ACTA ARITHMETICA
XXI (1972)

New theorems concerning the diophantine equation $x^2+D = 4y^q$

by

W. LJUNGGREN (Oslo)

1. Introduction. Let D denote a positive integer $\equiv 3 \pmod 4$, without any square factor > 1. Let further q denote any odd prime. It is the purpose of this paper to prove some new theorems concerning the solvability in positive rational integers x, y and q of the equation

$$x^2+D = 4y^q \tag{1}$$

for given values of D.

Putting $x = 2z+1$ the equation (1) can also be written

$$z^2+z+\tfrac{1}{4}(D+1) = y^q. \tag{1'}$$

For $D = 3$ and $q \neq 3$ T. Nagell [8] showed that (1′) has no solutions with $y \neq 1$. In case $D = 3$ and $q = 3$ Ljunggren [4] proved that $y = 1$ and $y = 7$ are the only solutions.

Let $h(\sqrt{-D}) = h$ be the number of classes of ideals in the algebraic number field $Q(\sqrt{-D})$. B. Persson [9] proved the following theorem:

The equation (1′) with $D > 3$ is solvable in integers x and y only for a finite number of integers D for a given q with $(h, q) = 1$, and the integral solutions y of (1′) are less than the number $\frac{1}{4}D\operatorname{cosec}^2\frac{\pi}{q}+1$. The equation has at most $\frac{1}{2}(q-1)$ solutions y when D and q are given.

The last part of this theorem was improved by B. Stolt [11] showing that there is at most one solution, except for the case $D \equiv 3 \pmod 8$ with $q \equiv 1 \pmod 6$, where there are at most three solutions.

In a recent paper [3] I investigated the equation (1) in case $D \equiv 7 \pmod 8$, and proved the following two theorems:

THEOREM 1. *Let $D \equiv 7 \pmod{24}$, and let at the same time one of the following two conditions be satisfied*

1° $q \equiv 3 \pmod 8$,

2° $q \equiv 5 \pmod 8$, $D-4 = 3^{2m+1}D_1$, $D_1 \not\equiv 0 \pmod 3$.

Then the equation (1) *has only a finite number of solutions in positive integers x and y and odd primes q, provided $(h, q) = 1$. If there are solutions these can be effectively found.*

THEOREM 2. *Let $D \equiv 15 \pmod{72}$. Then the equation* (1) *has only a finite number of solutions in positive integers x and y and odd primes q, provided $(h, q) = 1$. If there are solutions these can be effectively found.*

As to the case $D \equiv 3 \pmod 8$ we proceed proving the following new theorems:

THEOREM 3. *The equation* (1) *has only a finite number of solutions in positive integers x, y and odd primes $q \not\equiv \pm 1 \pmod{24}$, provided $(h, q) = 1$. If there are solutions these can be effectively found.*

THEOREM 4. *Let $D \equiv 51 \pmod{72}$. Then the equation* (1) *has only a finite number of solutions in positive integers x, y and odd primes $q \not\equiv -1 \pmod{24}$. If there are solutions these can be effectively found.*

In the proof use is made of the following lemma due to J. W. S. Cassels [2]: Let Π be a finite set of rational primes and let P be the set of positive integers all of whose prime factors are in Π. Let $F > 0$ and $E \neq 0$ be rational integers and suppose that no prime factors of E is in Π. Then there are only a finite number of solutions Z, Y of the equation

$$Z^2 - FY^2 = E,$$

where Z is a rational integer and $Y \epsilon P$. These can all be obtained in a finite number of steps.

2. If $(h, q) = 1$ it is easily proved that (1) implies

$$\tfrac{1}{2}(x + \sqrt{-D}) = \left(\tfrac{1}{2}(a + b\sqrt{-D})\right)^q, \tag{2}$$

a, b denoting rational integers, $a \equiv b \pmod 2$. See for instance Ljunggren [3]. Here is also proved that

$$b = \left(\frac{-D}{q}\right) = \pm 1$$

(equation (2) impossible for $D \equiv 0 \pmod q$).

From (2) it then follows

$$2^{q-1} b = q a^{q-1} - \binom{q}{3} a^{q-3} D + \ldots + (-D)^{\frac{1}{2}(q-1)}. \tag{3}$$

Putting

$$\lambda = \tfrac{1}{2}(a + \sqrt{-D}), \quad \lambda' = \tfrac{1}{2}(a - \sqrt{-D}), \quad a > 0$$

(2) may be written

$$\frac{\lambda^q - \lambda'^q}{\lambda - \lambda'} = b = \pm 1. \tag{2'}$$

For brevity we introduce the notations

$$T_m = \frac{\lambda^m - \lambda'^m}{\lambda - \lambda'}, \quad S_m = \lambda^m + \lambda'^m, \quad R_m = \frac{\lambda^m + \lambda'^m}{\lambda + \lambda'}, \tag{4}$$

where m denotes any positive integer in the expressions for T_m and S_m, and any odd positive integer in R_m.

The following formulas are easily verified

$$T^2_{\frac{1}{2}(q+1)} - \lambda\lambda' T_{\frac{1}{2}(q-1)} = b, \tag{5}$$

$$T_{\frac{1}{2}(q+1)} \cdot S_{\frac{1}{2}(q-1)} = b + (\lambda\lambda')^{\frac{1}{2}(q-1)}, \tag{6}$$

$$T_{\frac{1}{2}(q-1)} \cdot S_{\frac{1}{2}(q+1)} = b - (\lambda\lambda')^{\frac{1}{2}(q-1)}. \tag{7}$$

These three identities have proved to be very useful in dealing with problems of the type investigated in this paper. See for instance Ljunggren [4], [5], [6] and Aigner [1].

In case $D \equiv 3 \pmod 8$ we have $\lambda\lambda' \equiv 1 \pmod 2$. From the formulas

$$T_m = (a^2 - \lambda\lambda') T_{m-2} - a\lambda\lambda' T_{m-3},$$
$$S_m = (a^2 - \lambda\lambda') S_{m-2} - a\lambda\lambda' S_{m-3}$$

we conclude that all T_m (S_m) are odd integers for $m \not\equiv 0 \pmod 3$ and even integers for $m \equiv 0 \pmod 3$. Making use of the identities

$$T_6 = a T_3 R_3, \quad T_{6h+3} = T_{6h} S_3 - (\lambda\lambda')^3 T_{6h-3},$$

and noticing $R_3 \equiv T_3 \equiv 0 \pmod 2$, we find

$$T_m \equiv 0 \pmod 4 \text{ if } m \equiv 0 \pmod 6, \quad T_m \equiv \pm T_3 \text{ if } m \equiv 3 \pmod 6. \tag{8}$$

For later use we note

$$\begin{aligned} T_3 &= \tfrac{1}{4}(3a^2 - D), & S_2 &= \tfrac{1}{2}(a^2 - D), \\ R_3 &= \tfrac{1}{4}(3a^2 - D), & T_3 + R_3 &= a^2 - D. \end{aligned} \tag{9}$$

In case $D \equiv 3 \pmod 8$, $D > 3$ we have

$$\left(\frac{a + b\sqrt{-D}}{2}\right)^3 \epsilon Z[\sqrt{-D}], \quad a, b \text{ odd integers}.$$

Consequently, *there are no solutions of* (1) *for $q = 3$ and $(h, 3) = 1$.*

3. In this section we prove some lemmas with $b = \pm 1$.

LEMMA 1. *If the equation* (3) *is satisfied with $a \equiv 0 \pmod 3$, then either*

(i) $q \equiv 1 \pmod 4$, $\quad b = 1$, $\quad D^2 - 16 = 3^{2m_1} D_2$, $\quad (D_2, 3) = 1$ *and* $q \equiv 2DD_2 \pmod 3$

or

(ii) $q \equiv 3 \pmod 4$, $b \equiv -D \pmod 3$, $D+4b = 3^{2m_2}D_3$,

$(D_3, 3) = 1$ *and* $q \equiv DD_3 \pmod 3$.

Proof. The equation (1) implies

$$D^{\frac{1}{2}(q-1)} - b(-4)^{\frac{1}{2}(q-1)} = \sum_{i=1}^{\frac{1}{2}(q-1)} (-1)^{i-1}\binom{q}{2i} D^{\frac{1}{2}(q-1)-i} a^{2i}. \tag{10}$$

Putting $a = 3^s a_1$, $(a_1, 3) = 1$, $s \geqslant 1$ we observe that the first term on the right-hand side of (10) is exactly divisible by $3^{\delta+2s}$, $q-1 = 2q_1 3^{\delta}$, $(q_1, 3) = 1$, $\delta \geqslant 0$. The general term in the sum in (10) may be written in the form

$$\binom{q}{2} a^2 \binom{q-2}{2i-2} \frac{a^{2i-2}}{i(2i-1)} D^{\frac{1}{2}(q-1)-i}.$$

Here we have

$$3^{2i-2} > i(2i-1) \quad \text{for} \quad i \geqslant 2,$$

and consequently this term is divisible by a power of 3 with exponent greater than $\delta+2s$. Hence the right-hand side of (10) is exactly divisible by $3^{\delta+2s}$. In case $q \equiv 1 \pmod 4$ it is easily seen that the condition $a \equiv 0 \pmod 3$ implies $b = 1$. Putting further $D^2-16 = 3^m D'$, $(D', 3) = 1$, $m \geqslant 1$, we find that the quantity on the left-hand side of (10) is exactly divisible by $3^{\delta+m}$. Hence $m = 2s$. As to the case $D^2-16 = 3^{2s}D_2$, we easily find, when both sides of (11) are divided by $3^{\delta+2s}$ that $q \equiv 2DD_2 \pmod 3$. We have then proved the first part of the lemma.

In case $q \equiv 3 \pmod 4$ the left-hand side of (10) may be written

$$D^{\frac{1}{2}(q-1)} + (4b)^{\frac{1}{2}(q-1)},$$

which implies $b \equiv -D \pmod 3$. The remainder of the proof is similar to the proof of the first part of the lemma.

LEMMA 2. *If* $a^2 \equiv D \equiv 1 \pmod 3$, *then the equation* (1) *has no solutions in rational integers* x, y *and odd primes* q, q *satisfying the condition that the Legendre symbol* $(2/q) = -b$.

Proof. We note that $S_2 = \lambda^2 + \lambda'^2 \equiv 0 \pmod 3$ and $\lambda\lambda' \equiv -1 \pmod 3$. By means of the recurrence formula

$$T_m = T_{m-2} \cdot S_2 - (\lambda\lambda')^2 T_{m-4}$$

we deduce that $T_m \equiv -T_{m-4} \equiv T_{m-8} \pmod 3$. Hence $T_m \equiv 0 \pmod 3$ if and only if $m \equiv 0 \pmod 4$. By (5) the lemma then follows since $\frac{1}{2}(q \pm 1) \equiv 0 \pmod 4$ for $b = 1$ and $\not\equiv 0 \pmod 4$ for $b = -1$.

4. In this section we prove some lemmas, assuming $b = -1$.

LEMMA 3. *If* $D \equiv -1 \pmod 3$, *then the equation* (10) *has no solutions in case* $b = -1$.

Proof. If $a \equiv 0 \pmod 3$, then (10) gives $b \equiv 1 \pmod 3$, contrary to the assumption. If $a^2 \equiv 1 \pmod 3$, then $\lambda\lambda' \equiv 0 \pmod 3$, but this contradicts (5).

LEMMA 4. *Necessary conditions for the solvability of* (3) *with* $b = -1$ *are* $q \equiv -1 \pmod 3$ *and*

(i) $a^2 \leqslant D-2$ *for* $q \equiv 1 \pmod 8$ *or* $q \equiv 3 \pmod 8$,

(ii) $a^2 \leqslant 3D-8$ *for* $q \equiv 5 \pmod 8$.

Proof. Treating (5) as a congruence mod 4 we conclude by (8) that $q \equiv -1 \pmod 3$. In the remainder of the proof we distinguish three cases and let p denote any odd prime.

1° $q = 8r+1$. Using formula (7) we obtain

$$S_{4r+1} \cdot T_{4r} = -1-(\lambda\lambda')^{4r}. \tag{11}$$

S_2 is a divisor of T_{4r}. Let $p \mid S_2$, $S_2 > 0$. Since $S_2 \equiv -1 \pmod 8$ this number contains at least one prime factor not of the form $p \equiv 1 \pmod 4$, which contradicts (11), where the right-hand side only has odd prime factors $p \equiv 1 \pmod 8$. Consequently, $S_2 \leqslant -1$ since $S_2 = 1$ cannot occur, i.e.

$$a^2 \leqslant D-2.$$

2° $q = 8r+3$. Using (7) we obtain

$$S_{4r+2} \cdot T_{4r+1} = -1-(\lambda\lambda')^{4r+1}. \tag{12}$$

Assume $p \mid S_2$. Then (12) implies

$$\left(\frac{-\lambda\lambda'}{p}\right) = \left(\frac{-a^2-D}{p}\right) = 1 = \left(\frac{-2}{p}\right)$$

since $D \equiv a^2 \pmod p$, i.e. $p = 8h+1$ or $8h+3$. As in 1° we conclude

$$a^2 \leqslant D-2.$$

3° $q = 8r+5$. By (6) we find $T_{\frac{1}{2}(q+1)} \equiv 0 \pmod 8$, implying $T_3 \equiv 0 \pmod 8$. Hence $R_3 \equiv -2 \pmod 8$. Now making use of (7) we get

$$S_{4r+3} \cdot T_{4r+2} = -1-(\lambda\lambda')^{4r+2}. \tag{13}$$

$\frac{1}{2}R_3$ is an odd divisor of T_{4r+2}. Let $p \mid \frac{1}{2}R_3$, $R_3 > 0$. By (13) we deduce $p \equiv 1 \pmod 4$, which contradicts the fact that $\frac{1}{2}R_3 \equiv -1 \pmod 4$. Hence, as before, $\frac{1}{2}R_3 \leqslant -1$, i.e.

$$a^2 \leqslant 3D-8.$$

Remark. I have not succeeded in proving a similar theorem for $q = 8r+7$.

LEMMA 5. *The equation* (3) *has no solutions if*

$$D \equiv b \equiv -1 \pmod 3.$$

Proof. If $a \equiv 0 \pmod 3$, then (10) gives $b = 1$, contrary to the assumption. If $a^2 \equiv 1 \pmod 3$, then $\lambda\lambda' \equiv 0 \pmod 3$, but this is impossible on account of (5).

LEMMA 6. *A necessary condition for the solvability of* (3) *for* $b = -1$ *and* $q = 8r+7$ *is that* $q \equiv -1 \pmod 3$ *and either*

(i) $$D \equiv 0 \pmod 3$$

or

(ii) $$a \equiv 0 \pmod 3, \quad D \equiv 1 \pmod 3.$$

Proof. In virtue of Lemma 2 we can not have $D \equiv 1 \pmod 3$ and $a \not\equiv 0 \pmod 3$. The assumption $D \equiv -1 \pmod 3$ contradicts Lemma 5. As in Lemma 4 it is shown that $q \equiv -1 \pmod 3$, i.e. $q \equiv -1 \pmod{24}$.

5. In this section we prove some further lemmas assuming $b = 1$.

LEMMA 7. *Let* $p > 3$ *be a prime dividing* D, *and assume that* $q-1 \equiv 0 \pmod{(p-1)}$. *Then the equation* (3) *is not satisfied with* $b = 1$.

For a proof see T. Nagell [7] or Ljunggren [3].

LEMMA 8. *If* $D \equiv 6 \pmod 9$ *then the equation* (3) *is not satisfied with* $b = 1$.

For a proof see Ljunggren [3].

The next lemma is partly similar to Lemma 4.

LEMMA 9. *Necessary conditions for the solvability of* (3) *with* $b = 1$ *and* $q \equiv \pm 3 \pmod 8$ *are*

(i) $$a^2 \leqslant 3D+8 \quad \textit{for} \quad q \equiv 3 \pmod 8,$$

(ii) $$a^2 \leqslant D-2 \quad \textit{for} \quad q \equiv -3 \pmod 8.$$

Proof. 1° $q = 8r+3$. At first we prove that $\lambda\lambda' \equiv 1 \pmod 4$. If $\lambda\lambda' \equiv -1 \pmod 4$ we would have $S_2 = 2\lambda\lambda' - D \equiv 3 \pmod 8$. By (7) we obtain

$$S_{4r+2} \cdot T_{4r+1} = 1 - (\lambda\lambda')^{4r+1}.$$

S_2 is a divisor of S_{4r+2}. Let p be an odd prime dividing S_2. This implies

$$\left(\frac{\lambda\lambda'}{p}\right) = \left(\frac{a^2+D}{p}\right) = \left(\frac{2}{p}\right) = 1,$$

i.e. $p = 8t \pm 1$, contradicting $S_2 \equiv 3 \pmod 8$.

We proceed now by assuming $\lambda\lambda' \equiv 1 \pmod 4$. Treating (5) as a congruence mod 4 we obtain $T_{4r+1} \equiv 0 \pmod 2$, and hence $q \equiv 1 \pmod 3$. By (6) we have further

$$T_{4r+2} \cdot S_{4r+1} = 1 + (\lambda\lambda')^{4r+1}.$$

Now $\frac{1}{2}R_3$ is an odd divisor of S_{4r+1}. If $p \mid \frac{1}{2}R_3$, $R_3 > 0$, we get

$$\left(\frac{-\lambda\lambda'}{p}\right) = \left(\frac{-a^2-D}{p}\right) = \left(\frac{-D}{p}\right) = 1,$$

since $a^2 \equiv 3D \pmod p$. This implies $\left(\frac{-3}{p}\right) = 1$, i.e. $p = 6k+1$. Consequently $a^2 - 3D = 8(6k_1+1)$, a contradiction mod 3, and then

$$a^2 \leqslant 3D+8.$$

Remark. The congruences $q \equiv 3 \pmod 8$ and $q \equiv 1 \pmod 3$ give $q \equiv 19 \pmod{24}$.

2° $q = 8r+5$. By (6) we obtain

(14) $$T_{4r+3} \cdot S_{4r+2} = 1 + (\lambda\lambda')^{4r+2}.$$

S_2 is a divisor of S_{4r+2}. Let the odd prime $p \mid S_2$, $S_2 > 0$. Then by (14) $p \equiv 1 \pmod 4$. But $S_2 \equiv -1 \pmod 4$, and hence

$$a^2 \leqslant D-2.$$

Remark. It is possible to obtain a somewhat better result in case $q \equiv -1 \pmod 3$. By (5) we then conclude $\lambda\lambda' \equiv 3 \pmod 8$, giving $\frac{1}{2}T_3 = \frac{1}{8}(3a^2-D) \equiv -1 \pmod 4$. Now $\frac{1}{2}T_3$ is a divisor of T_{4r+3}, and therefore $\frac{1}{2}T_3 \equiv 1 \pmod 4$ in virtue of (14). Consequently,

$$a^2 \leqslant \tfrac{1}{3}(D+8) \quad \text{for} \quad q \equiv 5 \pmod{24}.$$

LEMMA 10. *A necessary condition for the solvability of* (3) *with* $b = 1$ *and* $q \equiv \pm 7 \pmod{24}$ *is*

$$a^2 < \tfrac{1}{3}(D-4) \quad \textit{for} \quad q \equiv -7 \pmod 8,$$

$$a^2 \leqslant D-2 \quad \textit{for} \quad q \equiv 7 \pmod 8.$$

Proof. 1° $q = 24h+17$. By (6) we have $T_3 \equiv 2 \pmod 4$, which by (5) implies $\lambda\lambda' \equiv 3 \pmod 8$. Hence $\frac{1}{2}T_3 \equiv -1 \pmod 4$. Let the odd prime $p \mid T_3$, $T_3 > 0$. On account of (6) we get further $\frac{1}{2}T_3 \equiv 1 \pmod 4$. Consequently, $a^2 < \frac{1}{3}(D-4)$.

2° $q = 24h+7$. Treating (5) as a congruence mod 8, we get by (9) that $T_{12h+3} \equiv T_3 \equiv 0 \pmod 4$. Equation (6) may be written

(15) $$T_{12h+4} \cdot S_{12h+3} = 1 - (\lambda\lambda')^{12h+3}.$$

S_2 is a divisor of T_{12h+4}. By (15) we get, putting $p|S_2$, $S_2 > 0$

$$1 = \left(\frac{-\lambda\lambda'}{p}\right) = \left(\frac{-a^2-D}{p}\right) = \left(\frac{-2}{p}\right),$$

i.e.

$$p = 8t+1 \quad \text{or} \quad 8t+3.$$

However, from $T_3 \equiv 0 \pmod 4$, we find $S_2 = \frac{1}{2}(a^2-D) \equiv -1 \pmod 8$. Consequently, $a^2 \leqslant D-2$, since $a^2-D = 2$ contradicts $D \equiv 3 \pmod 8$.

6. Proof of the two theorems. At first we prove Theorem 1. There are only a finite number of possibilites for a if $q \equiv \pm 1 \pmod{24}$. In case $b = -1$ this fact follows from Lemmas 4 and 6, and in case $b = 1$ from Lemmas 9 and 10. If there are any solutions these can be effectively found by use of Cassels' theorem mentioned in the introduction. We have only to write (1) in the form

$$x^2-(a^2+D)\left(\frac{a^2+D}{4}\right)^{q-1} = -D.$$

In order to prove Theorem 2 we note that $D \equiv 51 \pmod{72}$ implies $D \equiv 6 \pmod 9$. Lemma 8 then shows that (3) cannot be satisfied for $b = 1$. In case $b = -1$ Lemmas 6 and 4 show that $q \equiv -1 \pmod{24}$ is the only possibility. Then we may use Cassels' lemma. Our two theorems are proved.

EXAMPLE. $x^2+11 = 4y^q$. Here $h = 1$. Since $D \equiv -1 \pmod 3$ Lemma 3 shows that $b = 1$. Utilizing Lemma 9 we obtain the following bounds for a^2: $a^2 \leqslant 25$ for $q \equiv 3 \pmod 8$ and $a^2 \leqslant 9$ for $q \equiv 5 \pmod 8$. Here $a = 3$ is excluded by Lemma 1. The value $a^2 = 25$ gives $\lambda\lambda' = 9$, i.e. $x^2+11 = 4\cdot 3^{2q}$, an equation with $q = 1$ as the only integers olution. It then remains $a^2 = 1$ with $\lambda\lambda' = 3$, i.e. $x^2+11 = 4\cdot 3^q$. In order to avoid the laborious calculations in using Cassels' lemma, we prefer an application of Skolem's p-adic method. Here we have

$$\lambda^4 = \left(\tfrac{1}{2}(1+\sqrt{-11})\right)^4 = 1+5\xi, \qquad \xi = \tfrac{1}{2}(1-\sqrt{-11}).$$

Then we must solve the equation

$$\tfrac{1}{2}(x+\sqrt{-11}) = \left(\tfrac{1}{2}(1+\sqrt{-11})\right)^{4z+r}, \quad r = 1 \text{ or } 3. \tag{16}$$

By (16) we get

$$\tfrac{1}{2}(x+\sqrt{-11}) = (1+5\xi)^z\left(\tfrac{1}{2}(1+\sqrt{-11})\right)^r.$$

It is easily seen that $r = 3$ can be excluded mod 5. In case $r = 1$ we obtain the following 5-adic development:

$$0 = z(z-1)+5(\ \ )+5^2(\ \ )+\ldots \tag{17}$$

According to a theorem due to Th. Skolem [10] this equation (17) has at most two solutions in rational integers z. Now $z = 0$ and $z = 1$ are solutions. For $z = 1$ we get $y = 5$ and

$$31^2+11 = 4\cdot 3^5$$

as the only solution in case $q \equiv \pm 3 \pmod 8$. Lemma 10 gives no further solutions. Hence:

The equation $x^2+11 = 4y^q$ has no solutions in rational integers x and y and primes $q \not\equiv \pm 1 \pmod{24}$, with the exception of $q = 5$, where $y = 3$ is the only solution.

References

[1] A. Aigner, *Die diophantische Gleichung $x^2+4D = y^p$ im Zusammenhang mit Klassenzahlen*, Mh. Math. 72 (1968), pp. 1–5.
[2] J. W. S. Cassels, *On a class of exponential equations*, Ark. Mat. 4 (1961), pp. 231–233.
[3] W. Ljunggren, *On the diophantine equation $x^2+D = 4y^q$*, Mh. Math. 75 (1971), pp. 136–143.
[4] — *Einige Bemerkungen über die Darstellung ganzer Zahlen durch binäre kubische Formen mit positiver Diskriminante*, Acta. Math. 75 (1942), pp. 1–21.
[5] — *On the diophantine equation $Cx^2+D = y^n$*, Pacific J. Math. 14 (1964), pp. 585–596.
[6] — *On the diophantine equation $Cx^2+D = 2y^n$*, Math. Scand. 18 (1966), pp. 69–86.
[7] T. Nagell, *Sur l'impossibilité de quelques équations à deux indéterminées*, Norsk Mat. Forenings Skrifter, Ser. I, No. 13 (1923), pp. 65–82.
[8] — *Des équations indéterminées $x^2+x+1 = y^q$ et $x^2+x+1 = 3y^q$*, Norsk Mat. Forenings Skrifter, Ser. I, No. 2 (1921), 14 pp.
[9] B. Persson, *On a diophantine equation in two unknowns*, Ark. Mat. 1 (1949), pp. 45–57.
[10] Th. Skolem, *Ein Verfahren zur Behandlung gewisser exponentialer Gleichungen und diophantischer Gleichungen*, 8de Skand. Mat. Kongress, Stockholm 1934, pp. 163–188.
[11] B. Stolt, *Die Anzahl von Lösungen gewisser diophantischer Gleichungen*, Arch. Math. 8 (1957), pp. 393–400.

Received on 15. 7. 1971 (195)