($|a|+|b|>0$) may have at least $n$ integer solutions ($n$ given) for $c = 2(ab)^{k+1}d^{2k+1}$ and appropriately chosen $d$, viz. $x = ap^2$, $y = bq^2$, $z = a^{k+1}p^{2k+1}+b^{k+1}q^{2k+1}$, where $p$, $q$ are complementary divisors of $d$, i.e. $d = pq$.

**4.** The equation

$$(x+k)\ldots(x+kn) = y(y+1)\ldots(y+n)$$

has a solution for every $k$ and $n$: $x = k^{n+1}$, $y = k^n$.

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# The representation of real numbers by infinite series of rationals

by

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*Wacław Sierpiński in memoriam*

**1.** In a recent note Galambos [1] has obtained some remarkable theorems about the ergodic properties of the denominators in the expansion

$$x \sim \frac{1}{d_1}+\frac{a_1}{b_1}\cdot\frac{1}{d_2}+\frac{a_1a_2}{b_1b_2}\cdot\frac{1}{d_3}+\ldots; \tag{1.1}$$

he refers (Ref. 9 in Galambos [1]) to unpublished work of mine on this expansion. It seems appropriate now to give detailed results.

The expansion for any $x>0$ (not necessarily confined to the interval $(0,1)$) derives from the algorithm

$$x = x_1,\quad d_i = 1+[1/x_i],\quad x_i = 1/d_i+(a_i/b_i)x_{i+1} \tag{1.2}$$

for $i = 1, 2, \ldots$ Herein

$$a_i = a_i(d_1, d_2, \ldots, d_i),\quad b_i = b_i(d_1, d_2, \ldots, d_i)$$

are positive numbers (usually integers).

Several questions arise:

(i) to give conditions to ensure that the infinite series (necessarily convergent) in (1.1) has sum $x$;

(ii) to obtain the conditions induced by the algorithm on the integers $d_i \geqslant 1$ (one such condition is

$$d_{i+1} > (a_i/b_i)d_i(d_i-1)); \tag{1.3}$$

(iii) to obtain necessary and sufficient conditions that a convergent infinite series (1.1) shall be the expansion of its sum by the algorithm. (A simple set of sufficient conditions is given by

$$d_{i+1}-1 \geqslant (a_i/b_i)d_i(d_i-1).) \tag{1.4}$$

If the sequences $\{a_i\}$, $\{b_i\}$ are both sequences of natural numbers, a further question may be proposed:

(iv) determine the kind of expansion which occurs for rational $x$.

It will be observed that the series of Sylvester, of Engel (both originally found by Lambert) and of Lüroth (for these series see Perron [3], 116–127), the infinite product of Cantor (see Perron [3], 116–127) and its generalisation (Oppenheim [2]) are all special cases of the expansion (1.1).

Two main cases arise: (i) when the sequences $\{a_i\}$, $\{b_i\}$ are prescribed, i.e. the same for all $x$; (ii) when the members of the sequences are chosen in succession as described above.

The first case itself splits up into two cases according as the series of positive terms

$$A = A_1 = 1+\frac{a_1}{b_1}+\frac{a_1 a_2}{b_1 b_2}+\ldots \tag{1.5}$$

is divergent or convergent.

If $A = \infty$, then the algorithmic expansion (1.1) always converges to $x$. If $A < \infty$ further conditions are needed. Let $A_1 = 1+(a_1/b_1)A_2$, $A_2 = 1+(a_2/b_2)A_3$, etc. Then provided that $A_i \geqslant 3/2$ $(i = 1, 2, \ldots)$ and $x \epsilon (0, A_1]$ the expansion (1.1) does have sum $x$.

In all cases (whether the $a_i, b_i$ are prescribed initially or not) we obtain

$$x = u_1+u_2+\ldots+u_i+v_{i+1} \tag{1.6}$$

where

$$\begin{aligned} u_i &= a_1 \ldots a_{i-1}/b_1 \ldots b_{i-1}d_i \quad (i = 1, 2, \ldots) \quad (a_0 = b_0 = 1), \\ v_i &= u_i d_i x_i, \quad\quad (i = 2, 3, \ldots). \end{aligned} \tag{1.7}$$

The series $\sum_1^\infty u_i$ converges to $u \leqslant x$, $u_i \to 0$, $v_i$ tends monotonically decreasing to $v \geqslant 0$ and $u+v = x$. If $v = 0$, which is certainly the case if the sequence $\{x_i\}$ has a bounded subsequence, then $u = x$, i.e. the series (1.1) has sum $x$.

Complete (but complicated) necessary and sufficient conditions can be given for the expansion. When for each $i$ the number $(a_i/b_i)d_i(d_i-1)$ is an integer these conditions reduce to the set (1.4).

The nature of the expansion for rational $x$ (when the sequences $\{a_i\}$, $\{b_i\}$ are both sequences of natural numbers) can be settled in each of the following cases

$$\begin{aligned} &b_i \mid d_i \quad (i = 1, 2, 3, \ldots), \\ &a_i = 1, \quad b_i = d_i e_i \quad (e = e(d) \text{ unique}) \quad (i = 1, 2, \ldots), \\ &b_{2i} = a_{2i-1}, \quad a_{2i} = b_{2i-1} = 1 \quad (i = 1, 2, \ldots). \end{aligned} \tag{1.8}$$



In the first and third of these cases we have ultimately

$$d_{i+1}-1 = (a_i/b_i)d_i(d_i-1);$$

in the second case the sequence $\{d_i\}$ is ultimately periodic. The first case $(b_i \mid d_i$ all $i)$ includes the series of Sylvester and of Engel; it includes also the infinite product of Cantor. The second case includes Engel's series and Lüroth's series. The third case appears to be quite new.

Nevertheless there are simple cases not included in the above which appear to be very difficult to decide. Thus take $a_i = 1$, $b_i = 2$ (all $i$): each $x \epsilon (0, 2]$ has the unique expansion

$$x = 1/d_1+1/2d_2+1/4d_3+1/8d_4+\ldots$$

where the positive integers $d_i$ satisfy the conditions

$$d_{i+1}-1 \geqslant \tfrac{1}{2}d_i(d_i-1),$$

these conditions being both necessary and sufficient. I believe but I cannot prove that for $x$ rational, $0 < x < 2$, we must have eventually $d_{i+1}-1 = \frac{1}{2}d_i(d_i-1)$.

Otherwise expressed, let $x = x_1 = p_1/q_1$, $x_i = p_i/q_i$ for coprime integers $p_i, q_i \geqslant 1$, $0 < x_1 < 2$; let

$$\frac{p_{i+1}}{q_{i+1}} = 2\left(\frac{p_i}{q_i}-\frac{1}{d_i}\right), \quad d_i = 1+\left[\frac{q_i}{p_i}\right] \quad (i \geqslant 1).$$

Then the conjecture is: eventually

$$p_i = 1, \quad q_{i+1} = \tfrac{1}{2}q_i(q_i+1).$$

**2.** I give now in detail various theorems relating to these expansions.

THEOREM 1. *The algorithm defined by*

$$d_i = 1+[1/x_i] \quad (i = 1, 2, \ldots), \quad x = x_1 > 0,$$
$$x_i = 1/d_i+(a_i/b_i)x_{i+1}$$

*always yields a series* (1.1) *convergent to a sum* $u \leqslant x$.

*The remainder after* $i$ *terms*

$$v_{i+1} = (a_1 a_2 \ldots a_i/b_1 b_2 \ldots b_i)x_{i+1}$$

*decreases monotonically to* $v \geqslant 0$ *and* $u+v = x$.

*If the sequence* $\{x_i\}$ *has a bounded subsequence or if infinitely many* $d_i \geqslant 2$, *then* $v = 0$ *and the expansion has sum* $x$.

Only the last statement requires proof. If $d_{i+1} \geqslant 2$ infinitely often then, for this subsequence of $i$,

$$v_{i+1} \leqslant \frac{a_1 \dots a_i}{b_1 \dots b_i} \cdot \frac{1}{d_{i+1}} \cdot \frac{d_{i+1}}{d_{i+1}-1} \leqslant 2u_{i+1}$$

which $\to 0$ as $i \to \infty$.

If however $d_i = 1$ (all large $i$) then $(a_1 \dots a_i/b_1 \dots b_i) \to 0$ as $i \to \infty$; $v_{i+1} \to 0$ if the sequence $\{x_i\}$ has a bounded subsequence.

THEOREM 2. *The integers $d_i$ satisfy the following inequalities: if $d_i > 1$*

$$(2.1) \quad \frac{1}{d_i-1} > \frac{1}{d_i} + \frac{a_i}{b_i d_{i+1}} + \dots + \frac{a_i \dots a_{i+j}}{b_i \dots b_{i+j} d_{i+j+1}} \quad (i \geqslant 1,\ j \geqslant 0).$$

*In particular, for $d_i > 1$,*

$$d_{i+1} > (a_i/b_i) d_i (d_i - 1),$$

*an inequality trivially true for $d_i = 1$.*

The conditions (2.1) are also sufficient to ensure that a convergent series (1.1) shall be the expansion of its sum by the algorithm.

A simple set of sufficient conditions is given by

$$d_{i+1} - 1 \geqslant (a_i/b_i) d_i (d_i - 1).$$

We need only prove the last two statements. Suppose that (1.1) is convergent and that (2.1) is true. With an obvious notation we require to prove that for $d_i \geqslant 2$

$$\frac{1}{d_i-1} \geqslant x_i > \frac{1}{d_i}$$

(the inequalities being trivially true if $d_i = 1$). But when $j \to \infty$, the right-hand side of (2.1) $\to x_i$ so that $1/(d_i-1) \geqslant x_i$ while $x_i > 1/d_i$ is trivial.

As for the last statement in Theorem 2 we need only consider $d_i > 1$ (and therefore all subsequent $d_i > 1$) and note that

$$\frac{1}{d_i-1} - x_i = \frac{1}{d_i(d_i-1)} - \frac{a_i}{b_i} x_{i+1} \geqslant \frac{a_i}{b_i}\left(\frac{1}{d_{i+1}-1} - x_{i+1}\right)$$

so that

$$\frac{1}{d_i-1} - x_i \geqslant \frac{a_i \dots a_{i+j}}{b_i \dots b_{i+j}} \cdot \frac{1}{d_{i+j+1}-1} - \frac{a_i \dots a_{i+j}}{b_i \dots b_{i+j}} x_{i+j+1},$$

but the right-hand side $\to 0$ when $j \to \infty$. And the result follows.

As a deduction from Theorem 1 we have

THEOREM 3. *Suppose that the sequences $\{a_i\}$, $\{b_i\}$ are prescribed and that the series $\sum a_1 \dots a_i/b_1 \dots b_i$ is divergent. Then for every $x > 0$ the series (1.1) given by the algorithm has sum $x$.*



For, if $d_i = 1$ all large $i$, the series reduces to the divergent series $\sum a_1 \dots a_i/b_1 \dots b_i$ whereas $\sum u_i \leqslant x$. Thus $d_i \geqslant 2$ infinitely often. Theorem 1 applies.

The next theorem considers the case $A < \infty$.

THEOREM 4. *Suppose that*

$$A = A_1 = 1 + \frac{a_1}{b_1} + \frac{a_1 a_2}{b_1 b_2} + \dots < \infty,$$

*let*

$$A_1 = 1 + (a_1/b_1) A_2, \quad A_2 = 1 + (a_2/b_2) A_3, \quad \dots$$

*Then the necessary and sufficient conditions that every $x \in (0, A_1]$ be represented by* (1.1) *are that*

$$A_i \geqslant 3/2 \quad (i = 1, 2, \dots).$$

If any one of these conditions is violated, there exist $x \in (0, A_1]$ which cannot be represented: $u(x) < x$.

Plainly the greatest number which can be represented is $A_1$. If $x_2 > A_2$, then $x_2$ cannot be represented, and so on. Now if $d_1 = 1$ we have

$$\frac{a_1}{b_1} x_2 = x_1 - 1 \leqslant A_1 - 1 = \frac{a_1}{b_1} A_2, \quad x_2 \leqslant A_2.$$

If however $d_1 \geqslant 2$ we obtain

$$\frac{a_1}{b_1} x_2 \leqslant \frac{1}{d_1(d_1-1)} \leqslant \frac{1}{2}$$

and since we can take the left member as close to $\frac{1}{2}$ as we wish we require

$$\frac{1}{2} \leqslant \frac{a_1}{b_1} A_2 = A_1 - 1, \quad \text{i.e.} \quad A_1 \geqslant \frac{3}{2}.$$

Similar argument apply to $A_2, A_3, \dots$

**3. Expansions for rational $x$.** I give separately the results for the cases mentioned in (1.8).

THEOREM 5. *Suppose that, for each $i$, $b_i \mid d_i$. Then for $x$ rational we have eventually*

$$d_{i+1} - 1 = (a_i/b_i) d_i (d_i - 1).$$

With the notation used earlier each $x_i$ is rational, $x_i = p_i/q_i$ ($p_i$, $q_i$ coprime positive integers) and

$$\frac{p_{i+1}}{q_{i+1}} = \frac{p_i d_i - q_i}{a_i (d_i/b_i) q_i}, \quad 0 < p_i d_i - q_i \leqslant p_i.$$

Hence $1 \leqslant p_{i+1} \leqslant p_i d_i - q_i \leqslant p_i$, $p_i = P \geqslant 1$ all large $i$,

$$P = P d_i - q_i, \quad P \mid q_i, \quad P = 1; \quad q_i = d_i - 1$$

and so

$$d_{i+1}-1 = a_i(d_i/b_i)(d_i-1)$$

as required.

The case $b_i = 1$ (all $i$) includes Sylvester's series. The case $b_i = d_i$ (all $i$) includes Engel's series. The case $b_i = d_i$, $a_i = 1+d_i$ yields Cantor's product.

THEOREM 6. *Suppose that, for each $i$, $a_i = 1$ and $b_i = d_i e_i$ where $e_i$ is given uniquely as a function of $d_i$. Then for rational $x$ the sequence $d_i$ is ultimately periodic.*

Here we have

$$\frac{p_{i+1}}{q_{i+1}} = \frac{e_i(p_i d_i - q_i)}{q_i}, \qquad 0 < p_i d_i - q_i \leqslant p_i.$$

Hence $1 \leqslant q_{i+1} \leqslant q_i$, $q_i = Q \geqslant 1$ all large $i$; $p_{i+1} = e_i(p_i d_i - q_i)$. Two cases arise: either $d_i = 1$ (all large $i$) (so that $\{d_i\}$ is ultimately periodic) or else $d_i \geqslant 2$ infinitely often. But in the later case $p_i(d_i-1) \leqslant q_i$ yields $p_i \leqslant Q$ infinitely often. Hence there exist suffixes $i, j$ such that

$$p_i = p_j, \qquad q_i = q_j = Q, \qquad i < j.$$

Clearly the sequence $\{d_i\}$ is now periodic.

The case $b_i = d_i$ yields Engel's series; the case $b_i = d_i - 1$ (for $0 < x < 1$ so that each $d_i \geqslant 2$) yields Lüroth's series.

THEOREM 7. *Suppose that*

$$b_{2i} = a_{2i-1}, \qquad a_{2i} = b_{2i-1} = 1 \qquad (i = 1, 2, \ldots)$$

*so that the expansion has the form*

$$1/d_1 + a_1/d_2 + 1/d_3 + a_3/d_4 + \ldots$$

*Then for rational $x$ from some point on*

$$d_{2i}-1 = a_{2i-1}d_{2i-1}(d_{2i-1}-1),$$
$$d_{2i+1}-1 = d_{2i}(d_{2i}-1)/a_{2i-1}.$$

In the proof of Theorem 5 we showed that the sequence $\{p_i\}$ is monotonic decreasing. Here we show first that the sequence $\{p_{2i-1}\}$ is monotonic decreasing. It is sufficient to consider $p_i/q_i$ $(i = 1, 2, 3)$. Here

$$\frac{p_2}{q_2} = \frac{1}{a_1}\cdot\frac{p_1 d_1 - q_1}{q_1 d_1}, \qquad 0 < p_1 d_1 - q_1 \leqslant p_1,$$
$$\frac{p_3}{q_3} = \frac{a_1(p_2 d_2 - q_2)}{q_2 d_2}, \qquad 0 < p_2 d_2 - q_2 \leqslant p_2.$$



Let $(a_1, q_2) = g \geqslant 1$; $a_1 = gA$, $q_2 = gQ$ where $(A, Q) = 1$. Then $A \mid (p_1 d_1 - q_1)$ and so for some integer $\lambda \geqslant 1$

$$\lambda A p_2 = p_1 d_1 - q_1 \leqslant p_1.$$

Also

$$\mu p_3 = A(p_2 d_2 - q_2) \leqslant A p_2$$

for some integer $\mu \geqslant 1$. Hence

$$p_3 \leqslant \lambda\mu p_3 \leqslant p_1.$$

Repetition of the argument yields $p_1 \geqslant p_3 \geqslant p_5 \ldots$ and so, for all large $i$, $p_{2i-1} = P \geqslant 1$. Renumber so that $p_{2i-1} = P$ for $i = 1, 2, \ldots$ Then $\lambda\mu P \leqslant P$ so that each $\lambda$ and each $\mu$ must be 1. And now $Ap_2 = P$, $P = Pd_1 - q_1$, $P = 1$, $A = 1$ (each $A = 1$), $p_2 = 1$ (each $p_{2i} = 1$), whence finally

$$q_i = d_i - 1, \qquad q_{2i} = a_{2i-1}q_{2i-1}d_{2i-1}, \qquad q_{2i+1}a_{2i-1} = q_{2i}d_{2i}$$

and Theorem 7 is proved.

As stated earlier this theorem appears to be new.

**4.** I conclude with a theorem about the expansion by the algorithm

$$1/d_1 + a_1/d_2 + 1/d_3 + a_3/d_4 + \ldots$$

where each integer $a_1, a_3, \ldots$ is chosen from the set $\{1, 2, 3, 4\}$.

THEOREM 8. *The necessary and sufficient conditions satisfied by the $d_i$ are*

$$d_{2i}-1 \geqslant a_{2i-1}d_{2i-1}(d_{2i-1}-1),$$
$$d_{2i+1}-1 \geqslant d_{2i}(d_{2i}-1)/a_{2i-1},$$

*except in the following cases:*

*if*

$$a_{2i-1} = 3 \text{ and } d_{2i} \equiv 2 \pmod 3 \text{ and } 3d_{2i+1} = 1 + d_{2i}(d_{2i}-1),$$

*then*

$$d_{2i+2}-1 \geqslant a_{2i+1}d_{2i+1}(3d_{2i+1}-1);$$

*if*

$$a_{2i-1} = 4 \text{ and } d_{2i} \equiv 2 \text{ or } 3 \pmod 4 \text{ and } 2d_{2i+1} = 1 + \tfrac{1}{2}d_{2i}(d_{2i}-1),$$

*then*

$$d_{2i+2}-1 \geqslant a_{2i+1}d_{2i+1}(2d_{2i+1}-1).$$

I omit the proof. It is plain that this result can be extended.

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# On a linear diophantine problem of Frobenius

by

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**Introduction.** Given integers $0 < a_1 < \ldots < a_n$ with $\gcd(a_1, \ldots, a_n) = 1$, it is well-known that the equation $N = \sum_{k=1}^{n} x_k a_k$ has a solution in non-negative integers $x_k$ provided $N$ is sufficiently large. Following [9], we let $G(a_1, \ldots, a_n)$ denote the greatest integer $N$ for which the preceding equation has no such solution.

The problem of determining $G(a_1, \ldots, a_n)$, or at least obtaining non-trivial estimates, was first raised by G. Frobenius (cf. [2]) and has been the subject of numerous papers (e.g., cf. [1], [2], [3], [4], [7], [8], [9], [11], [12], [13]). It is known that:

$$G(a_1, a_2) = (a_1-1)(a_2-1)-1 \qquad ([2], [11]);$$

$$G(a_1, \ldots, a_n) \leqslant (a_1-1)(a_n-1)-1 \qquad ([2], [4]);$$

$$G(a_1, \ldots, a_n) \leqslant \sum_{k=1}^{n-1} a_{k+1} d_k / d_{k+1}$$

where $d_k = \gcd(a_1, \ldots, a_k)$ ([2]). The exact value of $G$ is also known for the case in which the $a_k$ form an arithmetic progression ([1], [13]).

In this paper, we obtain the bound

$$G(a_1, \ldots, a_n) \leqslant 2a_{n-1}\left[\frac{a_n}{n}\right] - a_n,$$

which in many cases is superior to previous bounds and which will be seen to be within a constant factor of the best possible bound. We also consider several related extremal problems and obtain an exact solution in the case that $a_n - 2n$ is small compared to $n^{1/2}$.

**A general bound.** As before, we consider integers $0 < a_1 < \ldots < a_n$ with $\gcd(a_1, \ldots, a_n) = 1$.

Theorem 1.

$$G(a_1, \ldots, a_n) \leqslant 2a_{n-1}\left[\frac{a_n}{n}\right] - a_n. \tag{1}$$