangle) = \gamma\left(\frac{a_1}{c_1}\right) F(S, z)$$

where $\gamma\left(\frac{a_1}{c_1}\right) = 1$ if $c_1 = 0$ and $a_1 = 1$ otherwise.

(42) $$\gamma\left(\frac{a_1}{c_1}\right) = e^{\frac{\pi i}{4}(2n-m)} |d|^{-1/2} c_1^{-m/2} \sum_{\underline{u} \,(\mathrm{mod}\, c_1)} e^{\frac{2\pi i a_1}{c_1} S[\underline{u}]}.$$

Also by the way the function $\varphi(S, z, s)$ has been defined we see that

(43) $$(c_1 z + d_1)^{-(n+s)2} (c_1 \bar{z} + d_1)^{-(m-n+s)/2} \varphi(S, M\langle z\rangle, s) = \gamma\left(\frac{a_1}{c_1}\right) \varphi(S, z, s).$$

By continuity therefore $\varphi(S, z)$ also satisfies equation (41) with $\varphi(S, z)$ in place of $F(S, z)$.

Let us denote the number $\tau(S)$ by $\tau_m(S)$ to show its dependence on m the order of the matrix S. A fundamental result due to Minkowski–Siegel is that for $m \geqslant 2$

$$\tau_m(S) = 2$$

whether S is definite or indefinite. In order to prove the theorem, we shall assume that by for binary matrices S, definite or indefinite

$$\tau_2(S) = \tau_2$$

is independent of S. Suppose therefore as inductive hypothesis that we have proved that for $m-1 \geqslant 2$, $\tau_{m-1}(T)$ is independent of the $(m-1)$-rowed symmetric semi-integral matrix T. Then it follows from the Gauss–Eisenstein–Dirichlet–Smith–Minkowski–Siegel formula [6] that for S of m rows, $t \neq 0$

(44) $$M(S, t) = \frac{\tau_{m-1}}{\tau_m(S)} \prod_p \alpha_p(S, t), \qquad m \geqslant 3.$$

Let us put

$$g(S, z) = F(S, z) - \frac{\tau_{m-1}}{\tau_m(S)} \varphi(S, z)$$

so that from (44), (39) and (32) we get

$$g(S, z) = u + v\eta^{1-\frac{m}{2}}$$

where

(45) $$u = 1 - \frac{\tau_{m-1}}{\tau_m(S)}, \qquad v = \Gamma\left(\frac{m}{2} - 1\right)(4\pi)^{1-\frac{m}{2}} \left(M(S, 0) - \frac{\tau_{m-1}}{\tau_m(S)} \prod_p \alpha_p(S, 0)\right).$$

Since $F(S, z)$ and $\varphi(S, z)$ satisfy the transformation formula (41) we see that $g(S, z)$ also satisfies the same formula. It should be noted that since $m \geqslant 3$, for $S > 0$ the second expression $v = 0$ since $A(0, 0, z) = 0$.

Let us now consider the case $S > 0$ and $m \geqslant 3$. Since $g = 1 - \frac{\tau_{m-1}}{\tau_m(S)}$ satisfies the transformation formula (41), it follows that

$$\tau_m(S) = \tau_{m-1} \tag{46}$$

and so for definite S

$$\tau_m = \tau_2, \quad m \geqslant 2 \tag{47}$$

and $g = 0$ which means that, from (47),

$$F(S, z) = \varphi(S, z).$$

Let us therefore assume that S is indefinite so that $n(m-n) \neq 0$. We also assume that S satisfies the conditions of Lemma 1. We can assume, without loss in generality that $n \geqslant m-n$. First let $n > m-n > 0$. Then

$$|c_1 z + d_1|^{-(m-n)}(c_1 z + d_1)^{-(2n-m)/2} g(S, M\langle z\rangle) = \gamma\left(\frac{a_1}{c_1}\right) g(S, z). \tag{48}$$

Since $(2n-m)/2 > 0$, all the terms in (48) except $(c_1 z + d_1)^{-(2n-m)/2}$ (taken with principal part) are non-analytic. Therefore

$$u = 0, \quad v = 0.$$

This again gives the theorem.

We have therefore to consider the case $n = m-n > 0$. Then

$$|c_1 z + d_1|^{-m/2}\left(u + v\frac{\eta^{1-\frac{m}{2}}}{|c_1 z + d_1|^{2-m}}\right) = \gamma\left(\frac{a_1}{c_1}\right)\left(u + v\eta^{1-\frac{m}{2}}\right).$$

Let us take $c_1 > 0$, $z = -\frac{d_1}{c_1} + i\eta$, $\eta > 0$ and take the limit as $\eta \to 0$. Then

$$c_1^{-m/2}\left(u\eta^{-1} + \frac{v}{c_1^{2-m}}\eta^{\frac{m}{2}-2}\right) = \gamma\left(\frac{a_1}{c_1}\right)\left(u\eta^{\frac{m}{2}-1} + v\right). \tag{49}$$

If $m > 4$ then when $\eta \to 0$ the left side of (49) tends to infinity if $u \neq 0$ whereas the right side is finite. Therefore

$$u = 0.$$

Again using (49) it follows that $v = 0$. Thus the theorem is proved in this case.

We are only left with the case $n = m-n = 2$. Again from (41) we get $u = 0$ and

$$v = \gamma\left(\frac{a_1}{c_1}\right) v. \tag{50}$$

We shall show that $\gamma\left(\frac{a_1}{c_1}\right) \neq 1$ at least for one M satisfying (40). Suppose that $\gamma\left(\frac{a_1}{c_1}\right) = 1$ for all M satisfying (40). Then by definition of $\gamma\left(\frac{a_1}{c_1}\right)$,

$$1 = \gamma\left(\frac{a_1}{c_1}\right) = d^{-1/2} c_1^{-2} \sum_{\underline{u} (\mathrm{mod}\, c_1)} e^{\frac{2\pi i a_1}{c_1} S[\underline{u}]}, \quad c_1 > 0.$$

Therefore

$$\sum_{\underline{u} (\mathrm{mod}\, c_1)} e^{\frac{2\pi i a_1}{c_1} S[\underline{u}]} = d^{1/2} c_1^2. \tag{51}$$

Given any integer $c_1 > 0$ satisfying (40) and any $a_1 > 0$ prime to c_1 one can always find a matrix M with $\binom{a_1}{c_1}$ as the first column. Thus (51) is true for any given c_1 and any a_1 prime to c_1. This shows that the left side of (51) is a rational integer. Thus d is a square. Since $n = m-n = 2$ this would mean that S is the matrix of a quaternionic zero form. This has been excluded in the statement of the theorem.

Our theorem is completely proved.

As a corollary we have

$$\tau_m(S) = \tau_2$$

for all $m \geqslant 2$ and all S. Further

$$M(S, 0) = \prod_p \alpha_p(S, 0), \quad m \geqslant 3. \tag{52}$$

As a matter of fact $\tau_m(S) = \tau_2 = 2$ as stated in the beginning. This can be proved by using the Dirichlet method as Minkowski and Siegel have done.

It is to be noticed however that Siegel has proved that $\tau_m(S) = 2$ for all $m \geqslant 2$ even in the cases of ternary and quaternionic zero forms omitted in Lemma 1. This can be very easily proved by using a very interesting idea due to Siegel (see [6], II, pages 255–256).

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10_2	[97], [118]	[97], [109], [118]
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