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Comments on some formulae of Ramanujan

by

EMIL GROSSWALD* (Philadelphia, Penn.)

1. Introduction. It has been shown (see [1]) that if $\zeta(s)$ stands for the Riemann zeta function, then one has for integral $n \geqslant 1$,

$$\zeta(2n+1) = \pi^{2n+1} r_{2n+1} - S_n, \tag{1}$$

with rational r_{2n+1} and

$$S_n = 2\sum_{m=1}^{\infty} m^{-(2n+1)}(e^{2\pi m}-1)^{-1} + (2\pi/n)\big(1+(-1)^n\big)\sum_{m=1}^{\infty} m^{-2n} e^{2\pi m}(e^{2\pi m}-1)^{-2}.$$

Formula (1) puts into evidence the relevance of the arithmetical nature of sums of the form

$$\sum_{m=1}^{\infty} (\pi m)^{-a} e^{2\pi m b}(e^{2\pi m}-1)^{-c}$$

and of some of their linear combinations. For $n = -k$, $0 < k \in \boldsymbol{Z}$ we still define S_{-k} formally as above and for $n = 0$ we understand by S_0 the sum

$$S_0 = \sum_{m=1}^{\infty} m(e^{2\pi m}-1)^{-1}.$$

It is rather remarkable that for $n = -k < -1$ one has the explicit formula

$$S_n = S_{-k} = 2\sum_{m=1}^{\infty} m^{2k-1}(e^{2\pi m}-1)^{-1} - 2\pi k^{-1}\big(1+(-1)^k\big)\sum_{m=1}^{\infty} m^{2k} e^{2\pi m}(e^{2\pi m}-1)^{-2} = B_{2k}/2k; \tag{2}$$

here the B_{2k}'s stand for the Bernoulli numbers in the customary normalization so that $B_{2k}/2k = -\zeta(1-2k)$ and is rational. No corresponding result seems to be known for positive n.

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For k odd, (2) has been stated by Ramanujan (see [4], v. II, p. 171, Cor. iv) and has been proven by Watson (see [6]). For k even, (2) does not seem to occur in the literature, but it is, like Watson's result, an easy consequence of the following Theorem A. Here and in what follows, α and β stand for real, positive numbers.

THEOREM A. *For $\alpha\beta = \pi^2$ and integral $k > 1$ one has*

$$(3)\qquad \alpha^k\Big(\tfrac{1}{2}\zeta(1-2k)+\sum_{m=1}^{\infty} m^{2k-1}(e^{2m\alpha}-1)^{-1}\Big)- \\ -(-\beta)^k\Big(\tfrac{1}{2}\zeta(1-2k)+\sum_{m=1}^{\infty} m^{2k-1}(e^{2m\beta}-1)^{-1}\Big)=0.$$

Theorem A has been stated by Ramanujan ([4], v. I, p. 259, No. 14) and has been proven by Hardy [3]. For k odd and $\alpha = \beta = \pi$ one obtains from (3) that

$$(2')\qquad \sum_{m=1}^{\infty}\frac{m^{2k-1}}{e^{2\pi m}-1}=\frac{B_{2k}}{4k},$$

in agreement with (2) for odd k.

For k even, and $\alpha = \beta = \pi$ the left hand member of (3) vanishes trivially; but if one first divides by $\alpha-\beta$ and then lets $\alpha\to\pi$, while observing that $\alpha\beta = \pi^2$, a simple computation yields

$$(2'')\qquad \sum_{m=1}^{\infty}\frac{m^{2k-1}}{e^{2\pi m}-1}-\frac{2\pi}{k}\sum_{m=1}^{\infty}\frac{m^{2k}e^{2\pi m}}{(e^{2\pi m}-1)^2}=\frac{B_{2k}}{4k},$$

i.e. (2) for even k.

A most remarkable result, involving precisely the sums S_n $(n > 0)$ here under consideration has been stated by Ramanujan (see [4], v. I, p. 259, No. 15 and v. II, p. 177, No. 21) and reads (in the present notations) as follows:

THEOREM B. *For $\alpha\beta = \pi^2$ and rational, integral $n > 1$,*

$$(4)\qquad (4\alpha)^{1-n}\Big\{\tfrac{1}{2}\zeta(2n-1)+\sum_{m=1}^{\infty} m^{1-2n}(e^{2\alpha m}-1)^{-1}\Big\}- \\ -(-4\beta)^{1-n}\Big\{\tfrac{1}{2}\zeta(2n-1)+\sum_{m=1}^{\infty} m^{1-2n}(e^{2\beta m}-1)^{-1}\Big\}+ \\ +\sum_{k=0}^{[n/2]}{}'(-1)^k\pi^{2k}\frac{B_{2k}}{(2k)!}\,\frac{B_{2n-2k}}{(2n-2k)!}\{(-\alpha)^{n-2k}+\beta^{n-2k}\}=0.$$

Here and in what follows $[x]$ stands for the greatest integer function and the dash $\sum'$ means that for even n, the last term is $(-1)^{n/2}\pi^n(B_n/n!)^2$ rather than twice that value.

2. Main results. All statements quoted so far are immediate corollaries of the main result of the present paper, which may be formulated as follows:

THEOREM 1. *For real positive α, β with $\alpha\beta = \pi^2$ and $s = \sigma+it$ (σ, t real) consider the function*

$$(5)\qquad F(s;\alpha,\beta)=(-\alpha)^{1-s}\Big\{\tfrac{1}{2}\zeta(2s-1)+\sum_{m=1}^{\infty} m^{1-2s}(e^{2\alpha m}-1)^{-1}\Big\}- \\ -\beta^{1-s}\Big\{\tfrac{1}{2}\zeta(2s-1)+\sum_{m=1}^{\infty} m^{1-2s}(e^{2\beta m}-1)^{-1}\Big\}- \\ -\tfrac{1}{2}\sum_{\nu=0}^{[\sigma]}(-1)^\nu\zeta(2\nu)\zeta(2s-2\nu)\pi^{2\nu-2s}\{(-\alpha)^{s-2\nu}+\beta^{s-2\nu}\};$$

where the last sum is empty (i.e., equal to zero) for $\sigma < 0$, and where $(-\alpha)^{1-s}$ is defined by $\exp\{(1-s)\log(-\alpha)\}$ with some definite (but arbitrary) determination of the logarithm. Then $F(n;\alpha,\beta) = 0$ for all rational integers $n \neq 1$.

The summation condition on ν shows that $F(s;\alpha,\beta)$ is not holomorphic (not even continuous) in s. In view of the fact that in the proof we shall consider separately the cases $n < 0$, $n = 0$, $n > 1$, one may still suspect that $F(1;\alpha,\beta)$, properly interpreted, would vanish. That this is in fact not the case is born out by

THEOREM 2. *For $\alpha\beta = \pi^2$ and real s,*

$$(6')\qquad \lim_{s\to 1^+}F(s;\alpha,\beta) \\ =\tfrac{1}{4}\log(-\beta/\alpha)+(\beta-\alpha)/12+\tfrac{1}{2}\sum_{m=1}^{\infty}m^{-1}(\coth\alpha m-\coth\beta m),$$

$$(6'')\qquad \lim_{s\to 1^-}F(s;\alpha,\beta) \\ =\tfrac{1}{4}\log(-\beta/\alpha)+(\beta-\alpha)/24+\tfrac{1}{2}\sum_{m=1}^{\infty}m^{-1}(\coth\alpha m-\coth\beta m).$$

For $\alpha = \beta = \pi$, in particular, $\lim_{s\to 1}F(s;\pi,\pi) = \frac{1}{4}\log(-1) \neq 0$ the determination of $\log(-1)$ being consistent with that in Theorem 1.

Theorem 2 may be formulated differently, by using the following, apparently new result, which may have some independent interest.

THEOREM 3. *For real, positive α, define*

$$G(\alpha)=\sum_{m=1}^{\infty}m^{-1}(e^{2m\alpha}-1)^{-1}-\tfrac{1}{4}\log\alpha+\alpha/12;$$

then, if $\alpha\beta = \pi^2$,

$$G(\alpha)=G(\beta).$$

Taking particular values for α and β one obtains any number of corollaries, such as, e.g., the following one which corresponds to $\beta = 2\pi$, $\alpha = \pi/2$;

COROLLARY 1.

$$\sum_{m=1}^{\infty} m^{-1}(e^{\pi m}-1)^{-1} - \sum_{m=1}^{\infty} m^{-1}(e^{4\pi m}-1)^{-1} = (\pi - \log 16)/8.$$

For $k = 0$, (2) is meaningless; instead, the following theorem holds.

THEOREM 4.

$$\sum_{m=1}^{\infty} (e^{2\pi m}-1)^{-1} + \sum_{m=1}^{\infty} (e^{2\pi m}-1)^{-2} = (\pi-3)/24\pi.$$

Using Theorem 3, Theorem 2 may be reformulated as

THEOREM 2'. *For real* s, $\alpha > 0$, $\beta > 0$, $\alpha\beta = \pi^2$,

$$\lim_{s\to 1^+} F(s;\alpha,\beta) = \frac{\beta-\alpha}{6} + \frac{1}{4}\log(-1),$$

$$\lim_{s\to 1^-} F(s;\alpha,\beta) = \frac{\beta-\alpha}{8} + \frac{1}{4}\log(-1),$$

the determination of $\log(-1)$ *being consistent with that in Theorem* 1.

The statements of Theorems 1–4 and some related ones will be proven in the following Sections 3 to 8. No significant contribution is made to the determination of the arithmetical nature of the sums S_n for $n > 0$, which was the original motivation of this investigation. The author gratefully acknowledges his indebtedness to Professor C. L. Siegel, who stimulated this work and was instrumental in the correction of some errors in an earlier version of this paper.

3. Proof of Theorem 1 (First Part). In what follows Z stands for the set of rational integers.

(a) Case 1, $0 > n \in Z$. For $s = n < 0$, the last sum in (5) is empty and, setting $s = 1-k$, $1 < k \in Z$, (5) reduces essentially to the left hand side of (3). The result now follows from Theorem A, proven by Hardy [3] and (2'), (2'') immediately follow from it, as already seen.

(b) Case 2, $s = 0$. The case $s = 0$, $k = 1$ is specifically excluded by Hardy. Formally, setting $s = 0$ in (5), the assertion of Theorem 1 becomes

$$\alpha\Big\{\tfrac{1}{2}\zeta(-1) + \sum_{m=1}^{\infty} m(e^{2\alpha m}-1)^{-1}\Big\} + \beta\Big\{\tfrac{1}{2}\zeta(-1) + \sum_{m=1}^{\infty} m(e^{2\beta m}-1)^{-1}\Big\} = -\tfrac{1}{4},$$

and, using $\zeta(-1) = -1/12$, the statement to be proven reads as follows:

THEOREM 5. *With* $\alpha > 0$, $\beta > 0$, $\alpha\beta = \pi^2$,

$$\alpha\sum_{m=1}^{\infty} m(e^{2\alpha m}-1)^{-1} + \beta\sum_{m=1}^{\infty} m(e^{2\beta m}-1)^{-1} = -\frac{1}{4} + \frac{\alpha+\beta}{24}. \tag{7}$$

Formula (7) occurs in Ramanujan (see [4], v. I, p. 257, No. 9 and v. II, p. 170, Cor. 1), as does also the

COROLLARY 2.

$$\sum_{m=1}^{\infty} m(e^{2\pi m}-1)^{-1} = (\pi-3)/24\pi.$$

Although Hardy's first method of [3] works without any difficulty, no proof of (7) seems to exist in the literature; for that reason one such proof will be sketched in the next section. Before doing this, however, we observe that Theorem 4 and Corollary 2 combine to yield the rather curiously looking.

THEOREM 6.

$$\sum_{m=1}^{\infty} (e^{2\pi m}-1)^{-1} + \sum_{m=1}^{\infty} (e^{2\pi m}-1)^{-2} - \sum_{m=1}^{\infty} m(e^{2\pi m}-1)^{-1} = 0.$$

In fact, Theorem 6 is quite trivial being nothing but the particular instance $x = e^{2\pi}$ of the easily verified formal identity

$$\sum_{m=1}^{\infty} (x^m-1)^{-2} = \sum_{m=1}^{\infty} (m-1)(x^m-1)^{-1}.$$

4. Proof of Theorem 5. Let $f(x) = 2x/(e^{2x}-1)$ for $x \neq 0$, $f(0) = 1$ and set

$$g(x) = (2/\pi)^{1/2}\int_0^{\infty} f(t)\cos tx\,dt.$$

Then

$$g(0) = (2/\pi)^{1/2}\int_0^{\infty} 2x(e^{2x}-1)^{-1}dx$$

$$= (2\pi)^{-1/2}\int_0^{\infty} y(e^{y}-1)^{-1}dy = (2\pi)^{-1/2}\Gamma(2)\zeta(2).$$

Here the last equality follows from the classical (see, e.g. [5], p. 18) formula

$$\Gamma(s)\zeta(s) = \int_0^{\infty} y^{s-1}(e^{y}-1)^{-1}dy.$$

It follows that

$$g(0) = (2\pi)^{3/2}/24,$$

in formal agreement with Hardy's formula with $k = 1$. In particular, for $ab = 2\pi$, one has

$$a^{1/2}\Big\{\tfrac{1}{2}+\sum_{m=1}^{\infty} f(ma)\Big\} = b^{1/2}\Big\{(2\pi)^{3/2}/48+\sum_{n=1}^{\infty} g(nb)\Big\}. \tag{8}$$

Proceeding essentially as in [3] (observe the extra factor 2),

$$g(x) = (8/\pi)^{1/2}\frac{d}{dx}\int_0^{\infty}\frac{\sin tx}{e^{2t}-1}\,dt = (2\pi)^{1/2}\frac{d}{dx}\{(e^{\pi x}-1)^{-1}-(\pi x)^{-1}+\tfrac{1}{2}\} = g_1(x)+g_2(x).$$

Here

$$g_2(x) = -(2\pi)^{1/2}\frac{d}{dx}\big((\pi x)^{-1}-\tfrac{1}{2}\big) = (2/\pi)^{1/2}x^{-2},$$

so that

$$\sum_{n=1}^{\infty} g_2(nb) = (2/\pi)^{1/2}b^{-2}\zeta(2)$$

and

$$b^{1/2}\sum_{n=1}^{\infty} g_2(nb) = (2\pi/b)^{3/2}2^{1/2}/2^{3/2}\cdot 6 = a^{3/2}/12.$$

Next,

$$g_1(x) = (2\pi)^{1/2}\frac{d}{dx}(e^{\pi x}-1)^{-1} = (2\pi)^{1/2}\frac{d}{dx}\sum_{m=1}^{\infty}e^{-\pi m x} = -\pi(2\pi)^{1/2}\sum_{m=1}^{\infty} m e^{-\pi m x}$$

and

$$b^{1/2}\sum_{n=1}^{\infty} g_1(nb) = -\pi(2\pi)^{1/2}b^{1/2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m e^{-mn\pi b}$$
$$= -2\pi^2 a^{-1/2}\sum_{m=1}^{\infty} m e^{-m\pi b}/(1-e^{-m\pi b}) = -2\pi^2 a^{-1/2}\sum_{m=1}^{\infty} m(e^{m\pi b}-1)^{-1}.$$

Substituting in (8) and using $ab = 2\pi$, we obtain,

$$a^{1/2}\Big\{\tfrac{1}{2}+2a\sum_{m=1}^{\infty} m(e^{2ma}-1)^{-1}\Big\} = \frac{\pi^2 a^{-1/2}}{12}+\frac{a^{3/2}}{12}-\frac{\pi a^{1/2}}{2}\cdot 2b\sum_{m=1}^{\infty} m(e^{m\pi b}-1)^{-1},$$

or

$$\frac{1}{2}+2a\sum_{m=1}^{\infty} m(e^{2ma}-1)^{-1} = (\pi^2+a^2)/12a-\pi b\sum_{m=1}^{\infty} m(e^{m\pi b}-1)^{-1}.$$

With $\alpha = a$, $\beta = \frac{1}{2}\pi b$, one has $\alpha\beta = \frac{1}{2}\pi ab = \pi^2$ and we proved that

$$\frac{1}{2}+2\alpha\sum_{m=1}^{\infty}\frac{m}{e^{2m\alpha}-1} = \frac{\pi^2+\alpha^2}{12\alpha}-\pi\cdot\frac{2\beta}{\pi}\sum_{m=1}^{\infty}\frac{m}{e^{2m\beta}-1} = \frac{\beta+\alpha}{12}-2\beta\sum_{m=1}^{\infty}\frac{m}{e^{2m\beta}-1},$$

i.e. (7), holds. Setting in (7) $\alpha = \beta = \pi$ we obtain Corollary 2.

5. Proof of Theorem 1 (End).

Case 3, $s = n > 1$. In order to prove (5) for $1 < s = n \in \mathbf{Z}$, we recall the following result of [1]. Let $\sigma_r(n) = \sum_{d|n} d^r$ and define $F(\tau) = \sum_{m=1}^{\infty}\sigma_{-a}(m)e^{2\pi i m\tau}$; then one has for $t > 0$, $\tau = it$ and odd $a = 2n+1$:

$$2\{F(it)-(-1)^n t^{a-1}F(i/t)\} = (-1)^{n-1}\Big\{\zeta(a)\Big(\operatorname{cosec}\frac{\pi a}{2}-t^{a-1}\Big)+ \tag{9}$$
$$+(-1)^n\frac{(2\pi)^a}{(a+1)!}\sum_{\nu=0}^{n+1}(-1)^{\nu}t^{a-2\nu}\binom{a+1}{2\nu}B_{2\nu}B_{a+1-2\nu}\Big\}.$$

One observes that

$$F(it) = \sum_{m=1}^{\infty} m^{-a}(e^{2\pi m t}-1)^{-1},$$

so that

$$F(i/t) = \sum_{m=1}^{\infty} m^{-a}(e^{2\pi m/t}-1)^{-1}.$$

If we write also $\pi t = \alpha$, $\pi/t = \beta$, so that $\alpha\beta = \pi^2$, (9) becomes

$$2\Big\{\sum_{m=1}^{\infty} m^{-a}(e^{2m\alpha}-1)^{-1}-(-1)^n(\alpha/\pi)^{2n}\sum_{m=1}^{\infty} m^{-a}(e^{2m\beta}-1)^{-1}\Big\} \tag{10}$$
$$= (-1)^{n-1}\Big\{\zeta(2n+1)\big((-1)^n-(\alpha/\pi)^{2n}\big)+$$
$$+(-1)^n\frac{(2\pi)^{2n+1}}{(2n+2)!}\sum_{\nu=0}^{n+1}(-1)^{\nu}(\alpha/\pi)^{2n-2\nu+1}\binom{2n+2}{2\nu}B_{2\nu}B_{2n-2\nu+2}\Big\}.$$

We now multiply (10) by $\frac{1}{2}\alpha^{-n}$, use $\pi^2/\alpha = \beta$, replace n by $n-1$ and obtain

$$\alpha^{1-n}\sum_{m=1}^{\infty} m^{1-2n}(e^{2m\alpha}-1)^{-1}-(-\beta)^{1-n}\sum_{m=1}^{\infty} m^{1-2n}(e^{2m\beta}-1)^{-1} \tag{11}$$
$$= (-1)^n\Big\{\tfrac{1}{2}\zeta(2n-1)\big((-\alpha)^{1-n}-\beta^{1-n}\big)+$$
$$+(-1)^{n-1}\frac{(2\pi)^{2n-1}}{2(2n)!}\sum_{\nu=0}^{n}(-1)^{\nu}\alpha^{n-2\nu}\pi^{1+2\nu-2n}\binom{2n}{2\nu}B_{2\nu}B_{2n-2\nu}\Big\}.$$

Clearly, $\sum_{\nu=0}^{n} = \frac{1}{2}\{\sum_{\nu=0}^{n}+\sum_{\nu=n}^{0}\}$ and the last term becomes

$$\frac{(-1)^{n-1}}{4}(2\pi)^{2n-1}\sum_{\nu=0}^{n}(-1)^{\nu}\big(\alpha^{n-2\nu}\pi^{1+2\nu-2n}+(-1)^n\alpha^{2\nu-n}\pi^{1-2\nu}\big)\frac{B_{2\nu}}{(2\nu)!}\frac{B_{2n-2\nu}}{(2n-2\nu)!}$$
$$= \frac{(-1)^{n-1}}{8}(2\pi)^{2n}\sum_{\nu=0}^{n}(-1)^{\nu}\pi^{-2\nu}\big(\beta^{2\nu-n}+(-1)^n\alpha^{2\nu-n}\big)\frac{B_{2\nu}}{(2\nu)!}\frac{B_{2n-2\nu}}{(2n-2\nu)!}.$$

If we now replace $B_{2\nu}/(2\nu)!$ by $(-1)^{\nu-1}2\zeta(2\nu)(2\pi)^{-2\nu}$ and similarly for $B_{2n-2\nu}/(2n-2\nu)!$, a simple regrouping shows that (11) is equivalent to (5) with $1<s\in\mathbf{Z}$. This finishes the proof of Theorem 1. Ramanujan's Theorem B (i.e. (4)) can be obtained directly from (11) by taking together the terms with ν and $n-\nu$ in the last sum, multiplication by 4^{1-n} and some trivial regrouping of terms.

6. Proof of Theorem 2. $F(s;\alpha,\beta)$ may be written as

$$\tfrac{1}{4}(s-1)^{-1}\big((-\alpha)^{1-s}-\beta^{1-s}\big)\{(2s-2)\zeta(2s-1)\}+(-\alpha)^{1-s}\sum_{m=1}^{\infty}m^{1-2s}(e^{2\alpha m}-1)^{-1}-$$

$$-\beta^{1-s}\sum_{m=1}^{\infty}m^{1-2s}(e^{2\beta m}-1)^{-1}-\tfrac{1}{2}\sum_{0\leqslant\nu\leqslant\sigma}(-1)^{\nu}\zeta(2\nu)\zeta(2s-2\nu)\pi^{2\nu-2s}\big((-\alpha)^{s-2\nu}+\beta^{s-2\nu}\big).$$

Letting $s\to 1^{+}$, one obtains

$$\lim_{s\to 1^{+}}\frac{(-\alpha)^{1-s}-\beta^{1-s}}{4(s-1)}+\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{1}{e^{2\alpha m}-1}-\frac{1}{e^{2\beta m}-1}\right)-$$

$$-\frac{1}{2}\zeta(0)\zeta(2)\{\pi^{-2}\big((-\alpha)+\beta\big)-\big((-\alpha)^{-1}+\beta^{-1}\big)\}$$

$$=\frac{1}{4}\log\left(-\frac{\beta}{\alpha}\right)+\sum_{m=1}^{\infty}\frac{1}{m}\,\frac{e^{2\beta m}-e^{2\alpha m}}{(e^{2\alpha m}-1)(e^{2\beta m}-1)}+$$

$$+\frac{1}{4}\cdot\frac{\pi^{2}}{6}\{-\beta^{-1}+\alpha^{-1}-\big((-\alpha)^{-1}+\beta^{-1}\big)\}$$

$$=\frac{1}{4}\log\left(-\frac{\beta}{\alpha}\right)+\frac{1}{2}\sum_{m=1}^{\infty}\frac{1}{m}(\coth\alpha m-\coth\beta m)+\frac{\beta-\alpha}{12},$$

thus proving (6′). We observe that for $s\to 1^{-}$, the last sum contains only the term corresponding to $\nu=0$; this yields (6″) and finishes the proof of Theorem 2.

7. Proof of Theorem 3 and Theorem 4. By Mellin's integral for the exponential one has (see [3]; the present formula is easily verified directly, although it corresponds to $k=0$, not covered in [3]; see also [7]) for $0<\varepsilon<1$,

$$\psi(\alpha)=\sum_{m=1}^{\infty}m^{-1}(e^{2m\alpha}-1)^{-1}=\frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty}\Gamma(s)\zeta(s)\zeta(s+1)(2\alpha)^{-s}ds.$$

The shift of the line of integration (with due regard to the residues of poles crossed) to $\sigma=-1-\varepsilon$ is easily justified. The poles are at $s=1$ (simple pole, residue $R_1=\zeta(2)/2\alpha=\pi^2/12\alpha$); at $s=0$ (double pole, residue $R_0=\frac{1}{2}\log(\alpha/\pi)$); and at $s=-1$ (simple pole, residue $R_{-1}=-\zeta(-1)\zeta(0)(2\alpha)=-\alpha/12$). Consequently,

$$\psi(\alpha)=\frac{\pi^2}{12\alpha}-\frac{\alpha}{12}+\frac{1}{2}\log\frac{\alpha}{\pi}+\frac{1}{2\pi i}\int_{-1-\varepsilon-i\infty}^{-1-\varepsilon+i\infty}\Gamma(s)\zeta(s)\zeta(s+1)(2\alpha)^{-s}ds. \tag{12}$$

In the last integral we replace s by $-s$ and use the functional equations of the Γ and the ζ-functions. Setting $\pi^2/\alpha=\beta$ the integral becomes

$$\frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty}\Gamma(s)\zeta(s)\zeta(s+1)(2\pi^2/\alpha)^{-s}ds=\psi(\beta)$$

and (12) yields

$$\sum_{m=1}^{\infty}m^{-1}(e^{2m\alpha}-1)^{-1}-\tfrac{1}{4}\log\alpha+\alpha/12=\sum_{m=1}^{\infty}m^{-1}(e^{2m\beta}-1)^{-1}-\tfrac{1}{4}\log\beta+\beta/12,$$

i.e. Theorem 3. One may rewrite the statement of Theorem 3 as

$$\sum_{m=1}^{\infty}m^{-1}(e^{2m\alpha}-1)^{-1}-\sum_{m=1}^{\infty}m^{-1}(e^{2m\beta}-1)^{-1}=-\tfrac{1}{4}\log(\beta/\alpha)+(\beta-\alpha)/12;$$

for $\alpha=\beta=\pi$, this holds trivially, but if we first divide both sides by $\beta-\alpha$ and then let $\alpha\to\pi$ (with $\beta=\pi^2/\alpha$), we obtain the statement of Theorem 4.

8. Proof of Theorem 2′. As seen in the proof of Theorem 2, for $\alpha\beta=\pi^2$

$$\tfrac{1}{2}\sum_{m=1}^{\infty}m^{-1}\{\coth\alpha m-\coth\beta m\}=\sum_{m=1}^{\infty}m^{-1}(e^{2\alpha m}-1)^{-1}-\sum_{m=1}^{\infty}m^{-1}(e^{2\beta m}-1)^{-1}.$$

By Theorem 3, the right-hand side equals $\frac{1}{4}\log(\alpha/\beta)+(\beta-\alpha)/12$; making the corresponding substitution in (6′) and (6″) we obtain Theorem 2′.

9. Final remarks. In the evaluation of the Riemann zeta function at odd, positive arguments ($\neq 1$) occur the sums S_n defined in (1), with $0<n\in\mathbf{Z}$, and one is interested in the arithmetic character of $\pi^{-(2n+1)}S_n$; this does not seem to be known for any $n\in\mathbf{Z}$ $(n>0)$. Reviewing preceding results, we can make the following, somewhat related remarks.

(a) For $-1>n\in\mathbf{Z}$, $S_n=-B_{-2n}/2n$ is rational.

(b) $S_0=\sum_{m=1}^{\infty}m(e^{2\pi m}-1)^{-1}$ $\big(=\sum_{m=1}^{\infty}e^{2\pi m}(e^{2\pi m}-1)^{-2}=\sum_{m=1}^{\infty}(e^{2\pi m}-1)^{-1}+$ $+\sum_{m=1}^{\infty}(e^{2\pi m}-1)^{-2}\big)$ is transcendental and so is $\pi^r S_0$ for any integer r. Indeed using Corollary 2, $\pi^r-3\pi^{r-1}-24(\pi^r S_0)=0$ and if $\pi^r S_0$ would be algebraic, also π would be algebraic, which is not the case.

At this point it may be appropriate to recall also the following known (see [2]) fact:

(c) For $n \geqslant 1$, $S_n = H_n(i)$ where $H_n(\tau) = \sum_{m=1}^{\infty} c_n(m) e^{2\pi i m\tau}$ with $c_n(m) = 2\big(1+\big(1+(-1)^n\big)\pi m/n\big)\sigma_{-(2n+1)}(m)$ (for n odd, $H_n(\tau) = 2F(\tau)$).

If we define, more generally, $H_n(\tau; \chi) = \sum_{m=1}^{\infty} \chi(m) c_n^*(m) e^{2\pi i m\tau/k}$ with $c_n^*(m) = 2\big(1+\big(1+(-1)^{n+\delta}\big)\pi m/kn\big)\sigma_{-(2n+1)}(m)$, where $\chi(m)$ is a real, primitive (non-principal!) congruence character modulo $k > 1$ and $\delta = \big(1-\chi(-1)\big)/2$, then $H_n(i, \chi)\pi^{-(2n+1)}$ belongs to the quadratic field generated by $\sqrt{k}$ over the rationals. This result apparently cannot be extended in any obvious way to $\chi(m)$ a principal character.

Neither of these remarks seems to have any direct bearing upon the rationality or transcendency of $\pi^{-(2n+1)}\zeta(2n+1)$.

Bibliography

[1] E. Grosswald, *Die Werte der Riemannschen Zetafunktion an ungeraden Argumentstellen*, Nachrichten der Akad. Wiss. Göttingen (1970), pp. 9–13.

[2] — *Remarks concerning the values of the Riemann zeta function at integral odd arguments* (to appear Journ. Number Theory 4 (1972)).

[3] G. H. Hardy, *A formula of Ramanujan*, Journ. London Math. Soc. 3 (1928), pp. 238–240; Collected Papers IV, pp. 537–539.

[4] S. Ramanujan, *Notebooks*, Facsimile edition in 2 volumes by The Tata Institute of Fundamental Research, Bombay 1957.

[5] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Oxford 1951.

[6] G. N. Watson, *Theorems stated by Ramanujan (II)*, Journ. London Math. Soc. 3 (1928), pp. 216–225.

[7] A. Weil, *Sur une formule classique*, Journ. Math. Soc. Japan 20 (1968), pp. 400–402.

DEPARTMENT OF MATHEMATICS
TEMPLE UNIVERSITY
Philadelphia, Pennsylvania

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Sur la résolubilité de l'équation $x^2+y^2+z^2 = 0$ dans un corps quadratique

par

TRYGVE NAGELL (Uppsala)

§ 1. Méthode non-constructive

1. Introduction. Pour que l'équation

$$x^2+y^2+z^2 = 0 \tag{1}$$

soit résoluble en nombres x, y, z d'un corps algébrique K (hors le cas $x = y = z = 0$) il faut évidemment que K soit totalement imaginaire ($=$ t. im.). Pour reconnaître si cette équation est résoluble ou non il est naturel d'appliquer le résultat suivant:

LEMME 1. *Soit donné le corps algebrique K t. im. Pour que l'équation* (1) *soit résoluble en nombres entiers x, y, z du corps K (le cas $x = y = z = 0$ étant exclu) il faut et il suffit que la congruence*

$$\xi^2+\eta^2+\zeta^2 \equiv 0 \pmod{\mathfrak{j}}$$

soit résoluble en entiers ξ, η, ζ de K pour tous les idéaux de K, tel qu'on ait $(\xi, \eta, \zeta, \mathfrak{j}) = 1$.

Ce résultat est, bien entendu, un cas particulier d'un théorème de Hilbert sur les formes quadratiques; voir [2] ([1]). Il est évident qu'on peut, dans ce lemme, remplacer l'idéal $\mathfrak{j}$ par l'idéal (N) où N parcourt tous les nombres naturels.

Or, si N est *impair* il est bien connu que la congruence

$$x^2+y^2+z^2 \equiv 0 \pmod{N}$$

est toujours résoluble dans le corps rationnel de manière qu'on ait $(x, y, z, N) = 1$; voir p. ex. [4], p. 192. Donc, on peut remplacer le Lemme 1 par le

([1]) Les numéros figurant entre crochets renvoient à la bibliographie placée à la fin de ce mémoire.