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Further developments in the comparative prime number theory, VII

by

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To the memory of W. Sierpiński

1. The aim of the present note is to prove the following theorem which was announced without proof in our paper [1]. Denoting by c explicitly calculable positive numerical constants (not necessarily the same in different occurences) there exist U_1, U_2, U_3, U_4 numbers for $T > c$ with (1)

$$\log_3 T \leqslant U_2 \exp(-\log^{15/16} U_2) \leqslant U_1 < U_2 \leqslant T, \tag{1.1}$$

$$\log_3 T \leqslant U_4 \exp(-\log^{15/16} U_4) \leqslant U_3 < U_4 \leqslant T \tag{1.2}$$

such that

$$\sum_{\substack{U_1<p<U_2 \\ p\equiv 1 \bmod 4}} \log p - \sum_{\substack{U_1<p<U_2 \\ p\equiv 3 \bmod 4}} \log p > \sqrt{U_2} \tag{1.3}$$

and

$$\sum_{\substack{U_3<p<U_4 \\ p\equiv 1 \bmod 4}} \log p - \sum_{\substack{U_3<p<U_4 \\ p\equiv 3 \bmod 4}} \log p < -\sqrt{U_4}. \tag{1.4}$$

The essential part is of course (1.1)–(1.3). As we mentioned this implies also for $T > c$ the existence of consecutive primes p_ν and $p_{\nu+1}$ both $\equiv 1$ mod 4 and satisfying the inequality

$$\log_3 T \leqslant p_\nu < p_{\nu+1} \leqslant T. \tag{1.5}$$

The somewhat weaker fact that we have infinitely often

$$p_l \equiv p_{l+1} \equiv 1 \bmod 4 \tag{1.6}$$

could have been derived from Littlewood's deep theorem

$$\overline{\lim_{x\to\infty}}\big(\pi(x, 4, 1) - \pi(x, 4, 3)\big) = +\infty \tag{1.7}$$

(1) $\log_\nu T$ stand for ν-times iterated logarithm.

but not cheaper; in particular no arithmetical approach can prove (1.6) at present. The natural further conjecture that for arbitrarily large ω we have for infinitely many ν's

$$p_\nu \equiv p_{\nu+1} \equiv \ldots \equiv p_{\nu+\omega} \equiv 1 \bmod 4$$

(to mention just one of the analogous conjectures) is at present beyond *all* possibilities, even for $\omega = 2$.

We want to emphasize again — as in [1] — the interest of (1.1)–(1.3) from the point of view of the facts that — as proved by Hardy-Littlewood and Landau — the assertion

$$\lim_{x\to+\infty} \sum_{p>2} (-1)^{(p-1)/2} \log p \cdot e^{-p/x} = -\infty$$

and — as proved in [3] — the assertion

$$\lim_{x\to+\infty} \sum_{p>2} (-1)^{(p-1)/2} \log p \cdot \exp\left(-\log^2 \frac{p}{x}\right) = -\infty$$

are equivalent to the assertion

$$f(s) \neq 0 \quad \text{for} \quad \sigma > \tfrac{1}{2}.$$

which is an unsolved special case of Riemann–Piltz conjecture.

2. For the proof we shall need three lemmas. Let $z_1, z_2, \ldots, z_n$, $n \leqslant N$ be complex numbers such that

$$|z_1| \geqslant |z_2| \geqslant \ldots \geqslant |z_n|$$

and a $0 < \varkappa \leqslant \pi/2$ such that

$$\varkappa \leqslant |\arc z_j| \leqslant \pi, \quad j = 1, 2, \ldots, n$$

and be given a positive number m. Then we assert the

LEMMA I. *There exist integer ν_1 and ν_2 such that*

$$m \leqslant \nu_1, \nu_2 \leqslant m + N(3+\pi/\varkappa)$$

and the inequalities

$$\operatorname{Re} \sum_{j=1}^{n} z_j^{\nu_1} \geqslant \frac{1}{3N} \left\{ \frac{N}{8e\big(m+N(3+\pi/\varkappa)\big)} \right\}^N |z_1|^{\nu_1},$$

$$\operatorname{Re} \sum_{j=1}^{n} z_j^{\nu_2} \leqslant -\frac{1}{3N} \left\{ \frac{N}{8e\big(m+N(3+\pi/\varkappa)\big)} \right\}^N |z_1|^{\nu_2}$$

hold.

For the proof see [2].

3. Further we shall need the

LEMMA II. *Let $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ two sequences of real constants for which, with fixed positive finite U, V and γ we have*

$$|\alpha_\nu| \geqslant U, \tag{3.1}$$

$$\sum \frac{1}{1+|\alpha_\nu|^\gamma} \leqslant V. \tag{3.2}$$

Then for real λ and $\Delta > 1/U$ we have in the interval

$$\lambda \leqslant x \leqslant \lambda + \Delta \tag{3.3}$$

a ξ-value such that the fractional part of $(\alpha_\nu \xi + \beta_\nu)$ is for all ν-indices between

$$\frac{1}{24V(1+|\alpha_\nu|^\gamma)} \quad \textit{and} \quad 1 - \frac{1}{24V(1+|\alpha_\nu|^\gamma)}. \tag{3.4}$$

For the proof see [3] (with the unnecessary restriction $\gamma > 1$.)

Let $s = \sigma + it$ and $f(s)$ should be defined for $\sigma > 0$ by

$$f(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}. \tag{3.5}$$

Then for $\sigma > 1$ we have

$$\frac{-f'}{f}(s) = \sum_{n \text{ odd}} \frac{\Lambda(n)(-1)^{(n-1)/2}}{n^s}. \tag{3.6}$$

We need the

LEMMA III. *There is a continuous broken line l, consisting of alternately horizontal and vertical segments running in the strip*

$$\tfrac{2}{5} \leqslant \sigma \leqslant \tfrac{5}{12}$$

from $-\infty$ to $+\infty$ such that on l the inequality

$$\left| \frac{f'}{f}(s) \right| < c \log(2+|t|) \tag{3.7}$$

holds.

The proof follows from standard theorems on L-functions.

We shall also use the integral formula

$$\frac{1}{2\pi i} \int_{(2)} \exp\{r(s+b)^2 - sx\}\, ds = \frac{e^{b^2 r}}{2\sqrt{\pi r}} \exp\left\{-\frac{(x-2br)^2}{4r}\right\} \tag{3.8}$$

if only $r \geqslant 4$, $b \geqslant 100$ say.

4. Now we can turn to the proof of our theorem. We have from (3.8)

$$(4.1)\qquad -\frac{1}{2\pi i}\int_{(2)}\frac{f'}{f}(s)\exp r(s+b)^2\,ds = \frac{e^{br^2}}{2\sqrt{\pi r}}\sum_{n\,\mathrm{odd}}\Lambda(n)(-1)^{(n-1)/2}\exp\left\{-\frac{(\log n-2br)^2}{4r}\right\},$$

r and b to be determined later. Shifting in (4.1) the line of integration to l a routine reasoning using Lemma III gives

$$(4.2)\qquad \sum_{n\,\mathrm{odd}}\Lambda(n)(-1)^{(n-1)/2}\exp\left\{-\frac{(\log n-2br)^2}{4r}\right\} = 2\sqrt{\pi r}\sum_{\varrho}{}'\exp\{r(\varrho^2+2b\varrho)\}+O(\sqrt{r}\log^3 b)\exp(\tfrac{11}{12}rb),$$

where $\sum'$ means that the summation on the right has to be extended to all nontrivial zeros

$$(4.3)\qquad \varrho=\sigma_\varrho+it_\varrho$$

of $f(s)$ right to l only. Since further the contribution of zeros with

$$|t_\varrho|>2\sqrt{b}$$

is evidently

$$<c\sum_{t_\varrho>2\sqrt{b}}\exp(1-t_\varrho^2+2b)r<c\sum_{t_\varrho>2\sqrt{b}}\exp(-\tfrac{1}{3}rt_\varrho^2)<c,$$

restricting b by

$$(4.4)\qquad \tau\leqslant b\leqslant\tau+1$$

we get from (4.2)

$$(4.5)\qquad \sum_{n\,\mathrm{odd}}\Lambda(n)(-1)^{(n-1)/2}\exp\left\{-\frac{(\log n-2br)^2}{4r}\right\} = 2\sqrt{\pi r}\,\mathrm{Re}\sum_{|t_\varrho|\leqslant 2\sqrt{\tau}}{}'\{\exp(\varrho^2+2b\varrho)\}^r+O(\sqrt{r})\log^3\tau\cdot\exp(\tfrac{11}{12}\tau r).$$

5. Next we determine b by applying Lemma II choosing the α_ν respectively β_ν numbers as

$$(5.1)\qquad \frac{1}{\pi}\mathrm{Im}\,\varrho\quad\text{respectively}\quad\frac{1}{2\pi}\mathrm{Im}(\varrho^2)$$

in the right-hand sum in (4.5), and by choosing

$$(5.2)\qquad \lambda=\tau\quad\text{and}\quad\Delta=1$$

with the τ in (4.4). Then we can put (as known)

$$U=2,$$

further

$$\gamma=\tfrac{11}{10},\qquad V=c.$$

Hence Lemma II gives a $b=b_0$ satisfying (4.4) such that for all of our remaining ϱ's the fractional part of

$$2b_0\left(\frac{t_\varrho}{2\pi}\right)+\frac{1}{2\pi}\mathrm{Im}(\varrho^2)$$

is between

$$(5.3)\qquad c\tau^{-11/20}\quad\text{and}\quad 1-c\tau^{-11/20}.$$

But this means that choosing as z_j's the numbers

$$(5.4)\qquad \exp(2b_0\varrho+\varrho^2)$$

we have

$$|\mathrm{arc}\,z_j|\geqslant\min_{\varrho\,\mathrm{in}\,(4.5)}2\pi\left\{\frac{b_0}{\pi}t_\varrho+\frac{1}{2\pi}\mathrm{Im}(\varrho^2)\right\}>c\tau^{-11/20},$$

i.e. we may choose for the $\varkappa$ in (2.2)

$$(5.5)\qquad \varkappa=c\tau^{-11/20}$$

in our case.

6. Putting

$$(6.1)\qquad Z(r)\doteq\mathrm{Re}\sum_{|t_\varrho|\leqslant 2\sqrt{\tau}}{}'\{\exp(2b_0\varrho+\varrho^2)\}^r$$

we shall estimate it from below by a positive (resp. from above by a negative) quantity by suitable choices of r. Choosing

$$(6.2)\qquad \tau=\log^{1/4}T$$

the sum in (6.1) is a power sum of fixed complex numbers and with the choice of $\varkappa$ in (5.5) Lemma I is applicable and for b_0 we have

$$(6.3)\qquad \log^{1/4}T\leqslant b_0\leqslant\log^{1/4}T+1.$$

For the number N we have

$$(6.4)\qquad N=c\sqrt{\tau}\log\tau\quad\text{or}\quad N=\log^{1/8}T(\log\log T)^2$$

for $T>c$ and we choose

$$(6.5)\qquad m=\frac{\log T}{2b_0}.$$

For $\varkappa$ we have for $T > c$ from (5.5), and (6.2)

$$\varkappa = \log^{-11/80} T (\log\log T)^{-1}; \tag{6.6}$$

hence using also (6.3)

$$\frac{N}{\varkappa} = \log^{21/80} T (\log\log T)^3 < \frac{\log T}{2(1+\log^{1/4} T)} \cdot \frac{1}{\log^{1/4} T} \leqslant \frac{\log^{3/4} T}{2b_0}.$$

Thus ν_1 and ν_2 from (2.3) will have the form

$$\big(1+o(1)\big)\tfrac{1}{2}\log^{3/4} T. \tag{6.7}$$

Further from (6.4), (6.5), (6.6) and (6.3) for $T > c$ we have

$$\left(\frac{N}{8e\big(m+N(3+\pi/\varkappa)\big)}\right)^N > \exp\{-\log^{1/8} T (\log\log T)^4\}. \tag{6.8}$$

Now let

$$\varrho^* = \sigma^* + it^* \tag{6.9}$$

be any zero of $f(s)$ among the ones in (6.1) or — a bit stronger — any zero of $f(s)$ with

$$\sigma^* \geqslant \tfrac{1}{2}, \quad |t^*| \leqslant \tfrac{1}{2}\log^{1/10} T. \tag{6.10}$$

Then

$$|z_1|^{\nu_1} \geqslant |\exp \nu_1(\varrho^{*2} + 2b_0\varrho^*)| = \{\exp(2b_0\nu_1)\}^{\sigma^*} \exp \nu_1(\sigma^{*2} - t^{*2})$$

and owing to $2b_0\nu_1 \geqslant 2b_0 m = \log T$, (6.7), and (6.10) the right side is

$$> T^{\sigma^*} \exp(-\tfrac{2}{3}\log^{19/20} T). \tag{6.11}$$

Thus choosing $r = \nu_1$ (2.4) gives for $T > c$

$$Z(\nu_1) < T^{\sigma^*} \exp(-\tfrac{2}{3}\log^{19/20} T) \tag{6.12}$$

and analogously (2.5) gives

$$Z(\nu_2) < -T^{\sigma^*} \exp(-\tfrac{2}{3}\log^{19/20} T). \tag{6.13}$$

Since in (4.5) owing to (6.2) and (6.7) for $T > c$, $j = 1, 2$,

$$\exp(\tfrac{11}{12}\tau\nu_j) < \exp\{\big(1+o(1)\big)\tfrac{11}{12}\cdot\tfrac{1}{2}\log T\} < T^{23/48}\log^{-4} T,$$

(4.5) gives for $T > c$ using also (6.12) and (6.13)

$$\sum_{n\,\mathrm{odd}} \Lambda(n)(-1)^{(n-1)/2} \exp\left(-\frac{(\log n - 2b_0\nu_1)^2}{4\nu_1}\right) > T^{\sigma^*} \exp(-\tfrac{3}{4}\log^{19/20} T) \tag{6.14}$$

and

$$\sum_{n\,\mathrm{odd}} \Lambda(n)(-1)^{(n-1)/2} \exp\left(-\frac{(\log n - 2b_0\nu_2)^2}{4\nu_2}\right) < -T^{\sigma^*} \exp(-\tfrac{3}{4}\log^{19/20} T). \tag{6.15}$$

7. Putting

$$2b_0\nu_j = \log x_j, \quad j = 1, 2, \tag{7.1}$$

(6.14) and (6.15) take the form

$$\sum_{n\,\mathrm{odd}} (-1)^{(n-1)/2} \Lambda(n) \exp\left(-\frac{1}{4\nu_1}\log^2\frac{n}{x_1}\right) > T^{\sigma^*} \exp(-\tfrac{3}{4}\log^{19/20} T) \tag{7.2}$$

respectively

$$\sum_{n\,\mathrm{odd}} (-1)^{(n-1)/2} \Lambda(n) \exp\left(-\frac{1}{4\nu_2}\log^2\frac{n}{x_2}\right) < -T^{\sigma^*} \exp(-\tfrac{3}{4}\log^{19/20} T). \tag{7.3}$$

What can be said on x_1 and x_2? From (2.3), (6.3), (6.6), (6.4) and (6.2) we have for $T > c$

$$\log T \leqslant 2b_0\nu_j \leqslant 2b_0\left\{\frac{\log T}{2b_0} + \log^{1/8} T (\log\log T)^2 (3 + \pi\log^{11/80} T \log\log T)\right\}$$
$$< \log T + \log^{21/40} T,$$

i.e. for $j = 1, 2$

$$T \leqslant x_j \leqslant T\exp(\log^{21/40} T). \tag{7.4}$$

8. Putting for $j = 1, 2$

$$\sum_{\substack{n\,\mathrm{odd}\\ n\leqslant x}} (-1)^{(n-1)/2} \Lambda(n) = G(x), \quad \exp\left(-\frac{1}{\nu_j}\log^2\frac{x}{x_j}\right) = H_j(x) \tag{8.1}$$

the left side of (7.2) and (7.3) can be written as

$$\int_1^\infty H_j(x)\,dG(x) = -\int_1^\infty G(x)H_j'(x)\,dx. \tag{8.2}$$

Since $G(x) = O(x)$, putting for $j = 1, 2$

$$\xi_j = x_j\exp(-3\sqrt{\nu_j \log x_j}), \quad \eta_j = x_j\exp(3\sqrt{\nu_j\log x_j}) \tag{8.3}$$

we can easily see that

$$\left|\int_1^{\xi_j} G(x)H_j'(x)\,dx\right| < H_j(\xi_j)O(\xi_j) = o(1), \qquad \left|\int_{\eta_j}^\infty G(x)H_j'(x)\,dx\right| = o(1). \tag{8.4}$$

Further

$$-\int_{\xi_1}^{\eta_1} G(x)H_1'(x)dx = -\int_{\xi_1}^{x_1} G(x)|H_1'(x)|dx + \int_{x_1}^{\eta_1} G(x)|H_1'(x)|dx$$

$$\leqslant -\min_{\xi_1\leqslant x\leqslant x_1} G(x)\int_{\xi_1}^{x_1}|H_1'(x)|dx + \max_{x_1\leqslant x\leqslant \eta_1} G(x)\int_{x_1}^{\eta_1}|H_1'(x)|dx$$

$$= \{\max_{x_1\leqslant x\leqslant\eta_1} G(x) - \min_{\xi_1\leqslant x\leqslant x_1} G(x)\}\int_{\xi_1}^{x_1}|H_1'(x)|dx - \max_{x_1\leqslant x\leqslant\eta_1} G(x)\int_{\xi_1}^{\eta_1} H_1'(x)dx$$

$$= O(1) + \{\max_{x_1\leqslant x\leqslant\eta_1} G(x) - \min_{\xi_1\leqslant x\leqslant x_1} G(x)\}\int_{\xi_1}^{x_1} H_1'(x)dx.$$

Since for $T > c$

$$\int_{\xi_1}^{x_1} H_1'(x)dx = 1 + o(1) < 2$$

and

(8.5) $$\max_{x_1\leqslant x\leqslant\eta_1} G(x) - \min_{\xi_1\leqslant x\leqslant x_1} G(x) \leqslant \max \sum_{\substack{n \text{ odd} \\ U_1\leqslant n\leqslant U_2}} (-1)^{(n-1)/2}\Lambda(n),$$

where the max refers to U_1, U_2's with

$$\xi_1 \leqslant U_1 < U_2 \leqslant \eta_1,$$

(7.2) gives with (8.2), (8.4) and (8.5) for $T > c$

(8.6) $$\max_{\xi_1\leqslant U_1<U_2\leqslant\eta_1} \sum_{\substack{n \text{ odd} \\ U_1\leqslant n\leqslant U_2}} (-1)^{(n-1)/2}\Lambda(n) > T^{\sigma^*}\exp(-\tfrac{4}{5}\log^{19/20}T)$$

and analogously

(8.7) $$\min_{\xi_2\leqslant U_3<U_4\leqslant\eta_2} \sum_{\substack{n \text{ odd} \\ U_3\leqslant n\leqslant U_4}} (-1)^{(n-1)/2}\Lambda(n) < -T^{\sigma^*}\exp(-\tfrac{4}{5}\log^{19/20}T).$$

We remark further that from (8.3), (7.4) and (6.7) we have for $j = 1, 2$

(8.8) $$\xi_j > T\exp(-\log^{15/16}T), \quad \eta_j < T\exp(\log^{15/16}T).$$

9. Now in order to complete the proof of our theorem we have to distinguish two cases.

Case I. $T > c$ and there is at least one zero of $f(s)$ in the parallelogram

(9.1) $$\sigma \geqslant \tfrac{1}{2} + \log^{-1/20}T, \quad |t| \leqslant T.$$

Choosing such a zero as ϱ^* the right side of (8.6) respectively (8.7) is

$$> \sqrt{T}\exp(\tfrac{1}{5}\log^{19/20}T)$$

respectively

$$< -\sqrt{T}\exp(\tfrac{1}{5}\log^{19/20}T).$$

Since owing to (8.8) we have for $T > c$

$$\Big|\sum_{\substack{p>2 \\ U_1\leqslant p^a\leqslant U_2 \\ a\geqslant 2}} (-1)^{(p^a-1)/2}\log p\Big| < c\sqrt{U_2}\log^2 U_2 < c\sqrt{\eta_1}\log^2\eta_1 < \sqrt{T}\exp(\log^{15/16}T),$$

(1.3) is proved for this case. Analogously (1.4).

We may remark that the localisation of U_1 and U_2 is in this case much sharper than in (1.1) and amounts to

(9.2) $$T\exp(-2\log^{15/16}T) \leqslant U_1 < U_2 \leqslant T.$$

We could also prove by small modifications in this case the corresponding theorem for

$$\sum_{\substack{U_1\leqslant p\leqslant U_2 \\ p\equiv 1 \bmod 4}} 1 - \sum_{\substack{U_1\leqslant p\leqslant U_2 \\ p\equiv 3 \bmod 4}} 1$$

with the (9.2)-localisation.

Case II. $T > c$ and all zeros of $f(s)$ in

$$\sigma \geqslant \tfrac{1}{2}, \quad |t| \leqslant T$$

satisfy

$$\sigma \leqslant \tfrac{1}{2} + \log^{-1/20}T.$$

Since the treatment of this case is rather long, it is based on ideas of Littlewood, Ingham and Skewes and it is similar to our treatment of the sign-changes of $\pi(x,4,1) - \pi(x,4,3)$ in [3] we shall postpone it to the forthcoming English version of the book [4] of the second named author.

References

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