la convergence de la série (S_2) entraîne la convergence de la série (S') pour tout t réel.

Par ailleurs, la convergence de la série (S_2) entraîne l'existence d'un $M > 0$ tel que

$$|f(aq^r)| \leqslant M \quad \text{pour } r \geqslant 0 \text{ et } 1 \leqslant a \leqslant q-1.$$

On a alors (9) et la convergence des séries (S_1) et (S_2) entraîne celle de la série (S'') pour tout t réel.

La série (S^*) est convergente pout tout t réel puisque (S') et (S'') le sont.

D'après le Théorème 2, pour tout t réel, F_t possède une valeur moyenne égale à

$$\prod_{r=0}^{+\infty} \frac{1}{q}\Big(1+\sum_{a=1}^{q-1} F_t(aq^r)\Big).$$

On va voir que cette valeur moyenne est une fonction continue de t. En effet, si l'on pose

$$\frac{1}{q}\Big(1+\sum_{a=1}^{q-1} F_t(aq^r)\Big) = 1+u_r(t), \quad \text{d'où } u_r(t) = \frac{1}{q}\sum_{a=1}^{q-1}(e^{itf(aq^r)}-1),$$

et

$$v_r(t) = \frac{it}{q}\sum_{a=1}^{q-1} f(aq^r),$$

le Lemme 4 (avec $k=1$, puis avec $k=2$) montre que l'on a pour $|t| \leqslant T$

$$|u_r(t)| \leqslant \frac{T}{q}\sum_{a=1}^{q-1}|f(aq^r)|, \quad \text{d'où } |u_r(t)|^2 \leqslant \frac{T^2(q-1)}{q^2}\sum_{a=1}^{q-1} f(aq^r)^2$$

et

$$|u_r(t)-v_r(t)| \leqslant \frac{T^2}{2q}\sum_{a=1}^{q-1} f(aq^r)^2.$$

Le produit infini $\prod_{r=0}^{+\infty}\big(1+u_r(t)\big)e^{-v_r(t)}$ est donc uniformément convergent sur tout intervalle $[-T, +T]$, d'après le Lemme 6, et par suite sa valeur est une fonction de t continue sur $\boldsymbol{R}$. Or il est égal à

$$\Big(\prod_{r=0}^{+\infty}\frac{1}{q}\Big(1+\sum_{a=1}^{q-1}F_t(aq^r)\Big)\Big)\exp\Big(-\frac{it}{q}\sum_{r=0}^{+\infty}\Big(\sum_{a=1}^{q-1} f(aq^r)\Big)\Big).$$

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On the diophantine equation $a(x^n-1)/(x-1) = y^m$

by

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1. Among the diophantine equations $p(x) = y^m$, where $p(x)$ is a polynomial with integer coefficients, the equation

$$(1) \qquad a(x^{n-1}+\ldots+x+1) = a\frac{x^n-1}{x-1} = y^m \qquad (n \geqslant 1,\ |x| > 1,\ m \geqslant 2)$$

is an interesting special case. In addition to x and y, also a, n, and m can be integral variables in (1). The problem of determining the solvability in integers of (1) when all these numbers are variables seems to be unfeasible. Various special cases arise by fixing, or specializing in some other way, one or more of the variables in (1). Particularly the case $a = 1$ with certain other restrictions has been treated by many authors. Obláth [12] has determined all solutions of (1) when $x = 10$, $1 < a < 10$. More accurately and by a slightly different route, he has proved that all numbers which are perfect powers and whose digits in the scale of 10 are identical and not equal to 1, are 4, 8 and 9. Sierpiński also has discussed this problem in his monograph ([14], p. 276)(1).

As, from our point of view, the cases $n = 1$ and $n = 2$ can be regarded as trivial, we assume in the following that $n \geqslant 3$. Without loss of generality, it can also be assumed that m is a prime.

We shall make use of the following results concerning the case $a = 1$ which were given by Nagell and Ljunggren.

(A) If $4\,|\,n$, then the only solution of (1) in integers is $n = 4$, $x = 7$, $m = 2$, $y = \pm 20$ (cf. [8]).

(B) If $m = 2$, (1) has only the solutions $n = 4$, $x = 7$, $y = \pm 20$ and $n = 5$, $x = 3$, $y = \pm 11$ (cf. [6], [9] and also, because of the used method, [2]).

(1) Note also the problem presented by Obláth (J.-ber. Deutsch. Math. Verein. 47 (1937), p. 64, Aufgabe 258) and three solutions of that problem (ibid. 50 (1940), pp. 3–5). The solution given by J. Eröd is not, however, complete, since the theorem of Siegel [13] has been applied incorrectly.

(C) If $3|n$, (1) has only the solutions $m = n = 3$, $x = 18$ or -19, $y = 7$ (cf. [6]).

(D) If $m = 3$ and $n \not\equiv -1 \pmod 6$, then (1) has only the solutions $n = 3$, $x = 18$ or -19, $y = 7$ (cf. [6]).

We add also the following theorem concerning the case $n = 3$.

(E) The equation $x^2+x+1 = y^m$ has only the solutions $m = 3$, $x = 18$ or -19, $y = 7$ (cf. (C) or [10] and [4]). The equation $x^2+x+1 = 3y^m$, where $m \geqslant 3$, $|x| > 2$, has no integer solutions (cf. [10]).

In this paper, we extend (A) (cf. Theorem 4) and slightly also (D) (Theorems 1 and 2) and improve Obláth's result by determining the solutions of (1) for $1 < a < x \leqslant 10$ (Theorem 7). We make use, in addition to the theorems (A), ..., (E), of some other rather deep results, but otherwise our treatment is elementary by nature.

2. Write

$$Q(n, x) = \frac{x^n-1}{x-1} \quad (x \neq 1).$$

If p is a prime, the greatest common divisor $(x-1, Q(p, x))$ is p or 1 according as $x-1$ is divisible by p or not. If p is odd, then $Q(x, p)$ is not divisible by p^2.

We shall generalize these facts in the following three lemmas.

LEMMA 1. *If p is a prime and x is an integer $\neq 1$, then*

$$(x^{p^i}-1, Q(p, x^{p^{j-1}})) = p \text{ or } 1 \quad (0 \leqslant i < j)$$

according as $p|x-1$ or not.

Proof. If $p|x-1$, then $p|x^{p^i}-1$, and vice versa, because, by Fermat's little theorem,

$$x^{p^i} \equiv x^{p^{i-1}} \equiv \ldots \equiv x \pmod p.$$

Further, $x^{p^i}-1|x^{p^j}-1$ for $i \leqslant j$ and

$$(x^{p^{j-1}}-1, Q(p, x^{p^{j-1}})) = p \text{ or } 1$$

according as $p|x-1$ or not.

LEMMA 2. *Let p be an odd prime and let x belong to the exponent $d \pmod p$. If*

$$p^a \| Q(d, x), \quad d|n, \quad p^b|n, \quad (a, b \geqslant 0),$$

then

$$p^{a+b} \| Q(n, x).$$

Remark. As usually, $p^a \| n$ $(a \geqslant 0)$ means that $p^a|n$, $p^{a+1} \nmid n$. Evidently, $p \nmid Q(n, x)$, if $p|x$ or $d \nmid n$.

Proof. Write $n = p^b t$. Then $p \nmid t$, $d|t$ and $p^a \| Q(t, x)$ since

$$Q(t, x) = Q(d, x)Q(m, x^d) \quad (m = t/d)$$

and

$$Q(m, x^d) = x^{d(m-1)}+\ldots+x^d+1 \equiv m \not\equiv 0 \pmod p.$$

We have

$$Q(n, x) = Q(t, x)Q(p, x^t) \ldots Q(p, x^{p^{b-1}t}),$$

where the first factor on the right-hand side is divisible exactly by the power p^a and the other factors by p^1. This proves the lemma.

LEMMA 3. *Suppose $2^s \| x+1$ and $2^t \| n$. Then*

$$2 \nmid Q(n, x) \quad \textit{if} \quad s = 0 \ (\textit{i.e. } 2|x) \textit{ or } t = 0, \tag{2}$$

and

$$2^{s+t-1} \| Q(n, x) \quad \textit{if} \quad s > 0 \textit{ and } t > 0. \tag{3}$$

Proof. The case (2) is clear since $Q(n, x)$ is odd if x is even or n is odd. Write $n = 2^t k$ and let $s > 0$, $t > 0$. In the equation

$$Q(n, x) = Q(k, x)(x^k+1)(x^{2k}+1) \ldots (x^{2^{t-1}k}+1)$$

$2 \nmid Q(k, x)$, $2^1 \| x^{2^i k}+1$ $(i = 1, 2, \ldots, t-1)$ and $2^s \| x^k+1$, because k is odd and

$$x^k+1 = (x+1)Q(k, -x),$$

where the latter factor on the right-hand side is odd. Hence the lemma follows.

3. We give now two theorems which slightly extend the result (D) concerning the equation

$$Q(n, x) = y^3 \quad (n > 2). \tag{4}$$

We begin with two lemmas.

LEMMA 4. *The diophantine equation* (4) *can not have solutions for which $x = z^3$ or z^3+1 with $|z| > 1$.*

Proof. The equation (4) assumes accordingly the forms

$$x(z^{n-1})^3-(x-1)y^3 = 1, \quad x^n = 1+(zy)^3.$$

Since for a given x ($|x| > 2$), the equation $xu^3-(x-1)v^3 = 1$ has, by a result of Nagell [11], only the solution $u = v = 1$, the former equation is impossible because $|z^{n-1}| > 1$. By another result of Nagell ([10], p. 14), also the latter equation is impossible.

LEMMA 5. *If p is a prime with $9 \nmid p-1$ and if x belongs to exponent 3 or 6 (mod p), then x does not satisfy the equation*

$$(x^{6k-1}-1)/(x-1) = y^3, \tag{5}$$

where $k > 0$ and y are integers. The same holds if x belongs to exponent 12 and $x^2(x+1)$ to an exponent divisible by 3.

Proof. In the former case, p is of the form $6l+1$, and the assumption referring to p is equivalent to the conditions

$$x^6 \equiv 1, \quad x^2 \not\equiv 1 \pmod{p}. \tag{6}$$

Suppose that x satisfies equation (5), which becomes

$$x^{6k}-x = x(x-1)y^3. \tag{7}$$

By (6), it follows that $1-x \equiv x(x-1)y^3 \pmod{p}$ and so, since $p \nmid x-1$, that

$$x^{(p-1)/3} \equiv 1 \pmod{p}. \tag{8}$$

However, this is impossible since $(p-1)/3$ is not divisible by 3.

In the latter case, the assumption that x belongs to exponent 12 (mod p) is equivalent to the conditions

$$x^6 \equiv -1, \quad x^2 \not\equiv -1 \pmod{p}. \tag{9}$$

Now p has the form $12l+1$. If k in (5) is even, we get from (7) again (8) and so a contradiction. If k is odd, equation (7) leads to

$$x^3(x+1)^3 \equiv x^2(x+1)(x^2-x^4)y^3 \pmod{p}.$$

But, by (9), $x^4-x^2+1 \equiv 0 \pmod{p}$ and $x \not\equiv -1 \pmod{p}$, so that

$$[x^2(x+1)]^{(p-1)/3} \equiv 1 \pmod{p}.$$

This is a contradiction by virtue of the assumption of the lemma. This completes the proof.

Let r be a primitive root of p. Put $p = 6l+1$. Then the solutions of (6) are

$$x \equiv r^{sl} \pmod{p} \quad (s = 1, 2, 4, 5). \tag{10}$$

For $p = 12l+1$, the solutions of (9) are

$$x \equiv r^{sl} \pmod{p} \quad (s = 1, 5, 7, 11). \tag{10'}$$

THEOREM 1. *Suppose that the integers $k \geqslant 1$, x, y satisfy the condition (5). Then $x \equiv 0, \pm 1 \pmod{9}$ and also (mod 7), and $x \not\equiv -2, \pm 3, \pm 4, -6 \pmod{13}$.*

Proof. Suppose that $x \not\equiv 0, \pm 1 \pmod{9}$.

The numbers x^2+x+1 and x^2-x+1 are relatively prime, so that at most one is divisible by 3, if it is divisible, but then not by 9. All the other prime factors of these numbers have the form $6l+1$. At least one of these prime factors, say p, fulfills the condition $9 \nmid p-1$, since, otherwise, either

$$x^2+x+1 \equiv 1, \quad x^2-x+1 \equiv 1 \pmod{9},$$

and so $x \equiv 0 \pmod{9}$, or

$$x^2 \pm x+1 \equiv 3, \quad x^2 \mp x+1 \equiv 1 \pmod{9},$$

and so $x \equiv \pm 1 \pmod{9}$.

We see directly that $x^3 \equiv \pm 1 \pmod{p}$ and $x^2 \not\equiv 1 \pmod{p}$. Hence x belongs to exponent 3 or 6 (mod p). Using Lemma 5 and the assumption of our theorem, we have a contradiction, which proves the first assertion.

Let us now apply Lemma 5 to the primes $p = 7$ and $p = 13$. When, in the first case, we omit the residue classes (10), the result mentioned in the theorem follows. In the case $p = 13$ $(r = 2)$, the solutions of (10) and (10′) are, respectively,

$$x \equiv \pm 3, \pm 4 \quad \text{and} \quad x \equiv \pm 2, \pm 6 \pmod{13}.$$

In the latter set, $x \equiv -2$ and $x \equiv -6 \pmod{13}$ satisfy the condition set for $x^2(x+1)$ and so the last result of the theorem follows immediately from Lemma 5.

THEOREM 2. *For $1 < x \leqslant 70$, the equation (5) is not solvable in integers $k > 0$, y, and the equation (4) has only the solution $n = 3$, $x = 18$, $y = 7$. For any given $x > 70$, the equation (5) has at most one solution in positive integers k, y.*

Proof. If k, x, y satisfy (5), there are according to Theorem 1 only the following possibilities:

$$x \equiv 0, 1, 8, 27, 28, 35, 36, 55, 62 \pmod{63}.$$

By the result concerning the prime $p = 13$, we see that at most the numbers $x = 8 = 2^3$, $27 = 3^3$, $28 = 3^3+1$ and $64 = 4^3$ in the interval $1 < x \leqslant 70$ can be solutions of (5). However, it follows from Lemma 4 that also these do not come into question. By (D), we conclude now that the equation (4) has only the solution $n = 3$, $x = 18$, $y = 7$ when $1 < x \leqslant 70$.

The equation (5) can also be written in the form

$$x^2(x^{2k-1})^3-(x-1)y^3 = 1.$$

Since for every fixed $x > 70$, we have $x^2(x-1) > 72 \cdot 70^2 > 4^2 \cdot 3^9$, the last assertion of the theorem follows from a known result of Siegel (cf. [13] or [7], p. 273). The same can be verified also by means of one of Nagell's results [11] mentioned already above. The proof is complete.

4. The following lemma is a collection of some well-known results, which we shall need later.

Lemma 6. *The only integer solutions of the equation*

(a) $x^4-2y^2 = 1$ *are* $x = \pm 1,\ y = 0$;

(b) $x^4-2y^2 = -1$ *are* $x = \pm 1,\ y = \pm 1$;

(c) $x^2-2y^4 = 1$ *are* $x = \pm 1,\ y = 0$;

(d) $x^2-2y^4 = -1$ *are* $x = \pm 1,\ y = \pm 1$ *and* $x = \pm 239,\ y = \pm 13$;

(e) $x^2-8y^4 = 1$ *are* $x = \pm 1,\ y = 0$ *and* $x = \pm 3,\ y = \pm 1$.

The deep result (d) was proved by Ljunggren [5]. The others have been known already a long time (cf. [16], p. 404 or [14], p. 98; (e) follows easily from (b) and (c) from (e)).

Theorem 3. *The only integer solutions of the system*

$$x+1 = 2^e py^2, \quad x^2+1 = 2z^2, \tag{11}$$

where p is a prime, $e \geqslant 0$ and y odd, are given by $x = 1, 7$ and $p = 2$.

Remark. In the case $p = 2$, this implies Genocchi's result (cf. [9], pp. 404–406) asserted already by Fermat.

Proof. Suppose that (11) holds and write the latter equation in (11) in the form

$$\left(\frac{x+1}{2}\right)^2+\left(\frac{x-1}{2}\right)^2 = z^2.$$

It suffices to consider only the case $x > 1$.

If $4\,|\,x-1$, there exist integers r and s such that

$$\tfrac{1}{2}(x+1) = r^2-s^2,\quad \tfrac{1}{2}(x-1) = 2rs,\quad r > s > 0,\ (r,s) = 1,\quad 2\,|\,rs. \tag{12}$$

From this and from (11), it follows that

$$(r-s)^2-2s^2 = 1,\quad (r+s)^2-2r^2 = -1,\quad r^2-s^2 = 2^{e-1}py^2, \tag{13}$$

where $p = 2$, $e = 0$ or $p \geqslant 3$, $e = 1$, since r^2-s^2 is odd. The numbers $r-s$ and $r+s$ are relatively prime, whence at least one of them is a square. By Lemma 6, a contradiction follows now from (13) and (12).

In the case $4\,|\,x+1$, the first two equations in (13) must be replaced by the conditions

$$\tfrac{1}{2}(x+1) = 2rs,\quad \tfrac{1}{2}(x-1) = r^2-s^2, \tag{12′}$$

so that now

$$(r+s)^2-2r^2 = 1,\quad (r-s)^2-2s^2 = -1,\quad x+1 = 4rs = 2^e py^2. \tag{13′}$$

By Lemma 6, it follows from the first equation in (13′) that r cannot be a square. From the same equation, it follows that $2\,|\,r$.

If $p\,|\,s$ and so $p \neq 2$, we obtain, from the latter condition in (13′), $r = 2r_1^2$, whence

$$(r+s)^2-8r_1^4 = 1.$$

This is of the form (e) so that $r+s = 3$ and, therefore, $r = 2$, $s = 1$, contradicting the condition $p\,|\,s$.

It remains to consider the case $p \nmid s$. From (13′), it follows that $s = s_1^2$ and so

$$(r-s)^2-2s_1^4 = -1.$$

By Lemma 6 (d), we have $s_1 = 1$ or 13. In the first case, $s = 1$, $r-s = 1$, $r = 2$, so that by virtue of the last relation in (13′), $x = 7$, $p = 2$. In case $s_1 = 13$, we have $s = 169$ and $r-s = 239$, whence $r = 408 = 8\cdot 3\cdot 17$. On the other hand, by (13′), $r = 2^{e-2}pr_2^2$, where r_2 is an integer. This contradiction establishes the theorem.

5. The following theorem is an extension of Nagell's result ([8], p. 78). However, we restrict ourselves to the case $x > 1$ since only this will be used later.

Theorem 4. *If the integers $n > 0$, $x > 1$, $y > 0$, $e \geqslant 0$, $f \geqslant 0$ and the primes $m \geqslant 2$, $p > 2$ satisfy the conditions*

$$Q(n,x) = 2^e p^f y^m,\quad 4\,|\,n,\quad (y, 2p^f) = 1, \tag{14}$$

then

(i) $n = 4$, $x = 7$, $e = 4$, $f = 0$, $m = 2$, $y = 5$ *or* $f = 2$, $p = 5$, $y = 1$,

or

(ii) $n = 8$, $x = 7$, $e = 5$, $f = 1$, $p = 1201$, $m = 2$, $y = 5$,

or the following conditions are fulfilled;

(iii) $n = 4l$, $(2,l) = (f,m) = 1$, $p \equiv 1 \pmod 4$, *and there are odd integers u, v, w such that $y = uvw$ and*

$$Q(l,x) = u^m,\quad x^l+1 = v^m,\quad v^{2l}+1 = p^f w^m \quad \text{if} \quad 2\,|\,x, \tag{15}$$

and

$$Q(l,x) = u^m,\quad x^l+1 = 2^{e-1}v^m,\quad x^{2l}+1 = 2p^f w^m \quad \text{if} \quad 2\nmid x. \tag{16}$$

Proof. We can suppose that $m \nmid (e,f)$ since otherwise it follows from (A) that (i) holds.

We write the equation (14) in the form

$$Q(l,x)(x^l+1)(x^{2l}+1)(x^{4l}+1)\dots(x^{2^{t-1}l}+1) = 2^e p^f y^m, \tag{17}$$

where $n = 2^t l$, $2 \nmid l$. The first factor on the left-hand side is odd and the g.c.d.

$$(x^{2^i l}-1, x^{2^i l}+1) = 1 \text{ or } 2 \quad (i \geqslant 0)$$

according as $e = 0$ $(2|x)$ or $e > 0$ $(2\nmid x)$.

Suppose first that $e = 0$. Then $2|x$. If $t \geqslant 3$, it follows from (17) that

$$(18) \qquad x^{2^s l}+1 = y_1^m \quad (s = 1 \text{ or } 2),$$

where y_1 is an integer. By a result of Lebesgue ([7], p. 301), this is impossible when m is odd. Clearly, the same holds in case $m = 2$. If $t = 2$, it follows likewise that $p^f|x^{2l}+1$. This leads to (iii) (15), since the condition $(f, m) = 1$ follows directly from what we just said about (18).

Let now $e > 0$. According to a result of Störmer [15], the condition

$$(19) \qquad x^{2^s l}+1 = 2z^m \quad (s \geqslant 1, x > 1)$$

is impossible if m is odd. Now it follows from (17), if m is odd, that $t = 2$, $p^f|x^{2l}+1$ and hence that (16) is valid. From Störmer's result it follows again that $(f, m) = 1$.

We must still consider case $m = 2$. At first, we note that $t \leqslant 3$ since, otherwise, it would follow from (17) that (19) holds for $s = 2$ or 3, contrary to Lemma 6 (b). If $t = 3$, $p^f|x^{4l}+1$ for the same reason. Hence there exist natural numbers g, v, w such that

$$(20) \qquad x^l+1 = 2^g v^2, \quad x^{2l}+1 = 2w^2, \quad 2\nmid v.$$

By Theorem 3, we find at once that $x = 7$, $l = 1$, $n = 8$. Since $Q(8, 7) = 2^5\cdot 5^2\cdot 1201$, this gives the solution (ii).

If, in case $t = 2$, $p^f|Q(l, x)$, we get from (16) again the conditions (20), whence $x = 7$, $l = 1$ and consequently $n = 4$. This leads to (i). If $p^f|x^l+1$, then there is an integer u such that

$$Q(l, x) = u^2.$$

Now, by (B), $l = 5$, $x = 3$ or $l = 1$. The former possibility cannot come into question since $3^{10}+1 = 2\cdot 5^2\cdot 1181$. If, in case $l = 1$ $(n = 4)$, $2|f$, we get again (20) and finally (i). If $2\nmid f$, we have, instead of (20),

$$x+1 = 2^{e-1}pv^2, \quad x^2+1 = 2w^2,$$

where v and w are odd. By Theorem 3, this system is impossible. There remains the case $p^f|x^{2l}+1$. Then (16) holds. The condition $(f, m) = 1$ can be clarified by using Theorem 3. This completes the proof.

Remark I. Every solution of system (15) and also of (16) gives, of course, a solution of equation (14).

Remark II. If $l = 1$, then (15) and (16) are reduced to the following systems

$$(21) \qquad x+1 = v^m, \quad x^2+1 = p^f w^m,$$

$$(21') \qquad x+1 = 2^{e-1}v^m, \quad x^2+1 = 2p^f w^m.$$

This is so at least when m is even. In fact, if $l \geqslant 3$, then from the first condition in (15) and (16) it would follow that $l = 5$, $x = 3$, However, this is impossible since $3^5+1 = 2^2\cdot 61$.

The only solutions of (21) in the interval $1 < x \leqslant 100$ are $x = 24, 26$ and those of (21′) $x = 3, 7, 15, 35, 49, 99$. These give solutions for equation (14) if $n = 4$.

Remark III. If $l \geqslant 3$ and $x \leqslant 10^{11}$, then the middle equation in (15), Catalan's equation, is impossible by a result of Hyyrö [1] and thus (15) drops out. If $x \leqslant 70$, it follows from (B), (C) and Theorem 2 that $m \geqslant 5$ and $3\nmid l$ in (16).

THEOREM 5. *If equation* (1) *has an integer solution with* $a = x-1$, $n \geqslant 3$, $x > 1$, *then* $x > 10^{11}$.

This follows immmediately from Hyyrö's result mentioned just above.

In the sequel, we still need

LEMMA 7. *The system*

$$(22) \qquad x^l-1 = (x-1)y^5, \quad x^l+1 = (x+1)z^5 \quad (l > 1)$$

where $x \not\equiv 0, \pm 1 \pmod{11}$ *and* $l \not\equiv 0, 1 \pmod 5$, *is not solvable in integers* x, y, z, l. *The same is true for* $x \geqslant 7$, $l \equiv 1 \pmod 5$ *and also for* $1 < x < 23$, $l \equiv 0 \pmod 5$.

Proof. Let x, y, z, l be a solution and let x belong to exponent $d \pmod{11}$. If $11|yz$, then $x^{2l} \equiv 1 \pmod{11}$. Thus $d|(10, 2l)$ and hence $d = 1$ or 2, so that $x^2 \equiv 1 \pmod{11}$, contradicting the assumption of Lemma 7. We obtain now from (22)

$$(x^l-1)^2 \equiv (x-1)^2, \quad (x^l+1)^2 \equiv (x+1)^2 \pmod{11}$$

and further from these congruences by subtraction

$$x^{l-1} \equiv 1 \pmod{11}.$$

Since $5\nmid l-1$, this leads to a contradiction in the same way as just above.

In case $l = 5k+1$ $(k > 0)$, by multiplying the equations (22), we get

$$x^2(x^{2k})^5-(x^2-1)(yz)^5 = 1.$$

The equation $x^2u^5-(x^2-1)v^5$ has, therefore, two solutions $u=v=1$ and $u=x^{2k}>1$, $v=yz$ provided that the system (22) has a solution with $x\geqslant 7$. This is impossible by Siegel's result [13] mentioned above, since, for $x\geqslant 7$,

$$[x^2(x^2-1)]^6>(50\cdot 45)^6=4^3\cdot 81^3\cdot 5^{18}>4^9\cdot 5^{21}>4^4\cdot 5^{25}.$$

It follows from a result of Lebesgue [3] that the equation $x^5+y^5=Az^5$ has no integer solution x, y, z, $|xyz|>1$, if A is not divisible by a prime factor of the form $5r+1$ and if $A\not\equiv \pm 1, \pm 7 \pmod{25}$. By this, we establish immediately that at least one of the equations (22) is not solvable when $l\equiv 0 \pmod 5$, $1<x<23$. The proof is now complete.

6. In this section, we restrict our considerations to the case $1<a<x<15$. Now the "digit" $a=2^cp^g$, where $p=3, 5, 7, 11, 13$ $(0\leqslant c\leqslant 3,\ 0\leqslant g\leqslant 2)$, and (1) can be written in the form

$$Q(n,x)=2^ep^fy^m,\qquad (y,2p^f)=1, \tag{23}$$

where $e=hm-c\geqslant 0$, $f=km-g\geqslant 0$ $(h,k\geqslant 0)$.

Let us consider first the case $a=2^c$ $(g=0,\ 1\leqslant c\leqslant 3)$.

THEOREM 6. *The only solution of the equation*

$$2^cQ(n,x)=y^m\qquad (1<2^c<x<15)$$

is $c=2$, $n=4$, $x=7$, $m=2$, $y=\pm 40$.

Proof. We consider (23) (from which p^f drops out).

In case $4|n$, we get directly, by Theorem 4, the solution mentioned in Theorem 6. Suppose now $4\nmid n$. If $m|e$, then $m|c$ and $c=m=2$ or $c=m=3$. By (B), in the former case, x must be 3, which gives the contradiction $c=1$. In the latter case, we have, by Theorem 2, $x\geqslant 18$.

Thus we need consider only the case $m\nmid e$. Then $e>0$ and, by Lemma 3, $n=2l$, where l is an odd number $\geqslant 3$. It follows now from (23) that

$$Q(l,x)=u^m,\qquad x^l+1=2^ev^m\qquad (y=uv). \tag{24}$$

In the case $m=2$, we have, by virtue of the first equation in (24), $l=5$, $x=3$, which, however, does not satisfy the second equation. As above, the condition $m=3$ leads to $x\geqslant 18$. The latter equation in (24) gives $2^e|x+1$. If $m\geqslant 7$, it follows from (23) that $e\geqslant 4$ and so $x\geqslant 15$. If, in case $m=5$, first $c=2$, then $e\geqslant 3$ and, necessarily, $x+1=8$. However, we see from (24) by means of Lemma 7 that this is impossible. (The case $5|l$ would be settled also by a result of Skolem ([7], p. 276), since the second equation in (24) becomes $(7^r)^5+1=8v^5$.)

If, next, $c=3$, then $e\geqslant 2$ and $x\geqslant 9$. Since $4|x+1$, we have $x=11$ and, by (24), $Q(l,-11)=3^4w^5$, where w is an integer. According to Lemma 2, $3|l$. Applying (C) to the first equation in (24), we come to a contradiction, which completes the proof.

We deal now with the case $a=2^cp^g$, $g>0$, where $g=1$ or 2 for $p=3$ and $g=1$ for other primes $p\leqslant 13$. From Theorem 4 and the Remarks II and III concerning this theorem it follows easily that in case $4|n$, it suffices to consider only (16) for $m\geqslant 5$ $(l\geqslant 5)$. By Theorem 5, the case $p=13$ is not possible, since then $a=13$, $x=14$. There remains only the case $p=5$. By the third equation in (16), it follows that x belongs to exponent $4 \pmod 5$, whence $x=7$ or 13. If $x=7$, we see from (14) (or from the second equation in (16)), by Lemma 3, that $e=4$. Since now $a=5$, we have $c=0$ and so, on the other hand, $e=hm\geqslant 5$. If $x=13$, we establish similarly that $e=2$. Since now $a=5$ or 10, we have $e\geqslant hm-1\geqslant 4$. Except for the solutions given by $n=4$, $x=7$, there do not exist any other solutions in case $4|n$.

Let now $4\nmid n$. If, in (23), $f=0$, then $p=3$, $a=9<x$, $c=0$, $m=2$. Thus $2|e$, which leads, by (B), to a contradiction. We can now suppose that $f>0$ and so $p\nmid x$.

We shall treat first the case $p=3$, $3|x-1$. By Lemma 2, it follows from (23) that $3|n$. If $n=6l$ and hence l is odd, we get from (23)

$$Q(3l,x)=3^fu^m,\qquad x^{3l}+1=2^ev^m\qquad (y=uv). \tag{25}$$

Further, from the first equation in (25),

$$Q(l,x)=3^{f-1}w^m,\qquad x^{2l}+x^l+1=3z^m, \tag{26}$$

since the g.c.d. $(x^l-1, Q(3,x^l))=3$ and $3^2\nmid Q(3,x^l)$. From the second equation in (26) it follows, by (E), that $m=2$, and further from the same equation that $l\geqslant 3$, since $x=22$ gives in the case $l=1$ the smallest solution $(x>1)$ of this equation. Either the first condition in (25) or the first condition in (26) gives now a contradiction, by (B).

In case $n=3l$, $2\nmid l$, we have $e=0$. From equation (23), we get now directly a pair of conditions which have the same forms as in (26). As above, we note that $m=2$, $l\geqslant 3$. Now either (23) or (26) gives, by (B), a contradiction.

Let x belong to exponent $d \pmod p$ and let first d be even. Then, by (23), $d|n$, $n=2l$ $(2\nmid l,\ l\geqslant 3)$ and

$$Q(l,x)=u^m,\qquad (x+1)Q(l,-x)=2^ep^fv^m\qquad (y=uv). \tag{27}$$

Since $3\leqslant p<x<15$, it follows again that $m\geqslant 5$, $3\nmid l$. If $p=3$, the second equation gives, by Lemma 2, $3^f|x+1$. Then $x\geqslant 3^f-1\geqslant 26$, since $f\geqslant m-2\geqslant 3$. In case $p\geqslant 5$, x cannot be odd, since otherwise $e>0$, and consequently, because now $c\leqslant 1$, $e\geqslant 4$ and $x\geqslant 2^4-1=15$. If $d=2$, i.e. $p|x+1$, then $x+1\geqslant 3p$, whence only $p=5$, $x=14$ is possible. Since now $3|y$, we have $3^5|Q(n,x)$ and, since $3^1\|Q(2,x)$, by Lemma 2,

$3\mid n$ and so $3\mid l$, contrary to the result found above. If $d=6$, then again $3\mid l$. There remains the case $d=10$, whereupon $p=11$. However, the even numbers in the interval $11<x<15$ do not belong to exponent 10 (mod 11). Let now d be odd. Then $p\geqslant 5$, since the case $p=3$, $3\mid x-1$ was treated already above. If $d=1$, it follows from Theorem 5 that the only possibility is $x=11$, $a=p=5$. If $d=3$, then correspondingly $p=7$ and $x=9$ or 11. If in the former case $n=2l$, (23) becomes

$$Q(4l,3)=2^{e+2}7^{t}y^{m},$$

which according to Theorem 4 is impossible. Since thus n is odd, we get, from (23), $Q(n,3)=u^m$. This is, however, false by (C) since $3\mid n$. The case $x=11$, $a=7$ remains open, likewise the case $x=14$, $p=a=11$ ($d=5$). Thus we have proved

THEOREM 7. *The only positive integer solution of the equation*

$$a\frac{x^n-1}{x-1}=y^m\qquad(1<a<x,\ n>2,\ m\geqslant 2)\tag{28}$$

when $x\leqslant 10$, *is* $a=4$, $n=4$, $x=7$, $m=2$, $y=40$. *If* $10<x<15$, *there are, if any, solutions only in the cases* $x=11$, $a=5$, 7 *and* $x=14$, $a=11$.

The only integer solutions of the equation (28), when $1\leqslant a<x<15$, $n=2$, are given by

$$x=3,\ a=1,2;\qquad x=7,\ a=1,2,4;\qquad x=8,\ a=1,3,4;$$
$$x=11,\ a=3.$$

For bases $x\leqslant 100$, the numbers with three identical digits which are perfect powers (i.e. the solutions of (28) with $1\leqslant a<x\leqslant 100$, $n=3$) are given by

$$x=18,\ a=1,7,8;\qquad x=22,\ a=3,12;\qquad x=30,\ a=19;$$
$$x=68,\ a=13,52,$$

and the numbers with four digits, by

$$x=7,\ a=1,4;\qquad x=41,\ a=21;\qquad x=99,\ a=58.$$

Of these, the first and the third are cubes, the second a biquadrate and all the other squares.

It can be easily shown that, if a solution mentioned in the latter part of Theorem 7 exists, then x and m satisfy the condition

$$Q(q,x)=z^m,$$

where q is a prime and z an integer. This reveals the decisive importance of the case $a=1$ in the study of the diophantine equation (1).

References

[1] S. Hyyrö, *Über das Catalansche Problem*, Ann. Univ. Turku A I, 79 (1964), 11 pp.

[2] H. J. Kanold, *Über eine spezielle Klasse von diophantischen Gleichungen*, J. Reine Angew. Math. 245 (1970), pp. 165–171.

[3] V. A. Lebesgue, *Sur l'équation indéterminée* $x^5+y^5=Az^5$, Journ. de Math. 8 (1843), pp. 49–70.

[4] W. Ljunggren, *Einige Bemerkungen über die Darstellung ganzer Zahlen durch binäre kubische Formen mit positiver Diskriminante*, Acta Math. 75 (1942), pp. 1–21.

[5] — *Zur Theorie der Gleichung* $x^2+1=Dy^4$, Avh. Norske Vid. Akad. Oslo I, no. 5 (1942), 27 pp.

[6] — *Noen setninger om ubestemte likninger av formen* $\frac{x^n-1}{x-1}=y^q$, Norsk. Mat. Tidsskr. 1. Hefte 25 (1943), pp. 17–20.

[7] L. J. Mordell, *Diophantine Equations*, London and New York 1969.

[8] T. Nagell, *Note sur l'équation indéterminée* $\frac{x^n-1}{x-1}=y^q$, Norsk. Mat. Tidsskr. 1920, pp. 75–78.

[9] — *Sur l'équation indéterminée* $\frac{x^n-1}{x-1}=y^2$, Norsk. Mat. Forenings Skrifter I No. 3 (1921), 17 pp.

[10] — *Des équations indéterminées* $x^2+x+1=y^n$ *et* $x^2+x+1=3y^n$, Norsk. Mat. Forenings Skrifter I No. 1 (1921), 14 pp.

[11] — *Solution complète de quelques équations cubiques à deux indéterminées*, Journ. de Math. (9) 4 (1925), pp. 209–270.

[12] R. Obláth, *Une propriété des puissances parfaites*, Mathesis 65 (1956), pp. 356–364.

[13] C. L. Siegel, *Die Gleichung* $ax^n-by^n=c$, Math. Ann. 114 (1937), pp. 57–68.

[14] W. Sierpiński, *Elementary Theory of Numbers*, Warszawa 1964.

[15] C. Störmer, *L'équation* $m\operatorname{arctang}\frac{1}{x}+n\operatorname{arctang}\frac{1}{y}=k\frac{\pi}{4}$, Bull. Soc. Math. France 27 (1899), pp. 160–170.

[16] J. V. Uspensky and M. A. Heaslet, *Elementary Number Theory*, New York and London 1939.

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