On reciprocally weighted partitions

by

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Dedicated to the memory of Wacław Sierpiński

Introduction. Let \( n \) be a positive integer and let

\[
P: a_1 + a_2 + a_3 + \ldots = n \quad (0 < a_2 \leq a_3 \leq a_4 \leq \ldots)
\]

be any partition of \( n \) into positive integer parts. We assign to this partition the weight

\[
w(P) = (a_2 a_3 a_4 \ldots)^{-1}.
\]

If the parts \( a_i \) are restricted to a set \( S \) of positive integers and if we add the weights of all the partitions of \( n \) into parts taken from \( S \) we obtain

\[
W_n(S) = \sum w(P),
\]

a rational number we call informally the weighted number of partitions of \( n \) into parts belonging to \( S \). If we require that the parts in (1) be distinct we obtain, in lieu of (2) a smaller sum, which we denote by \( W^*_n(S) \), called the weighted number of partitions of \( n \) into distinct parts belonging to \( S \).

For example if \( S \) is unrestricted and if \( n = 6 \) we have eleven partitions.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( w(P) )</th>
<th>( P )</th>
<th>( w(P) )</th>
</tr>
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<tbody>
<tr>
<td>1+1+4+4</td>
<td>1/8</td>
<td>1+1+1+1+1+1</td>
<td>1/3</td>
</tr>
<tr>
<td>1+1+5</td>
<td>1/5</td>
<td>2+2+2</td>
<td>1/8</td>
</tr>
<tr>
<td>2+4</td>
<td>1/4</td>
<td>1+1+2+2</td>
<td>1/4</td>
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<tr>
<td>3+3</td>
<td>1/9</td>
<td>1+1+1+1</td>
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<tr>
<td>1+2+3</td>
<td>1/6</td>
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</tbody>
</table>

Hence

\[
W_6(S) = \frac{581}{180} \quad \text{and} \quad W^*_6(S) = \frac{1}{6} + \frac{1}{5} + \frac{1}{8} + \frac{1}{6} = \frac{79}{120}.
\]
If we multiply \( W_n(S) \) and \( W^*_n(S) \), by \( n! \) we obtain two non-negative integers
\[
A_n(S) = n! \cdot W_n(S), \quad A^*_n(S) = n! \cdot W^*_n(S).
\]
That these are indeed integers follows from the fact that every term \( w(P) \) becomes an integer when multiplied by \( n! \). In fact even
\[
\frac{n!}{a_1 a_2 \cdots} \cdot \frac{n!}{w(P)} = \frac{n!}{(a_1 - 1)! \cdots (a_2 - 1)! \cdots}
\]
is an integer since it is a multinomial coefficient.

We adopt the convention
\[
A_n(S) = A^*_n(S) = 1.
\]

Generating functions. If we expand the following products into power series we see that \( A \) and \( A^* \) are generated by
\[
F(x) = F^*(x, S) = \prod_{m \in S} \left( 1 - x/m \right)^{-1} = \sum_{n=0}^{\infty} A_n(S) x^n/n!,
\]
and
\[
F^*(x) = F^*(x, S) = \prod_{m \in S} \left( 1 + x/m \right)^{-1} = \sum_{n=0}^{\infty} A^*_n(S) x^n/n!.
\]
The integers \( A_n \) and \( A^*_n \) may be computed recursively thus avoiding the generation of all the corresponding partitions of \( n \) in terms of which they are defined. To this effect we have

**Theorem 1.** Define \( \Gamma_n(S) = \Gamma_n \) and \( \Gamma^*_n(S) = \Gamma^*_n \) by

\[
\Gamma_0 = \Gamma^*_0 = 0, \quad \Gamma_n = n! \sum_{d \in S} \delta_{1-d,n/d}, \quad \Gamma^*_n = n! \sum_{d \in S} (-d)^{1-d,n/d},
\]
where the sums extend over all those divisors of \( n \) which belong to \( S \).

Then, symbolically,
\[
nA_n = (A + \Gamma)^n \quad \text{and} \quad nA^*_n = (A^* + \Gamma^*)^n.
\]
In other words,
\[
nA_n = \sum_{k=1}^{n} A_{n-k} \binom{n}{k} \Gamma_k, \quad nA^*_n = \sum_{k=1}^{n} A^*_{n-k} \binom{n}{k} \Gamma^*_k.
\]

**Proof.** To prove the first of these conclusions we take the logarithmic derivative of \( F(x) \), thus
\[
\frac{\text{d} F'}{\text{d} F}(x) = \sum_{m \in S} \frac{x^m}{1 - x^m/m} = \sum_{m \in S} x^m \sum_{n=0}^{\infty} x^{mn/m - n} = \sum_{m \in S} \sum_{n=1}^{\infty} x^{mn/m - n} = \sum_{k=1}^{\infty} x^k \Gamma_k/k!.
\]

Multiplying by \( F(x) \) and identifying coefficients of \( x^n \) on both sides gives
\[
nA_n/n! = \sum_{k=0}^{n} \Gamma_k A_{n-k}\left( \left[ \left( k! \right) \left( n-k \right) ! \right] \right)
\]
or
\[
nA_n = \sum_{k=0}^{n} \Gamma_k \binom{n}{k} A_{n-k} = (A + \Gamma)^n.
\]
The formula \( nA^*_n = (A^* + \Gamma^*)^n \) is proved in the same way using \( F^*(x) \).

**Corollary.** \( \Gamma_n \) and \( \Gamma^*_n \) are integers.

**Proof.** By Theorem 1
\[
\Gamma_n = nA_n - \sum_{k=1}^{n-1} \Gamma_k \binom{n}{k} A_{n-k}.
\]

Hence, by complete induction, \( \Gamma_n \) is an integer since its predecessors are. Similar reasoning applies to \( \Gamma^*_k \).

**Simple limit theorems.** Table I gives the first ten values of the six functions \( A_n, A^*_n, \Gamma_n, \Gamma^*_n, W_n, W^*_n \) in the unrestricted case, i.e., when \( S \) is the set of all positive integers.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \Gamma_n )</th>
<th>( A_n )</th>
<th>( W_n )</th>
<th>( W_n^* )</th>
<th>( \Gamma^*_n )</th>
<th>( A^*_n )</th>
<th>( W^*_n )</th>
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<td>1.00000</td>
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</table>

**Inspection** of this small table of \( W^*_n \) leads one to guess that \( W^*_n \) tends to some limit. This is confirmed by

**Theorem 2.** The weighted number of partitions of \( n \) into distinct parts tends to \( e^\gamma \) as \( n \to \infty \). That is
\[
\lim_{n \to \infty} W^*_n = e^\gamma = 0.56145948 \ldots,
\]
where \( \gamma \) is Euler's constant.
Proof. With \( S \) the set of all positive integers let
\[
G^*(x) = (1 - x) \ell_n = \sum_{n=1}^{\infty} b_n x^n = 1 - \frac{1}{2} x + \frac{1}{3} x^2 - \frac{1}{4} x^3 + \frac{1}{5} x^4 \ldots
\]
so that
\[
W_n = A_n ! = \sum_{k=1}^{n} b_k.
\]
Then, by (4),
\[
\log G^*(x) = \log (1 - x) + \sum_{n=1}^{\infty} \log (1 + x^n / n)
\]
\[
= - \sum_{n=1}^{\infty} x^n / n + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (-1)^{m-1} \frac{x^{mn}}{mn^m} = - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^m}{mn^m} x^{mn}.
\]
This series converges for \( x = 1 \) and
\[
-\log G^*(1) = \sum_{n=1}^{\infty} \sum_{m=2}^{\infty} \frac{(-1)^m}{mn^m} = \lim_{N \to \infty} \left( \sum_{n=1}^{N} \left[ \log \left(1 + \frac{1}{n}\right) - \frac{1}{n} \right] \right)
\]
\[
= \lim_{N \to \infty} \left( \log (1 + N) - \sum_{n=1}^{N} 1/n \right) = \gamma.
\]
Hence
\[
e^{-\nu} = G^*(1) = \lim_{N \to \infty} \sum_{n=0}^{N} b_n = \lim_{N \to \infty} W_n,
\]
which proves the theorem.

Further inspection of Table I suggests that \( W_n/n \) also tends to a limit. This is confirmed by

**Theorem 3.** The weighted number of unrestricted partitions of \( n \) is asymptotic to \( e^{-\nu} n \). That is
\[
\lim_{n \to \infty} W_n/n = e^{-\nu}.
\]

Before proving this theorem it is convenient to prove

**Theorem 4.** The weighted number of partitions of \( n \) into parts \( > 1 \) tends to \( e^{-\nu} \) as \( n \to \infty \).

Proof. Let \( S_1 \) be the set of all integers \( \geq 2 \) and let \( F(x) \) mean \( F(x, S_1) \), then
\[
G(x) = (1 - x) \ell_n = \sum_{n=1}^{\infty} c_n x^n = \frac{x^2}{2} - \frac{x^2}{6} + 
\]
so that
\[
W_n(S_1) = A_n(S_1) / n! = \sum_{k=1}^{n} c_k.
\]
Then by (3)
\[
\log G(x) = \log (1 - x) - \sum_{n=2}^{\infty} \frac{\log (1 - x^n / n)}{n}
\]
\[
= \sum_{n=2}^{\infty} \sum_{m=1}^{\infty} \frac{c_m}{mn^m} - \sum_{n=1}^{\infty} x^n = -x + \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{c_m}{mn^m}
\]
a series that converges at \( x = 1 \).

Hence
\[
\log G(1) = -1 + \lim_{N \to \infty} \sum_{n=2}^{N} \sum_{m=2}^{N} \frac{c_m}{mn^m} = -1 + \lim_{N \to \infty} \sum_{n=2}^{N} \left( \frac{\sum_{m=1}^{\infty} (n^{-m}) - 1}{n} \right)
\]
\[
= \lim_{N \to \infty} \left( \sum_{n=2}^{N} \log \left(1 + \frac{1}{n}\right) - \frac{1}{n} \right) = \lim_{N \to \infty} \left( \log N - \sum_{n=1}^{N} \frac{1}{n} \right) = -\gamma.
\]
Hence
\[
e^{-\nu} = G(1) = \lim_{N \to \infty} \sum_{n=0}^{N} c_n = \lim_{N \to \infty} W_n(S_1),
\]
which proves Theorem 4.

To prove Theorem 3 we observe that unrestricted partitions of \( n \) are of two types:

(a) Those involving the part 1.

(b) Those with all parts \( > 1 \).

Those of type (a) correspond uniquely to an unrestricted partition of \( n-1 \) simply by suppressing one unit part. These two partitions have the same weight. Hence
\[
W_n = W_{n-1} = W_{n}(S_1).
\]

Therefore
\[
\lim_{N \to \infty} \frac{W_n}{N} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (W_n - W_{n-1}) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} W_n(S_1).
\]

By Theorem 4 this average must tend to \( e^{-\nu} \). This proves Theorem 3.

**Arithmetical progressions.** In order to treat partitions whose parts lie in an arithmetical progression we need two lemmas.

**Lemma 1.** Let \( c_n \geq 0 \) and let \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) be such that for some \( \lambda > 0 \)
\[
\lim_{x \to 1 - 0} (1 - x)^{\lambda} f(x) = C.
\]
Then
\[
\lim_{n \to \infty} n \sum_{k=0}^{n-1} c_k = C/\Gamma(\lambda).
\]
Proof. For \( \lambda = 1 \) this is a Tauberian theorem of Hardy ([1], Theorem 96, p. 155). For a general \( \lambda \) we need to observe that as \( x \to 1^- \)

\[
f(x) \sim O(1-x)^{-1} = O \sum_{n=0}^{\infty} \left( \frac{n+\lambda -1}{n} \right) x^n.
\]

Furthermore

\[
\left( \frac{n+\lambda -1}{n} \right) = \frac{\Gamma(n+\lambda)}{\Gamma(n+1)\Gamma(\lambda)} = \frac{1}{\Gamma(\lambda)} (n+\lambda-1)(n+\lambda-2) \cdots (n+1) \sim n^{\lambda-1}/\Gamma(\lambda).
\]

With these modifications the proof goes through.

**Lemma 2.** If \( a > 0 \),

\[
\lim_{x \to 1^-} (1-x)^a \prod_{n=1}^{\infty} \left( 1- \frac{a^{n+b}}{an+b} \right)^{-1} = e^{-\gamma a}(\gamma/a) \left( \frac{1+\frac{b-1}{a}}{1+\frac{b}{a}} \right)^{-1} \Gamma \left( 1+\frac{b-1}{a} \right)/\Gamma \left( 1+\frac{b}{a} \right).
\]

**Proof.** Let

\[
G(x) = (1-x^a)^{-1} \prod_{n=1}^{\infty} \left( 1- \frac{a^{n+b}}{an+b} \right).
\]

For typographical simplicity set

\[
y = x^a, \quad \alpha = 1/a, \quad ba = c,
\]

then

\[
(1-x^a)^{1/a} = (1-y) = \exp \{ a \log (1-y) \} = \exp \left( \sum_{n=0}^{\infty} \frac{-ay^n}{n} \right) = \prod_{n=1}^{\infty} e^{-ay^n/n}.
\]

Hence

\[
G(x) = \prod_{n=1}^{\infty} \left( 1- \frac{a^{n+c}}{n+c} \right) e^{ay^n/n}
\]

\[
= e^{ay} \left( 1- \frac{a^{n+c}}{n+c+1} \right) \prod_{n=1}^{\infty} \left( 1- \frac{a^{n+c+1}}{n+c+1} \right)e^{ay^{n+1}/(n+1)}
\]

\[
= e^{ay} \frac{c+1-ay^{n+1}}{c+1} \prod_{n=1}^{\infty} \left( 1+ \frac{c+1-ay^{n+1}}{n} \right) e^{-\gamma (c+1-ay^{n+1})/n} \times \prod_{n=1}^{\infty} \left( 1+ \frac{c+1}{n} \right) e^{-\gamma (1+y^n)/n} \right)^{-1} \left[ \prod_{n=1}^{\infty} \exp \left( \frac{y^n}{n} - \frac{1}{n+1} y^{n+1} \right) \right]^{-1}.
\]

Since

\[
\Gamma(x) = e^{-\gamma x} \prod_{n=1}^{\infty} \left( 1+ \frac{x}{n} \right)^{-1} e^{\gamma/n} \Gamma(n+1)
\]

we have

\[
G(x) = \Gamma(1+c) e^{\gamma+c}(c+1-ay^{c+1}) \times \exp \left[ a \left[ y + \sum_{n=1}^{\infty} \frac{y^{n+1}}{n+1} - \frac{y^{n+1}}{n+1} \right] \right] P(y)
\]

where

\[
P(y) = \prod_{n=1}^{\infty} \left( 1+ \frac{c+1-ay^{n+1}}{n} \right) e^{-\gamma (c+1-ay^{n+1})/n}.
\]

The expression inside the square brackets is simply

\[-(1-\gamma y^{c+1}) \log (1-y)\]

and this vanishes as \( y \) tends to 1.

Next we consider

\[
\lim_{y \to 1^-} P(y).
\]

The logarithm of the \( n \)th factor is

\[
T_n = \frac{c+1-ay^{n+1}}{n} \log \left( 1+ \frac{c+1-ay^{n+1}}{n} \right)
\]

\[
= \sum_{m=1}^{\infty} \frac{(-1)^{m+1}(c+1-ay^{n+1})^m}{mn^m}.
\]

Let

\[v = c+1-a \]

and let

\[N > 2v.\]

Then for \( n > N \)

\[
|T_n| < \frac{v^2}{n^2} \left( 1+ \frac{|v|}{n} + \frac{|v|^2}{n^2} + \ldots \right) < \frac{N^2}{4v^2} \sum_{n=1}^{\infty} 2^{-1} = \frac{1}{2} \left( \frac{N}{v} \right)^2.
\]

Since

\[
\sum_{n=N}^{\infty} N^2/n^2
\]
converges it follows that \( \log P(y) \) is analytic for \( |y| \leq 1 \), and hence \( P(y) \) tends to
\[
P(1) = \prod_{n=1}^{\infty} \left( 1 + \frac{y}{n} \right) e^{-\gamma n} = e^{-\gamma/(\Gamma'(1))}.
\]
Hence
\[
\lim_{x \to 0} G(x) = \Gamma(a+1) e^{(a+1)x} P(1) = e^{\gamma/2} \Gamma'(a+1) / \Gamma'(a+1 - a).
\]
It remains to observe that
\[
\lim_{x \to 0} \frac{(1 - x)^{\frac{1}{2}}}{(1 - x)^{a+1/2}} = a^{-1/2}
\]
and to restore the variables \( a \) and \( b \) in terms of \( a \) and \( c \) to obtain the theorem.

**Theorem 5.** Let \( S_a \) denote the set of all odd numbers. Then for the weighted number \( W_n(S_a) \) of partitions of \( n \) into odd parts we have
\[
W_n(S_a) \sim \frac{2}{\pi} \sqrt{2e^{-\gamma}n} = .6746124\sqrt{n}.
\]

**Proof.** In Lemma 2 we put \( a = 2 \) and \( b = 1 \), we find
\[
\lim_{x \to 0} (1 - x)^{1/2} \prod_{n=1}^{\infty} \left( 1 - \frac{x^{2n+1}}{2n+1} \right)^{-1} = 2^{-1/2} e^{-\gamma/2} \Gamma(3/2) = \sqrt{\frac{2}{e^{\gamma}}}.
\]
Since the first factor \( (1 - x)^{-1} \) of the generator of \( W_n(S_a) \) is missing from the above product we have
\[
\lim_{x \to 0} (1 - x)^{1/2} \sum_{n=0}^{\infty} W_n(S_a) x^n = \sqrt{\frac{2}{e^{\gamma}}}
\]
By Lemma 1 therefore
\[
\sum_{n=0}^{\infty} W_n(S_a) \sim n^{1/2} \sqrt{\frac{2}{e^{\gamma}}} \Gamma(3/2) = n^{1/2} \sqrt{8e^{-\gamma}/\pi}.
\]
Since \( W_n(S_a) \) is an increasing function of \( n \)
\[
\lim_{n \to \infty} W_n(S_a) n^{-1/2} = \lim_{n \to \infty} n^{-1/2} \sum_{n=0}^{\infty} W_n(S_a) = \sqrt{8e^{-\gamma}/\pi}.
\]
leisurely rates. The five functions of the following table refer to partitions into distinct parts, parts $>1$, unrestricted parts, odd parts and even parts respectively.

<table>
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<tr>
<th>$n$</th>
<th>$W_n$</th>
<th>$W_n(S_i)$</th>
<th>$W_n(S_{o})$</th>
<th>$W_n(S_{e})$</th>
<th>$W_n(S_{o})/W_n(S_{e})$</th>
<th>$v_nW_n(S_{o})$</th>
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</table>

The slight irregularities in these functions are not due to inaccuracy. They reflect the existence of an asymptotic, or possibly convergent, series for each entry.

Reference


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Some diophantine equations solvable by identities

by

A. Mąkowski (Warszawa)

Dedicated to the memory of my teacher Wacław Sierpiński

1. W. Sierpiński in many of his papers investigated the triangular numbers $t_n = \frac{1}{2}n(n+1)$ and tetrahedral numbers $T_n = \frac{1}{3}n(n+1)(n+2)$.

   From the identity given by A. Gérardin [1] we get immediately the following identity

   $$(27n^3)^3 - 1 = (9n^3 - 3n)^3 + (9n^3 - 1)^3 = (9n^3 + 3n)^3 - (9n^3 + 1)^3.$$  

   With $a$ odd and positive the last identity provides infinitely many integer solutions of the equation

   $$t_n - (2x + 1)^3 = (2y)^3 + (2z)^3 = (2a)^3 - (2v)^3$$  

   which is equivalent to

   $$t_n = y^3 + z^3 = u^3 - v^3.$$  

   Thus there exist infinitely many triangular numbers which are simultaneously representable as sums and differences of two positive cubes.

   We have the identity $3aT_{n-1} = t_{a-1}$. Since there exist infinitely many tetrahedral numbers divisible by 3; $T_n = 3a$ we infer that there exist infinitely many triangular numbers which are products of two tetrahedral numbers $> 1$.

2. The numbers $x = 6^3 p r^3 n^3 + 6^3 p r^3 n^3$, $y = 6^3 p r^3 n^3 - 6^3 p r^3 n^3$, $z = 6^3 p r^3 n^3$ satisfy the equation

   $$p(x^3 + y^3 - z^3) = r(x - y).$$

   This answers a question posed by A. Oppenheim in [3].

3. L. J. Mordell [2] investigated the equation $x^2 = ax^3 + by^3 + c$.

   It may be noticed that the equation

   $$x^2 = ax^{2n+1} + by^{2k+1} + c$$