Now (4) is true by induction for our particular value of \( t \), and hence each of the four parts has the lower bound \( 4^j 2^{-j} t \), and the sum (14) itself is at least \( 4 j 2^{-j} t = 2^{-3} 4^j t \). The same lower bound holds if \( h(j', a_j) \) in (14) is replaced by \( h(j, a_j) \), \( h(j', a_j) \) or \( h(j, a_j) \). We now take the sum of (13) over \( j = 1, 2, \ldots, d^{t+1} \), and we obtain

\[
2 \sum_{j=1}^{d^{t+1}} a_j \geq \sum_{j=1}^{d^{t+1}} a_j + \frac{1}{4} 2^{-3} 4^j t = 4^{t+1} t - 2^{-5} (t+1)
\]

by (12). Dividing by \( 2 \cdot 4^{t+1} \) and recalling the definition of \( a_j \) we obtain (4) with \( t \) replaced by \( t+1 \).

References


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An application of Minkowski's theorem in the geometry of numbers

by

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In memory of Professor Waclaw Sierpinski

The classical result in the geometry of numbers is given by Minkowski's

**Theorem 1.** An \( n \)-dimensional closed convex region symmetrical around the origin and of volume not less than \( 2^n \), contains a point other than the origin of every lattice \( L \) in \( n \) variables of determinant one.

Very few applications of this theorem are to be found in the usual literature. They are mostly concerned with sums of powers of linear forms or separate linear forms. As problems are rather scarce, I notice another application which may be of interest and which is given by

**Theorem 2.** Let \( L \) be a lattice in \( 3n \) variables \( (x_1, \ldots, x_{3n}) \) of determinant one. Then the region given by

\[
|x_r| \leq a (r = 1, \ldots, 2n), \quad \sum_{r=1}^{3n} |x_{2r-1} - x_{2r}| \leq 2b, \quad a > b,
\]

contains a point other than the origin of \( L \) if

\[
\frac{a^n}{n!} \left( \frac{1}{n!} \right) + \frac{a^{n-1}}{1 \cdot (n+1)!} + \frac{a^{n-2}}{2 \cdot (n+2)!} + \ldots + \frac{(-1)^m}{(2m)!} \right) \geq 2^{-n}.
\]

The condition \( a > b \) is imposed to exclude the case when the lattice \( L \) contains a sublattice \( a \) of determinant one in \( x_1, x_2 \); for if \( a < b \), a trivial solution may exist in which \( x_1 = \ldots = x_{2n} = 0 \).

We have to express the condition that the volume \( V \) of (1) is \( \geq 2^n \).

Make the substitution

\[
\sqrt{2}x_{2r-1} \rightarrow x_{2r-1} + x_{2r}, \quad \sqrt{2}x_{2r} \rightarrow x_{2r-1} - x_{2r} \quad (r = 1, 2, \ldots, n).
\]
Then,
\[
\sum_{r=1}^{n} |x_{r-1}| \leq b\sqrt{2},
\]
(3)
\[
|x_{r+1} - x_r| \leq a\sqrt{2},
\]
(4)
\[
|x_{r+1} - x_{r-1}| \leq a\sqrt{2} \quad (r = 1, \ldots, n).
\]
(5)
Clearly (4) is equivalent to
\[
|x_{r-1}| + |x_r| \leq a\sqrt{2} \quad (r = 1, \ldots, n).
\]
(6)
Make the substitution \(x \to a\sqrt{2}x\). Then
\[
V = 2^{an}(a\sqrt{2})^{2a} V_1,
\]
where \(V_1\) is the volume of the region
\[
\sum_{r=1}^{n} x_{r-1} \leq b/a = c, \quad x_r \geq 0 \quad (r = 1, \ldots, 2n),
\]
(7)
\[
x_{r-1} + x_r \leq 1 \quad (r = 1, \ldots, n).
\]
Hence, integrating for \(x_r\), we have
\[
V_1 = \int \cdots \int (1 - x_1)(1 - x_2) \cdots (1 - x_{2n-1}) \, dx_1 \cdots dx_{2n-1}
\]
taken over the region
\[
0 \leq x_{r-1} \leq 1 \quad (r = 1, \ldots, n), \quad x_1 + x_2 + \cdots + x_{2n-1} \leq c.
\]
Since \(c < 1\), the inequality \(x_{r-1} \leq 1\) is redundant. Write \(x \to cx\). Then
\[
V_1 = c^n \int \cdots \int (1 - cx_1) \cdots (1 - cx_{2n-1}) \, dx_1 \cdots dx_{2n-1}, \quad \sum_{r=1}^{n} x_{r-1} \leq 1.
\]
(8)
Apply Dirichlet's integral
\[
\int \cdots \int x_1^{l_1-1} \cdots x_{s-1}^{l_{s-1}} \, dx_1 \cdots dx_s = \frac{\Gamma(l_1) \cdots \Gamma(l_s)}{\Gamma(l_1 + \cdots + l_s + 1)}.
\]
We have now
\[
V_1 = c^n \left( \frac{1}{n!} - \frac{ne}{(n+1)!} + \frac{n^2 - 1}{2!(n+2)!} \right.
\]
\[
- \frac{n^3}{3!(n+3)!} + \cdots + (-1)^{n-1} \frac{c^n}{(2n)!},
\]
and so we require
\[
V = 2^{an} a^{an} V_1 \geq 2^{an},
\]
or
\[
b^n \left( \frac{a^n}{n!} - \frac{na^{n-1}}{1!(n+1)!} b + \cdots + (-1)^n \frac{b^n}{(2n)!} \right) \geq 2^{-n}.
\]
(8)
We show now that the minimum value of (8) for given \(b\) occurs when \(a = b\). It suffices to show that if
\[
f(u) = \frac{u^n}{n!} - \frac{nu^{n-1}}{1!(n+1)!} + \frac{n(n-1)u^{n-2}}{2!(n+2)!} + \cdots + (-1)^n \frac{b^n}{(2n)!},
\]
then \(f(u) \geq f(v)\) if \(u \geq v > 1\). Write
\[
f(u) - f(v) = \frac{u^n - v^n}{n!} - \frac{n(u^{n-1} - v^{n-1})}{1!(n+1)!} + \cdots
\]
(10)
The terms in this series taken positively are steadily decreasing. This requires
\[
\frac{n(n-1)\cdots(n-r+1)}{r!(n+r)!} (u^{n-r} - v^{n-r}) > \frac{n(n-1)\cdots(n-r)}{(r+1)!(n+r+1)!} (u^{n-r-1} - v^{n-r-1}),
\]
or
\[
u^{n-r} - v^{n-r} > \frac{n-r}{(r+1)(n+r+1)} (u^{n-r-1} - v^{n-r-1}).
\]
The result is obvious since
\[
u^{n-r} - v^{n-r} = u^{n-r-1} - v^{n-r-1} + v^{n-r-1} (u - v).
\]
Hence \(f(u) \geq f(1)\) if \(u > 1\) and so (8) is satisfied if
\[
\frac{1}{n!} - \frac{1}{1!(n+1)!} + \frac{n}{2!(n+2)!} + \cdots + (-1)^n \frac{1}{(2n)!} \geq 2^{-n}.
\]
This completes the proof.

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