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Reducibility of quadrinomials

by

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This paper is based on [8] and the notation of that paper is retained. In particular if

$$\Phi(y_1, \ldots, y_k) = y_1^{a_1} \ldots y_k^{a_k} f(y_1, \ldots, y_k),$$

where a_i are integers and f is a polynomial not divisible by y_i $(1 \leqslant i \leqslant k)$ then

$$J\Phi(y_1, \ldots, y_k) = f(y_1, \ldots, y_k).$$

A polynomial $g(y_1, \ldots, y_k)$ is called *reciprocal* if

$$Jg(y_1^{-1}, \ldots, y_k^{-1}) = \pm g(y_1, \ldots, y_k).$$

Reducibility means reducibility over the rational field $\boldsymbol{Q}$ unless stated to the contrary.

$L\Phi(y_1, \ldots, y_k)$ is $J\Phi(y_1, \ldots, y_k)$ deprived of all its irreducible reciprocal factors and $K\Phi(x)$ is $J\Phi(x)$ deprived of all its cyclotomic factors.

Ljunggren [5] has proved the irreducibility of $K(x^m + \varepsilon_1 x^n + \varepsilon_2 x^p + \varepsilon_3)$ where $m > n > p$, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are ± 1 and the case $m = n+p$, $\varepsilon_3 = \varepsilon_1\varepsilon_2$ is excluded. He has also proved [6] the irreducibility of $K(x^m + \varepsilon_1 x^n + \varepsilon_2 x^p + \varepsilon_3 r)$, where r is a prime. The aim of this paper is to treat a general quadrinomial $q(x) = ax^m + bx^n + cx^p + d$ by means of Theorem 2 of [8]. In order to apply this theorem it is necessary to investigate first the reducibility of a quadrinomial in two variables. The result of the investigation is given below as Theorem 1. Combining this theorem with Theorem 2 of [8] we obtain a necessary and sufficient condition for the reducibility of $Lq(x)$ (Theorem 2). In general we have no such condition for the reducibility of $Kq(x)$ but in the case $a = 1$, $b = \varepsilon_1$, $0 < |c| \leqslant |d|$ (c, d integers) $Kq(x) = Lq(x)$ which leads to a generalization of the results of Ljunggren (Theorem 3). We prove

THEOREM 1. *A quadrinomial* $Q(y_1, y_2) = J(a_0 + \sum_{i=1}^{3} a_i y_1^{\nu_{1i}} y_2^{\nu_{2i}})$, *where* $a_i \neq 0$ $(0 \leqslant i \leqslant 3)$, $[\nu_{1i}, \nu_{2i}]$ *distinct and different from* $[0, 0]$, $[\nu_{ij}]$ *of rank 2* *is reducible in a field K of characteristic zero if and only if either it can be divided into two parts with the highest common factor $D(y_1, y_2)$ being a binomial or it can be represented in one of the forms*

$$k(U^3+V^3+W^3-3UVW) = k(U+V+W)(U^2+V^2+W^2-UV-UW-VW),$$
$$(1)\qquad k(U^2-4TUVW-T^2V^4-4T^2W^4) = k(U-TV^2-2TVW-2TW^2)(U+TV^2-2TVW+2TW^2),$$
$$k(U^2+2UV+V^2-W^2) = k(U+V+W)(U+V-W),$$

where $k\in K$ and T, U, V, W are monomials in $K[y_1, y_2]$. In the former case QD^{-1} is either irreducible in K and non-reciprocal or binomial. In the latter case the factors on the right hand side of (1) *are irreducible in K and non-reciprocal unless $\zeta_3\in K$ when*

$$U^2+V^2+W^2-UV-UW-VW = (U+\zeta_3V+\zeta_3^2W)(U+\zeta_3^2V+\zeta_3W).$$

THEOREM 2. *Let a, b, c, d be any non-zero integers, $m>n>p$ any positive integers and assume that $q(x) = ax^m+bx^n+cx^p+d$ is not a product of two binomials. $Lq(x)$ is reducible if and only if either $q(x)$ can be divided into two parts which have a non-reciprocal common factor or it can be represented in one of the forms* (1) *where $k\in Q$; T, U, V, W are monomials in $Q[x]$ and the factors on the right hand side of* (1) *are not reciprocal or finally $m = vm_1$, $n = vn_1$, $p = vp_1$,*

$$m_1 < C(a, b, c, d) = \exp_2(3\cdot 2^{a^2+b^2+c^2+d^2+2}\log(a^2+b^2+c^2+d^2))$$

and $L(ax^{m_1}+bx^{n_1}+cx^{p_1}+d)$ is reducible.

THEOREM 3. *Let $\varepsilon = \pm1$, c, d be integers, $0<|c|\leqslant|d|$, $m>n>p$ be positive integers and assume that $q(x) = x^m+\varepsilon x^n+cx^p+d$ is not a product of two binomials. $Kq(x)$ is reducible if and only if either there occurs one of the cases*

$$(-\varepsilon d)^{(m-p)/\delta_1} = (-c)^{n/\delta_1} \neq \pm1,\qquad \delta_1 = (m-p, n);$$
$$(-\varepsilon c)^{m/\delta_2} = (-d)^{(n-p)/\delta_2} \neq \pm1,\qquad \delta_2 = (m, n-p);$$
$$m = 2m_1,\ n = 2p,\ \varepsilon = -1,\ c^2 = -4d,$$
$$(2)\qquad m = 2p,\ n = 2n_1,\ \varepsilon = -1,\ c^2 = 4d,$$
$$m = 3m_1,\ n = 3n_1,\ p = m_1+n_1,\ c^3 = -27\varepsilon d,$$
$$m = 2m_1,\ n = 4n_1,\ p = m_1+n_1,\ \varepsilon = -1,\ c^4 = -64d,$$
$$m = 4m_1,\ n = 2n_1,\ p = m_1+n_1,\ \varepsilon = -1,\ c^4 = 64d$$

or $m = vm_1$, $n = vn_1$, $p = vp_1$,

$$m_1 < C(1, \varepsilon, c, d)$$

and $K(x^{m_1}+\varepsilon x^{n_1}+cx^{p_1}+d)$ is reducible.

COROLLARY. *Under the assumptions of Theorem 3 the quadrinomial $x^m+\varepsilon x^n+cx^p+d$ is reducible if and only if either there occurs one of the cases* (2) *or we have one of the equalities*

$$(-\varepsilon d)^{(m-p)/\delta_1} = (-c)^{n/\delta_1} = \pm1,\qquad \delta_1 = (m-p, n);$$
$$(-\varepsilon c)^{m/\delta_2} = (-d)^{(n-p)/\delta_2} = \pm1,\qquad \delta_2 = (m, n-p);$$
$$(-\varepsilon)^{p/\delta_3} = (-d/c)^{(m-n)/\delta_3},\qquad \delta_3 = (m-n, p);$$
$$\zeta^{m/\delta}+\zeta^{n/\delta}+c\zeta^{p/\delta}+d = 0,\qquad \zeta^6 = 1,\qquad \delta = (m, n, p)$$

or $m = vm_1$, $n = vn_1$, $p = vp_1$,

$$m_1 < C(1, \varepsilon, c, d),$$

and $x^{m_1}+\varepsilon x^{n_1}+cx^{p_1}+d$ is reducible.

LEMMA 1. *If $m>n$ non-zero integers, $ab\neq0$ and*

$$ax^m+bx^n = f_1(f_2(x)),$$

where f_1, f_2 rational functions, then for a suitable homography h we have either

$$f_1h(x) = ax,\qquad h^{-1}f_2(x) = x^m+\frac{b}{a}x^n$$

or

$$f_1h(x) = ax^{m/\delta}+bx^{n/\delta},\qquad h^{-1}f_2(x) = x^\delta$$

or

$$m = -n,\qquad f_1h(x) = 2ac^{m/\delta}T_{m/\delta}(\tfrac12 c^{-1}x),\qquad h^{-1}f_2(x) = x^\delta+c^2x^{-\delta},$$

where $c^{2m/\delta} = b/a$ and T_m is the mth Čebyšev polynomial.

Proof. Assume first that $n>0$. Then by a known lemma (see [2]) for suitable homography h, f_1h and $h^{-1}f_2$ are polynomials. We may assume the same about f_1, f_2 and suppose moreover that f_2 is monic with $f_2(0) = 0$. Let

$$f_1(x) = a\prod_{i=1}^{k}(x-x_i)^{a_i},\qquad x_i \text{ distinct},\ a_1+\ldots+a_k = a.$$

Since $f_2(x)-x_i$ are relatively prime in pairs exactly one factor, say $f_2(x)-x_1$ is divisible by x and we have $f_2(x)-x_1 = x^lg(x)$, where $la_1 = n$. However $g(x)^{a_1}\mid ax^{m-n}+b$, hence either $g(x) = 1$ or $a_1 = 1$.

In the first case the lemma follows, one obtains also $x_1 = 0$. In the second case $l = n$; if now $g(x) = x^\gamma+a_1x^{\gamma_1}+\ldots$, where $\gamma>\gamma_1>\ldots$ and $a_1\neq0$, then $f_1(f_2(x))$ begins with two non-zero terms

$$ax^{(\gamma+n)a}+aaa_1x^{a(\gamma+n)+\gamma_1-\gamma}.$$

It follows that $a(\gamma+n)+\gamma_1-\gamma = n$; $a = 1$, $\gamma = m-n$, $\gamma_1 = 0$, $f_1(x) = ax$, $f_2(x) = x^m+\dfrac{b}{a}x^n$.

The case $n < 0$, $m < 0$ can be reduced to the former by substitution $x \to 1/x$.

Assume now that $m > 0$, $n < 0$. Set

$$f_1(x) = \frac{R(x)}{S(x)}, \quad f_2(x) = \frac{P(x)}{Q(x)},$$

where P, Q, R, S are polynomials of degrees p, q, r, s respectively and $(P, Q) = (R, S) = 1$. Applying to P/Q a suitable homography we can achieve that $p > q$, $r > s$ and that P, Q are monic. Consider the identity

$$\frac{ax^{m-n}+b}{x^{-n}} = \frac{R(P, Q)}{S(P, Q)Q^{r-s}},$$

where $R(P, Q) = Q^r R(P/Q)$, etc. Since $R(P, Q), S(P, Q), Q$ are relatively prime in pairs we have either

$$S(P, Q) = cx^{-n}, \quad Q^{r-s} = 1 \quad \text{or} \quad S(P, Q) = c, \quad Q^{r-s} = x^{-n}.$$

In the first case $Q = 1$, by a suitable linear transformation we can achieve $P(0) = 0$ and thus $P(x) = x^{\delta}$, $S(x) = cx^{-n/\delta}$,

$$f_1(x) = ax^{m/\delta} + bx^{n/\delta}, \quad f_2(x) = x^{\delta}.$$

In the second case it follows in view of $p > q$ that $Q = x^{-n/r}$, $s = 0$, f_1 is a polynomial and we have $p = \dfrac{m-n}{r}$,

$$f_1(x^{n/r}P) = ax^m + bx^n.$$

If P contains terms $c_1 x^{p_1}$ with $c_1 \neq 0$, $p > p_1 > -n/r$ then taking the largest possible p_1 we get on the left hand side a term $arc_1 x^{m+p_1-p}$ lacking on the right hand side. Similarly we get a contradiction if P contains a term $c_2 x^{p_2}$ with $-n/r > p_2 > 0$. Therefore, $P = x^{(m-n)/r} + c_3 x^{-n/r} + c_4$ and applying to f_2 a suitable linear transformation we obtain $P = x^{(m-n)/r} + c_4$.

Let β be any $(m-n)$th root of $-b/a$. Then $c_4 = \beta^{(m-n)/r} \zeta_{2r}^{2h+1}$ for suitable h. Moreover

$$f_1\big(\beta^{(m-n)/r}(\zeta_{m-n}^{im/r} + \zeta_{2r}^{2h+1}\zeta_{m-n}^{in/r})\big) = 0$$

for all $i = 1, 2, \ldots, m-n$.

Suppose that for two values of i we get the same zero of f_1, i.e.

$$\zeta_{m-n}^{im/r} + \zeta_{2r}^{2h+1}\zeta_{m-n}^{in/r} = \zeta_{m-n}^{jm/r} + \zeta_{2r}^{2h+1}\zeta_{m-n}^{jn/r}.$$

It follows hence (see [7]) that either both sums are zero, or the terms are equal in pairs, i.e. either

$$\zeta_{m-n}^{i(m-n)/r} = \zeta_{m-n}^{j(m-n)/r} = \zeta_{2r}^{2h+r+1}$$

or

$$\zeta_{m-n}^{(i-j)m/r} = \zeta_{m-n}^{(i-j)n/r} = 1$$

or

$$\zeta_{m-n}^{(im-jn)/r} = \zeta_{m-n}^{(jm-in)/r} = \zeta_{2r}^{2h+1}.$$

The first equality implies $2i \equiv 2j \equiv 2h' + r + 1 \bmod 2r$ (h' fixed, determined by h and the choice of ζ_{m-n}, ζ_{2r}), the second $i \equiv j \bmod r(m-n)/(m, n)$, the third $i \equiv j \bmod r(m-n)/(m-n, m+n)$. Thus all but at most $\dfrac{m-n}{(m-n, m+n)} - 1$ zeros of f_1 obtained for $i \leqslant r\dfrac{m-n}{(m-n, m+n)}$ are distinct. Hence

$$r\frac{m-n}{(m-n, m+n)} \leqslant r + \frac{m-n}{(m-n, m+n)} - 1$$

and either $r = 1$ or $m-n \mid m+n$ thus $m+n = 0$. In the former case we get $f_1(x) = ax$, $f_2(x) = x^m + (b/a)x^n$, in the latter case

$$f_1(x) = 2a(\sqrt{c_4})^r T_r\left(\frac{x}{2\sqrt{c_4}}\right), \quad f_2(x) = x^{m/r} + c_4 x^{-m/r}.$$

Lemma 2. *Let m_i be integers different from zero, $m_0 \neq m_1$, $m_0 + m_1 \geqslant 0$; $m_2 \neq m_3$, $m_2 + m_3 \geqslant 0$, a_i $(i = 0, 1, 2, 3)$ complex numbers different from zero and the case $m_0 + m_1 = m_2 + m_3 = 0$, $a_0 a_1 = a_2 a_3$ be excluded. If the quadrinomial*

$$q(x, y) = J(a_0 x^{m_0} + a_1 x^{m_1} + a_2 y^{m_2} + a_3 y^{m_3})$$

is reducible in the complex field $\mathbf{C}$ then either it can be divided into two parts with the highest common factor $d(x, y)$ being a binomial or it can be represented in one of the forms

$$\begin{aligned} &u^3 + v^3 + w^3 - 3uvw = (u+v+w)(u + \zeta_3 v + \zeta_3^2 w)(u + \zeta_3^2 v + \zeta_3 w), \\ (3) \qquad & \\ &u^2 - 4tuvw - t^2v^4 - 4t^2w^4 = (u - tv^2 - 2tvw - 2tw^2)(u + tv^2 - 2tvw + 2tw^2), \end{aligned}$$

where t, u, v, w are monomials in $\mathbf{C}[x, y]$.

In the former case qd^{-1} is irreducible in $\mathbf{C}$ and non-reciprocal, in the latter case the factors on the right hand side of (3) are irreducible in $\mathbf{C}$ and non-reciprocal. Moreover if (3_1) holds, $u^2 + v^2 + w^2 - uv - uw - vw$ is also not reciprocal.

Proof. In view of symmetry we may assume that $m_0 \geqslant |m_1|$, $m_2 \geqslant |m_3|$. Set $f(x) = a_0 x^{m_0} + a_1 x^{m_1}$, $g(y) = -a_2 y^{m_2} - a_3 y^{m_3}$ and denote by Ω_{f-z} the splitting field of $f(x) - z$ over $\mathbf{C}(z)$. By proposition 2 of [4] there exist rational functions f_1, f_2, g_1, g_2 such that $f = f_1(f_2)$, $g = g_1(g_2)$, $\Omega_{f_1-z} = \Omega_{g_1-z}$

and $f-g$, f_1-g_1 have the same number of irreducible factors in C. (The number of irreducible factors of $F_1/F_2-G_1/G_2$, where $F_i\in C[x]$, $G_i\in C[y]$, $(F_1, F_2)=1=(G_1, G_2)$ is defined as the number of irreducible factors of $F_1G_2-F_2G_1$.) Since both conditions are invariant with respect to transformations $f_1\to f_1h$, $g_1\to g_1j$ where h, j are homographies we can apply Lemma 1 and infer that there occurs one of the cases

1. $f_1=a_0x^{n_0}+a_1x^{n_1}, \quad -g_1=a_2y^{n_2}+a_3y^{n_3}$,
2. $f_1=a_0x^{n_0}+a_1x^{n_1}, \quad -g_1=2\sqrt{a_2a_3}T_{n_2}(y), \ n_3=-n_2$,
3. $f_1=2\sqrt{a_0a_1}T_{n_0}(x), \quad -g_1=a_2y^{n_2}+a_3y^{n_3}, \ n_1=-n_0$,
4. $f_1=2\sqrt{a_0a_1}T_{n_0}(x), \quad -g_1=2\sqrt{a_2a_3}T_{n_2}(y), \ n_1=n_0, \ n_3=-n_2$,

where $n_i=m_i/\delta$ $(i=0,1)$, $n_i=m_i/\varepsilon$ $(i=2,3)$. Set

$$n_i'=n_i/(n_0, n_i) \ (i=0,1); \qquad n_i'=n_i/(n_2, n_3) \ (i=2,3).$$

Let $\sigma_a(f_1)$ be the branch permutation for the Riemann surface for $f_1(x)-z$ over the place $z=a$ on the z sphere and let ω be a generator of the extension $\Omega_{f_1-z}/C(z)$. ω is expressible rationally in terms of z and of $x^{(i)}(z)$'s $(i=1,\ldots,k)$, where

$$f_1(x)-z=F(x)^{-1}\prod_{i=1}^{k}\left(x-x^{(i)}(z)\right), \qquad F(x)\in C[x].$$

$|\sigma_a(f_1)|$, the order of $\sigma_a(f_1)$, is the least positive integer M such that each $x^{(i)}(z)$ is expressible as Laurent series in $(z-a)^{1/M}$ in the neighbourhood of $z=a$. It follows that ω is expressible as such series in $(z-a)^{1/|\sigma_a(f_1)|}$. On the other hand, if ω is expressible as a Laurent series in $(z-a)^{1/N}$ then all $x^{(i)}(z)$ are so expressible and hence $|\sigma_a(f_1)|\leqslant N$. Thus $|\sigma_a(f_1)|$ is the least integer N such that ω is expressible as a Laurent series in $(z-a)^{1/N}$ and therefore it is determined by Ω_{f_1-z}. From $\Omega_{f_1-z}=\Omega_{g_1-z}$ we have

$$|\sigma_a(f_1)|=|\sigma_a(g_1)|.$$

We use this observation separately in each of the cases 1–4.

1. If $n_1>0$ a simple computation shows that the branch permutations for Ω_{f_1-z} are σ_0 (an n_1 cycle), σ_∞ (an n_0 cycle), and n_0-n_1 other finite branch permutations (of order 2 and type $\sigma=\underbrace{(2)(2)\ldots(2)}_{(n_0,n_1)\text{ times}}$) corresponding to the branch points

$$z_i=\zeta_{n_0-n_1}^{in_1}\left(\frac{a_1(n_0-n_1)}{n_0}\right)\left(-\frac{a_1n_1}{a_0n_0}\right)^{n_1/(n_0-n_1)}, \qquad i=0,1,\ldots,\frac{n_0-n_1}{(n_0, n_1)}-1.$$

If $n_1<0$, $\sigma_\infty(f_1)$ is a product $\gamma_1\gamma_2$, where γ_1, γ_2 are disjoint cycles of length n_0 and $|n_1|$ respectively. The finite branch points are again z_i and the corresponding permutations are of type $\sigma=\underbrace{(2)(2)\ldots(2)}_{(n_0,n_1)\text{ times}}$. We have to consider several cases.

A. $n_1>0$, $n_3>0$. From $|\sigma_0(f_1)|=|\sigma_0(g_1)|$ we get $n_1=n_3$, from $|\sigma_\infty(f_1)|=|\sigma_\infty(g_1)|$ we get $n_0=n_2$. Also the branch points must be the same, which implies

$$\left(\frac{-a_0}{a_2}\right)^{n_1'}=\left(\frac{-a_1}{a_3}\right)^{n_0'}.$$

Since $(n_0', n_1')=1$ there exists a unique number r such that

$$r^{n_0'}=-a_2/a_0, \qquad r^{n_1'}=-a_3/a_1.$$

On substitution $x=zy$ the quadrinomial $f_1(x)-g_1(y)$ takes the form

$$f_1(x)-g_1(y)=a_0y^{n_0}(z^{n_0}-r^{n_0'})+a_1y^{n_1}(z^{n_1}-r^{n_1'}).$$

Since $\dfrac{z^{n_0}-r^{n_0'}}{z^{n_1}-r^{n_1'}}$ is not a power in $C(z)$ and

$$(z^{n_0}-r^{n_0'}, z^{n_1}-r^{n_1'})=z^{(n_0,n_1)}-r$$

we infer in virtue of Capelli's theorem that

$$f_1(x)-g_1(y)=y^{n_1}(z^{(n_0,n_1)}-r)F(z, y)$$

where F is irreducible in C. It follows that

$$f_1(x)-g_1(y)=(x^{(n_0,n_1)}-ry^{(n_0,n_1)})G(x, y)$$

where G is irreducible in C. Thus the number of irreducible factors of f_1-g_1 is $(n_0, n_1)+1$. On the other hand

$$d(x, y)=(a_0x^{m_0}+a_2y^{m_2}, a_1x^{m_1}+a_3y^{m_3})=x^{(n_0,n_1)}-ry^{(n_0,n_1)},$$

thus the number of irreducible factors of $f(x)-g(y)$ is at least $(n_0, n_1)(\delta, \varepsilon)+\nu$, where ν is the number of irreducible factors of qd^{-1} (q has no multiple factors). It follows that

$$(n_0, n_1)(\delta, \varepsilon)+\nu\leqslant(n_0, n_1)+1, \qquad \nu=1,$$

hence qd^{-1} is irreducible in C. Moreover it is not reciprocal since the degree of $Jq(x^{-1}, y^{-1})$ is greater than the degree of q and the degrees of $Jd(x^{-1}, y^{-1})$ and of d are equal.

B. $n_1 n_3 < 0$. In view of symmetry we may assume $n_1 > 0$. From $|\sigma_0(f_1)| = |\sigma_0(g_1)|$ we get $n_1 = 1$, from $|\sigma_\infty(f_1)| = |\sigma_\infty(g_1)|$, $n_0 = [n_2, n_3]$. Counting the number of remaining finite branch points we get

$$n_0 - 1 = \frac{n_2 + |n_3|}{(n_2, n_3)} \quad \text{or} \quad [n_2, n_3] - 1 = \frac{n_2 + |n_3|}{(n_2, n_3)}$$

or

$$n_2|n_3| - n_2 - n_3 - (n_2, n_3) = 0; \qquad (n_2 - 1)(|n_3| - 1) = (n_2, n_3) + 1.$$

This equation has three solutions with $n_2 \geqslant -n_3 > 0$:

$$(n_2, n_3) = (3, -2),\ (3, -3),\ (4, -2).$$

The first solution gives $n_0 = 6$,

$$f_1(x) - g_1(y) = a_0 x^6 + a_1 x + a_2 y^3 + a_3 y^{-2} = \big(a_2 y^5 + (a_0 x^6 + a_1 x) y^2 + a_3\big) y^{-2}$$

and the numerator of the fraction obtained is irreducible in C. Indeed, it clearly has no factor linear in y, thus a possible factorization would have the form

$$a_2 y^5 + (a_0 x^6 + a_1 x) y^2 + a_3 = a_2\big(y^2 + f_1(x) y + c_1\big)\big(y^3 + f_2(x) y^2 + f_3(x) y + c_2\big).$$

It follows hence

$$f_2(x) + f_1(x) = 0,$$
$$f_3(x) + f_1(x) f_2(x) + c_1 = 0,$$
$$c_2 f_1(x) + c_1 f_3(x) = 0,$$
$$-c_1 f_1^2(x) - c_2 f_1(x) + c_1^2 = 0;$$

$f_1(x) = -f_2(x) = \text{const}$, $f_3(x) = \text{const}$, which is impossible.

The second solution $(n_2, n_3) = (3, -3)$ gives $n_0 = 3$. Since the branch points must be the same

$$\pm \frac{2}{3} a_1 \sqrt{\frac{-a_1}{3a_0}} = \mp 2a_3 \sqrt{\frac{a_2}{a_3}}; \qquad a_1^3 = -27 a_0 a_2 a_3.$$

It follows that

$$q(x, y) = J\big(f(x) - g(y)\big) = (a_0 x^{3\delta} + a_1 x^{\delta}) y^{3\varepsilon} + a_2 y^{6\varepsilon} + a_3$$
$$= u^3 + v^3 + w^3 - 3uvw = (u + v + w)(u + \zeta_3 v + \zeta_3^{-1} w)(u + \zeta_3^{-1} v + \zeta_3 w),$$

where

$$u = a_0^{1/3} x^{\delta} y^{\varepsilon}, \qquad v = a_2^{1/3} y^{2\varepsilon}, \qquad w = a_3^{1/3}$$

and suitable values of the cubic roots are taken. The trinomials $u + \zeta_3^i v + \zeta_3^{-i} w$ are irreducible in C in virtue of Capelli's theorem since $\zeta_3^i a_2^{1/3} y^{\varepsilon} + \zeta_3^{-i} a_3^{1/3} y^{-\varepsilon}$ is not a power in $C(y)$. Moreover one verifies directly that $u + \zeta_3^i v + \zeta_3^{-i} w$ $(i = 0, \pm 1)$ and $u^2 + v^2 + w^2 - uv - uw - vw$ are not reciprocal.

The third solution $(n_2, n_3) = (4, -2)$ gives $n_0 = 4$. Since the branch points must be the same we have for suitable values of the cubic roots

$$\frac{3}{4} a_1 \left(\frac{-a_1}{4a_0}\right)^{1/3} = -\frac{6}{4} a_3 \left(\frac{2a_3}{4a_2}\right)^{1/3}; \qquad a_1^4 = 64 a_0 a_1 a_3^2.$$

If follows that

$$q(x, y) = J\big(f(x) - g(y)\big) = (a_0 x^{4\delta} + a_1 x) y^{2\varepsilon} + a_2 y^{6\varepsilon} + a_3$$
$$= u^2 - 4tuvw - t^2 v^4 - 4t^2 w^4$$
$$= (u - tv^2 - 2tvw - 2tw^2)(u + tv^2 - 2tvw + 2tw^2),$$

where

$$t = y^{\varepsilon}, \quad u = (-a_3)^{1/2}, \quad v = (-a_2)^{1/4} y^{\varepsilon}, \quad w = (-a_0/y)^{1/4} x^{\delta}$$

and suitable values of the quadratic and the quartic roots are taken. The quadrinomials $u \pm tv^2 - 2tvw \pm 2tw^2$ are irreducible in C since after the substitution

$$x = x_1 y_1^{\varepsilon}, \qquad y = y_1^{\delta}$$

we obtain

$$u \pm tv^2 - 2tvw \pm 2tw^2$$
$$= (-a_3)^{1/2} + y_1^{3\delta\varepsilon}\big[\pm(-a_2)^{1/2} - 2(a_0 a_2/4)^{1/4} x_1^{\delta} \pm 2(-a_0/4)^{1/2} x_1^{2\delta}\big]$$

and the expression in the brackets is not a power in $C[x_1]$. Moreover, one verifies directly that the quadrinomials $u \pm tv^2 - 2tvw \pm 2tw^2$ are not reciprocal.

C. $n_1 < 0$, $n_3 < 0$. From $|\sigma_\infty(f_1)| = |\sigma_\infty(g_1)|$ we get

$$[n_0, n_1] = [n_2, n_3]. \tag{4}$$

Counting the number of finite branch points we get

$$\frac{n_0 + |n_1|}{(n_0, n_1)} = \frac{n_2 + |n_3|}{(n_2, n_3)}. \tag{5}$$

If $(n_0, n_1) = (n_2, n_3) = 1$ we infer from (4), (5) and the inequalities $n_0 \geqslant -n_1 > 0$, $n_2 \geqslant -n_3 > 0$ that $n_0 = n_2$, $n_1 = n_3$. The same conclusion holds if

$$\frac{n_0 + |n_1|}{(n_0, n_1)} = \frac{n_2 + |n_3|}{(n_2, n_3)} = 2,\ 3 \text{ or } 4$$

since 2, 3 and 4 have only one partition into sum of two coprime positive integers. Since the branch points must be the same we get

$$\left(\frac{-a_2}{a_0}\right)^{n_1'} = \left(\frac{-a_3}{a_1}\right)^{n_0'}$$

and there exists a unique r such that

$$r^{n_0'} = -a_2/a_0, \qquad r^{n_1'} = -a_3/a_1.$$

On substitution $x = zy$ the quadrinomial $J\big(f_1(x)-g_1(y)\big)$ takes the form

$$J\big(f_1(x)-g_1(y)\big) = a_0 y^{n_0+2|n_1|} z^{|n_1|}(z^{n_0}-r^{n_0'}) + a_1 y^{|n_1|}(1-r^{n_1'}z^{|n_1|}).$$

Since the case $m_0+m_1 = m_2+m_3 = 0$, $a_0a_1 = a_2a_3$ has been excluded

$$\frac{1-r^{n_1'}z^{|n_1|}}{z^{n_0}-r^{n_0'}}$$

is not a power in $C(z)$. Also

$$(z^{n_0}-r^{n_0'},\, 1-r^{n_1}z^{|n_1|}) = z^{(n_0,n_1)}-r.$$

Thus in virtue of Capelli's theorem

$$J\big(f_1(x)-g_1(y)\big) = y^{|n_1|}(z^{(n_0,n_1)}-r)F(z,y)$$

where F is irreducible in $\boldsymbol{C}$.

It follows hence like in the case A that

$$d(x,y) = (a_0x^{m_0}+a_2y^{m_2},\, a_1x^{m_1}+a_3y^{m_3}) = x^{(n_0,n_1)\delta}-ry^{(n_0,n_1)\varepsilon}$$

and qd^{-1} is irreducible in $\boldsymbol{C}$.

Since the case $m_0+m_1 = m_2+m_3 = 0$, $a_0a_1 = a_2a_3$ has been excluded the degree of $Jq(x^{-1}, y^{-1})$ is greater than that of q. The degrees of $Jd(x^{-1}, y^{-1})$ and of d are equal, thus qd^{-1} is not reciprocal.

Assume therefore that

$$\frac{n_0+|n_1|}{(n_0,n_1)} = n_0'+n_1' > 4 \tag{6}$$

and set

$$f_3(x) = a_0x^{n_0'}+a_1x^{n_1'}, \qquad g_3(y) = -a_2y^{n_2'}-a_3y^{n_3'}.$$

If $\boldsymbol{\Omega}_{f_3-z} = \boldsymbol{\Omega}_{g_3-z}$ we get the assertion of the lemma by the previous argument. Without loss of generality we may assume that

$$\boldsymbol{\Omega}_{f_3-z} \neq \boldsymbol{\Omega}_{f_3-z}\boldsymbol{\Omega}_{g_3-z}.$$

By Lemma 1, f_3 is indecomposable. It follows by Lüroth theorem that there is no field between $\boldsymbol{C}(z)$ and $\boldsymbol{C}(x_1)$, where $f_3(x_1) = z$, thus the monodromy group $G\big(\boldsymbol{\Omega}_{f_3-z}/\boldsymbol{C}(z)\big)$ is primitive (cf. [3], Lemma 2).

On the other hand, this group contains a 2 cycle, thus it must be the symmetric group $\mathfrak{S}_{n_0'+n_1'}$ (see [9], p. 35). $\boldsymbol{\Omega}_{f_3-z}\cap\boldsymbol{\Omega}_{g_3-z}$ is a normal proper subfield of $\boldsymbol{\Omega}^f{}_{3-z}$ which corresponds to a normal subgroup of $G\big(\boldsymbol{\Omega}_{f_3-z}/\boldsymbol{C}(z)\big)$. It follows from the well known property of symmetric groups that this subgroup is $\mathfrak{S}_{n_0'+|n_1'|}$ or $\mathfrak{A}_{n_0'+|n_1'|}$ (see [9], p. 67). By the theorem of natural irrationalities

$$G\big(\boldsymbol{\Omega}_{f_3-z}/(\boldsymbol{\Omega}_{f_3-z}\cap\boldsymbol{\Omega}_{g_3-z})\big) \cong G(\boldsymbol{\Omega}_{f_3-z}\boldsymbol{\Omega}_{g_3-z}/\boldsymbol{\Omega}_{g_3-z}).$$

However $G(\boldsymbol{\Omega}_{f_3-z}\boldsymbol{\Omega}_{g_3-z}/\boldsymbol{\Omega}_{g_3-z})$ is a quotient group of $G(\boldsymbol{\Omega}_{g-z}/\boldsymbol{\Omega}_{g_3-z})$.

Since $g = g_3(x^{\varepsilon_1})$ we easily see that $G(\boldsymbol{\Omega}_{g-z}/\boldsymbol{\Omega}_{g_3-z})$ is a cyclic group and since by (6) none of the groups $\mathfrak{S}_{n_0'+|n_1'|}$, $\mathfrak{A}_{n_0'+|n_1'|}$ is cyclic we get a contradiction.

2. Riemann surface $2\sqrt{a_2a_3}T_{n_2}(x) = z$ has an n_2 cycle at ∞ and two branch points $2\varepsilon\sqrt{a_2a_3}$ with the permutations of type $\underbrace{(2)(2)\ldots(2)}_{(n_2-1)/2\text{ times}}$ if n_2 is odd and $\underbrace{(2)(2)\ldots(2)}_{(n_2-1-\varepsilon)/2\text{ times}}$ if n_2 is even ($\varepsilon = \pm 1$).

A. $n_0 > n_1 > 0$. Then

$$n_0 = n_2, \quad n_1 = 1, \quad n_0-1 = 2;$$

$$\pm\frac{2}{3}a_1\sqrt{\frac{-a_1}{3a_0}} = \pm 2\sqrt{a_2a_3}, \qquad a_1^3 = -27a_0a_2a_3,$$

the case considered under 1B.

B. $n_0 > 0 > n_1$. Then

$$[n_0, n_1] = n_2, \quad \frac{n_0+|n_1|}{(n_0,n_1)} = 2, \quad n_0 = -n_1 = n_2 = -n_3;$$

$$\pm 2a_1\sqrt{\frac{a_0}{a_1}} = \pm 2\sqrt{a_2a_3}, \qquad a_0a_1 = a_2a_3,$$

$m_0+m_1 = m_2+m_3 = 0$, the case excluded.

3. This case is symmetric to the former.

4. Then $n_0 = -n_1 = n_2 = -n_3$, $\pm 2\sqrt{a_0a_1} = \pm 2\sqrt{a_2a_3}$, $a_0a_1 = a_2a_3$, $m_0+m_1 = m_2+m_3 = 0$, the case excluded.

Lemma 3. *Let $\boldsymbol{K}$ be any field of characteristic zero, $a_i\in\boldsymbol{K}$, $a_i \neq 0$ $(i = 0, 1, 2, 3)$, m_i integers, $m_0+m_1 \geqslant 0$, $m_0 \neq m_1$, $m_2+m_3 \geqslant 0$, $m_2 \neq m_3$ and exactly one among m_i be zero. If $q(x,y) = J(a_0x^{m_0}+a_1x^{m_1}+a_2y^{m_2}+a_3y^{m_3})$ is reducible in $\boldsymbol{K}$ then it can be represented in the form*

$$t(u^2+2uv+v^2-w^2) = t(u+v+w)(u+v-w) \tag{7}$$

where $t\in\boldsymbol{K}$ and u, v, w are monomials in $\boldsymbol{K}[x,y]$. The factors on the right hand side of (7) are irreducible in $\boldsymbol{K}$ and non-reciprocal.

Proof. We may assume without loss of generality that $m_3 = 0$. Then $q(x,y)$ is a binomial over $\boldsymbol{K}(x)$. By Capelli's theorem, it is reducible only if either for some prime $l\,|\,m_2$, $a_2^{-1}(a_0x^{m_0}+a_1x^{m_1}+a_3) = -g(x)^l$

or $4 \mid m_2$, $a_2^{-1}(a_0x^{m_0}+a_1x^{m_1}+a_3) = 4g(x)^4$, $g(x) \in K(x)$. However $a_0x^{m_0}+a_1x^{m_1}+a_3$ may have at most double zero, therefore $l = 2$, $a_0x^{m_0}+a_1x^{m_1}+a_3 = -a_2g(x)^2$ and $g(x)$ has only simple zeros. Moreover $g(x)$ must have only two terms and taking

$$k = -a_2, \qquad u+v = Jg(x), \qquad w = \frac{J\big(g(x)\big)}{g(x)}y^{m_2/2}$$

we get the representation of $q(x,y)$ in the form (7). Again by Capelli's theorem the trinomials

$$u+v\pm w = J\big(g(x)\pm y^{m_2/2}\big)$$

are irreducible in K. One verifies directly that they are not reciprocal.

Lemma 4. *If any of the equations*

$$Q(y_1,y_2) = Z_0(U_0^2+2U_0V_0+V_0^2-1), \tag{8}$$

$$Q(y_1,y_2) = Z_0(U_0^3+V_0^3+1-3U_0V_0), \tag{9}$$

$$Q(y_1,y_2) = Z_0(U_0^2-4U_0V_0-V_0^4-4) \tag{10}$$

is satisfied by rational functions U_0, V_0, Z_0 of the type $cy_1^{a_1}y_2^{a_2}$, $c \in K$, then $Q(y_1,y_2)$ is representable in the corresponding form (1), *where $k \in K$, T, U, V, W are monomials over K and moreover*

$$UU_0^{-1} = VV_0^{-1} = W \quad \textit{if} \quad (8) \textit{ or } (9),$$
$$UU_0^{-1} = W^2T, \quad VV_0^{-1} = W \quad \textit{if} \quad (10).$$

Proof. Let y_i divide U_0, V_0, Z_0 with the exponent u_i, v_i, z_i. Since $\big(Q(y_1,y_2), y_1y_2\big) = 1$ we have

$$z_i = \begin{cases} -\min(2u_i, u_i+v_i, 2v_i, 0) & \text{if} \quad (8),\\ -\min(3u_i, 3v_i, u_i+v_i, 0) & \text{if} \quad (9),\\ -\min(2u_i, u_i+v_i, 4v_i, 0) & \text{if} \quad (10).\end{cases}$$

Since

$$u_i+v_i \geqslant \min(2u_i, 2v_i),$$
$$u_i+v_i \geqslant \min(3u_i, 3v_i, 0),$$
$$u_i+v_i \geqslant \min(2u_i, 4v_i, 0),$$

it follows that

$$z_i = \begin{cases} -\min(2u_i, 2v_i) = 2z_i' & \text{if} \quad (8),\\ -\min(3u_i, 3v_i, 0) = 3z_i' & \text{if} \quad (9),\\ -\min(2u_i, 4v_i, 0) = 2z_i' & \text{if} \quad (10),\end{cases}$$

where $z_i' \geqslant 0$ is an integer. We set in case (8) and (9)

$$k = Zy_1^{-z_1}y_2^{-z_2}, \quad W = y_1^{z_1'}y_2^{z_2'}, \quad U = U_0W, \quad V = V_0W;$$

in case (10)

$$k = Zy_1^{-z_1}y_2^{-z_2},\ W = y_1^{[z_1'/2]}y_2^{[z_2'/2]},\ T = y_1^{z_1'}y_2^{z_2'}W^{-2},\ U = U_0W^2T,\ V = V_0W$$

and the conditions of the lemma are satisfied.

Proof of Theorem 1. The sufficiency of the condition is obvious. In order to prove the necessity and the other assertions of the theorem set

$$\varDelta_1 = \begin{vmatrix}\nu_{11} & \nu_{12}\\ \nu_{21} & \nu_{22}\end{vmatrix}, \qquad \varDelta_2 = \begin{vmatrix}\nu_{12} & \nu_{13}\\ \nu_{22} & \nu_{23}\end{vmatrix}, \qquad \varDelta_3 = \begin{vmatrix}\nu_{13} & \nu_{11}\\ \nu_{23} & \nu_{21}\end{vmatrix};$$

$$\delta = \begin{cases} 1 & \text{if} \quad \varDelta_1+2\varDelta_2+\varDelta_3 \geqslant 0,\\ -1 & \text{if} \quad \varDelta_1+2\varDelta_2+\varDelta_3 < 0;\end{cases} \qquad \varepsilon = \begin{cases} 1 & \text{if} \quad \varDelta_1-\varDelta_3 \geqslant 0,\\ -1 & \text{if} \quad \varDelta_1-\varDelta_3 < 0.\end{cases}$$

Since the matrix $[\nu_{ij}]$ is of rank 2 we may assume without loss of generality that $\varDelta_1+\varDelta_3 \neq 0$. On substitution

$$y_1 = x^{\delta(\nu_{22}-\nu_{23})}y^{-\varepsilon\nu_{21}}, \qquad y_2 = x^{\delta(\nu_{13}-\nu_{12})}y^{\varepsilon\nu_{11}}$$

we get

$$\begin{aligned}\varPhi(y_1,y_2) &= a_0+\sum_{i=1}^{3}a_iy_1^{\nu_{1i}}y_2^{\nu_{2i}}\\ &= x^{-\delta\varDelta_2}(a_0x^{m_0}+a_1x^{m_1}+a_2y^{m_2}+a_3y^{m_3}) = x^{-\delta\varDelta_2}\varphi(x,y),\end{aligned}$$

where

$$m_0 = \delta\varDelta_2, \qquad m_1 = \delta(\varDelta_1+\varDelta_2+\varDelta_3), \qquad m_2 = \varepsilon\varDelta_1, \qquad m_3 = -\varepsilon\varDelta_3.$$

We have $m_0 \neq m_1$, $m_2 \neq m_3$ and by the choice of δ and ε, $m_0+m_1 \geqslant 0$, $m_2+m_3 \geqslant 0$.

Moreover setting $q(x,y) = J\varphi(x,y)$ we get

$$Q(y_1,y_2) = x^{A}y^{B}q(x,y). \tag{11}$$

Assume that

$$Q(y_1,y_2) = F_1(y_1,y_2)F_2(y_1,y_2),$$

where F_1, F_2 are non-constant polynomials over K. It follows that

$$\begin{aligned}q(x,y) = JF_1(x^{\delta(\nu_{22}-\nu_{23})}y^{-\varepsilon\nu_{21}},\ x^{\delta(\nu_{13}-\nu_{12})}y^{\varepsilon\nu_{11}})\times\\ \times JF_2(x^{\delta(\nu_{22}-\nu_{23})}y^{-\varepsilon\nu_{12}},\ x^{\delta(\nu_{13}-\nu_{12})}y^{\varepsilon\nu_{11}}),\end{aligned} \tag{12}$$

where the factors on the right hand side are non-constant. We distinguish three cases

(i) $m_0 = -m_1$, $m_2 = -m_3$, $a_0a_1 = a_2a_3$;
(ii) $m_0m_1m_2m_3 \neq 0$ and (i) does not hold;
(iii) $m_0m_1m_2m_3 = 0$.

(i) We have here $\varDelta_1 = -\varDelta_2 = \varDelta_3$. Hence

$$\nu_{i1} = \nu_{i2}+\nu_{i3} \qquad (i = 1, 2)$$

and

$$\Phi(y_1, y_2) = (a_0 + a_2 y_1^{\nu_{12}} y_2^{\nu_{22}}) \left(1 + \frac{a_3}{a_0} y_1^{\nu_{13}} y_2^{\nu_{23}}\right),$$

thus $Q(y_1, y_2)$ can be divided into two parts with the highest common factor

$$D = J(a_0 + a_2 y_1^{\nu_{12}} y_2^{\nu_{22}})$$

being a binomial. The complementary factor

$$QD^{-1} = J\left(1 + \frac{a_3}{a_0} y_1^{\nu_{13}} y_2^{\nu_{23}}\right)$$

is also a binomial.

(ii) Here we can apply Lemma 2 and we infer that either $q(x, y)$ can be divided into two parts with the highest common factor $d(x, y)$ being a binomial or $q(x, y)$ can be represented in one of the forms (2), where t, u, v, w are monomials in $C[x, y]$. In the former case qd^{-1} is irreducible in C and non-reciprocal, in the latter case the factors on the right hand side of (2) are irreducible in C and non-reciprocal. Now, if

$$d_i(x, y) = \left(J(a_0 x^{m_0} + a_i y^{m_i}), J(a_1 x^{m_1} + a_{5-i} y^{m_{5-i}})\right) \qquad (i = 2 \text{ or } 3),$$

$$D_i(x, y) = \left(J(a_0 + a_i y_1^{\nu_{1i}} y_2^{\nu_{2i}}), J(a_1 y_1^{\nu_{11}} y_2^{\nu_{21}} + a_{5-i} y_1^{\nu_{1,5-i}} y_2^{\nu_{2,5-i}})\right)$$

then

$$d_i(x, y) = JD_i(x^{\delta(\nu_{22}-\nu_{23})} y^{-\varepsilon\nu_{21}}, x^{\delta(\nu_{13}-\nu_{12})} y^{\varepsilon\nu_{11}}),$$

thus the properties of d_i imply the corresponding properties of D_i.

If

$$q(x, y) = u^3 + v^3 + w^3 - 3uvw = (u+v+w)(u + \zeta_3 v + \zeta_3^{-1} w)(u + \zeta_3^{-1} v + \zeta_3 w) \tag{13}$$

then by the absolute irreducibility of the factors on the right hand side and by (12) we have for suitable $i = 1$ or 2, suitable $j = 0$ or ± 1 and suitable α, β, γ

$$F_i(y_1, y_2) = \gamma x^\alpha y^\beta (u + \zeta_3^j v + \zeta_3^{-j} w).$$

We may assume without loss of generality that $j = 0$. It follows that

$$U_0 = uw^{-1} \in K(y_1, y_2), \qquad V_0 = vw^{-1} \in K(y_1, y_2) \tag{14}$$

and by (11) and (13)

$$Q(y_1, y_2) = x^A y^B w^3 (U_0^3 + V_0^3 + 1 - 3U_0 V_0).$$

Since u, v, w are monomials in $C[x, y]$, U_0, V_0 and $Z = x^A y^B w^3$ are of the form $cy_1^{\alpha_1} y_2^{\alpha_2}$, $c \in K$. By Lemma 4 there exist monomials U, V, W in $K[y_1, y_2]$ and $k \in K$ such that

$$Q(y_1, y_2) = k(U^3 + V^3 + W^3 - 3UVW), \qquad UU_0^{-1} = VV_0^{-1} = W.$$

It follows by (14) that

$$Uu^{-1} = Vv^{-1} = Ww^{-1},$$

$$J(U + \zeta_3^j V + \zeta_3^{-j} W)(x^{\delta(\nu_{22}-\nu_{23})} y^{-\varepsilon\nu_{21}}, x^{\delta(\nu_{13}-\nu_{12})} y^{\varepsilon\nu_{11}}) = \eta(u + \zeta_3^j v + \zeta_3^{-j} w)$$
$$(\eta \in C,\ j = 0,\ \pm 1)$$

and since $u + \zeta_3^j v + \zeta_3^{-j} w$ is irreducible in C and non-reciprocal, $U + \zeta_3^j V + \zeta_3^{-j} W$ has the same property. If $\zeta_3 \notin K$

$$U^2 + V^2 + W^2 - UV - UW - VW = (U + \zeta_3 V + \zeta_3^{-1} W)(U + \zeta_3^{-1} V + \zeta_3 W)$$

is irreducible in K. It is also non-reciprocal by the corresponding property of $u^2 + v^2 + w^2 - uv - uw - vw$.

Assume now that

$$q(x, y) = u^2 - 4tuvw - t^2 v^4 - 4t^2 w^4 = (u - tv^2 - 2tvw - 2tw^2)(u + tv^2 - 2tvw + 2tw^2).$$

Then by the absolute irreducibility of the factors on the right hand side and by (12) we have for a suitable sign and suitable α, β, γ

$$F_1(y_1, y_2) = \gamma x^\alpha y^\beta (u \pm tv^2 - 2tvw \pm 2tw^2).$$

It follows that

$$U_0 = ut^{-1} w^{-2} \in K(y_1, y_2), \qquad V_0 = vw^{-1} \in K(y_1, y_2) \tag{15}$$

and by (11)

$$Q(y_1, y_2) = x^A y^B t^2 w^4 (U_0^2 - 4U_0 V_0 - V_0^4 - 4).$$

By Lemma 4 there exist monomials T, U, V, W in $K[x, y]$ and $k \in K$ such that

$$Q(y_1, y_2) = k(U^2 - 4TUVW - V^4 - 4T^2 W^4), \qquad UU_0^{-1} = TW^2, \qquad VV_0^{-1} = W.$$

It follows by (15) that

$$Uu^{-1} = TV^2 t^{-1} v^{-2} = TVWt^{-1} v^{-1} w^{-1} = TW^2 t^{-1} w^{-2},$$

$$J(U \pm TV^2 - 2TVW \pm 2TW^2)(x^{\delta(\nu_{22}-\nu_{23})} y^{-\varepsilon\nu_{21}}, x^{\delta(\nu_{13}-\nu_{12})} y^{\varepsilon\nu_{11}}) = \eta(u \pm tv^2 - 2tvw \pm 2tw^2) \qquad (\eta \in C)$$

and since $u \pm tv^2 - 2tvw + 2tw^2$ is irreducible in C and non-reciprocal, $U \pm TV^2 - 2TVW \pm 2TW^2$ has the same property.

(iii) If two of the numbers m_0, m_1, m_2, m_3 were equal zero, two of the vectors $[0, 0]$, $[\nu_{1i}, \nu_{2i}]$ $(i \leqslant 3)$ would be equal. Thus exactly one m_i is zero, we can apply Lemma 3 and infer that $q(x, y)$ is representable in the form (7), where $k \in K$, u, v, w are monomials in $K[x, y]$, the trinomials $u + v \pm w$ are irreducible in K and non-reciprocal.

It follows from (12) that for a suitable sign and suitable α, β, γ

$$F_1(y_1, y_2) = \gamma x^{\alpha} y^{\beta}(u+v\pm w).$$

Thus

$$U_0 = uw^{-1} \in \boldsymbol{K}(y_1, y_2), \quad V_0 = vw^{-1} \in \boldsymbol{K}(y_1, y_2) \tag{16}$$

and by (11)

$$Q(y_1, y_2) = x^A y^B w^2 (U_0^2 + 2U_0V_0 + V_0^2 - 1).$$

By Lemma 4 there exist monomials U, V, W in $\boldsymbol{K}[x, y]$ and $k \in \boldsymbol{K}$ such that

$$Q(y_1, y_2) = k(U^2 + 2UV + V^2 - W^2), \quad UU_0^{-1} = VV_0^{-1} = W.$$

It follows by (16) that

$$Uu^{-1} = Vv^{-1} = Ww^{-1},$$

$$J(U+V\pm W)(x^{\delta(\nu_{21}-\nu_{23})}y^{-\varepsilon\nu_{21}}, x^{\delta(\nu_{13}-\nu_{12})}y^{\varepsilon\nu_{11}}) = \eta(u+v\pm w) \quad (\eta \in \boldsymbol{K})$$

and since $u+v\pm w$ is irreducible in $\boldsymbol{K}$ and non-reciprocal, $U+V\pm W$ has the same property. The proof of Theorem 1 is complete.

Proof of Theorem 2. In order to prove the necessity of the condition we apply Theorem 2 of [8] setting there

$$F(x_1, x_2, x_3) = ax_1 + bx_2 + cx_3 + d$$

so that

$$g(x) = F(x^m, x^n, x^p).$$

By the said theorem there exists a matrix $N = [\nu_{ij}]_{\substack{i \leq r \\ j \leq 3}}$ of rank $r \leq 3$ such that

$$0 < \max|\nu_{ij}| < c_r(F), \tag{17}$$

$$[m, n, p] = [v_1, \ldots, v_r]N, \tag{18}$$

$$L\Big(a\prod_{i=1}^{r} y_i^{\nu_{i1}} + b\prod_{i=1}^{r} y_i^{\nu_{i2}} + c\prod_{i=1}^{r} y_i^{\nu_{i3}} + d\Big) \overset{\text{can}}{=} \text{const}\prod_{\sigma=1}^{s} F_\sigma(y_1, \ldots, y_r)^{e_\sigma} \tag{19}$$

implies

$$Lq(x) \overset{\text{can}}{=} \text{const}\prod_{\sigma=1}^{s} LF_\sigma(x^{v_1}, \ldots, x^{v_r})^{e_\sigma}. \tag{20}$$

Therefore, if $Lq(x)$ is reducible then the left hand side of (19) is reducible. It follows by Lemma 14 of [8] that $r < 3$.

If $r = 2$, set in Theorem 1: $a_0 = d$, $a_1 = a$, $a_2 = b$, $a_3 = c$ so that the left hand side of (19) becomes $LQ(y_1, y_2)$ in the notation of that theorem. The vectors $[0, 0]$, $[\nu_{1i}, \nu_{2i}]$ $(i \leq 3)$ are all different in view of (18) and of the assumption $m > n > p > 0$. If $Q(y_1, y_2)$ is a product of two binomials, $q(x) = JQ(x^{v_1}, x^{v_2})$ is also such a product. This case has been excluded, but the condition is satisfied also here, since one of the binomials must be non-reciprocal and it is the desired non-reciprocal common factor of two parts of $q(x)$. Apart from this case, in virtue of Theorem 1, $LQ(y_1, y_2)$ is reducible if and only if either Q can be divided into two parts which have a non-reciprocal common factor or it can be represented in one of the forms (1), where $k \in \boldsymbol{Q}$ and T, U, V, W are monomials in $\boldsymbol{Q}[y_1, y_2]$. If $F_\sigma(y_1, y_2)$ is an irreducible non-reciprocal factor of $Q(y_1, y_2)$, $LF_\sigma(x^{v_1}, x^{v_2})$ is by (20) an irreducible non-reciprocal factor of $q(x)$. Therefore, we get either a partition of $q(x)$ into two parts which have a common non-reciprocal factor or a representation of $q(x)$ in one of the forms (1), where T, U, V, W are monomials in $\boldsymbol{Q}[x]$ and the factors on the right hand side are non-reciprocal.

Finally, if $r = 1$ then $m = vm_1$, $n = vn_1$, $p = vp_1$,

$$m_1 < c_1(F) = \exp_2\big(24 \cdot 2^{a^2+b^2+c^2+d^2-1}\log(a^2+b^2+c^2+d^2)\big) = C(a, b, c, d)$$

by (17), (18) and the formula for $c_1(F)$ given in [8]. Moreover

$$L(ax^{m_1} + bx^{n_1} + cx^{p_1} + d) \overset{\text{can}}{=} \text{const}\prod_{\sigma=1}^{s} F_\sigma(x)^{e_\sigma}$$

implies

$$L(ax^m + bx^n + cx^p + d) \overset{\text{can}}{=} \text{const}\prod_{\sigma=1}^{s} LF_\sigma(x^v)^{e_\sigma}.$$

Thus the necessity of the condition is proved. In order to prove the sufficiency it is enough to consider the case where $q(x)$ can be divided into two parts which have a common non-reciprocal factor $\delta(x)$. Since the highest common factor of two binomials is either 1 or a binomial and since binomial with a non-reciprocal factor is itself non-reciprocal we may assume that $\delta(x)$ is the highest common factor of two parts of $q(x)$ and hence a binomial. We prove that $q\delta^{-1}$ is non-reciprocal. Indeed, otherwise, we should have

$$\delta(x) = x^r + e, \quad e \neq \pm 1, \tag{21}$$

$$\pm(ax^m + bx^n + cx^p + d)(ex^r + 1) = (dx^m + cx^{m-p} + bx^{m-n} + a)(x^r + e) \tag{22}$$

and either

$$\delta(x) = (ax^m + bx^n, cx^p + d) \tag{23}$$

or

$$\delta(x) = (ax^m + cx^p, bx^n + d) \tag{24}$$

or

$$\delta(x) = (ax^m + d, bx^n + cx^p). \tag{25}$$

It follows from (22) that

$$\pm ae = d \tag{26}$$

thus by (21) $\delta(x)$ cannot divide $ax^m + d$ and (25) is excluded. If (23) or (24) holds we have $m \neq n+p$, since otherwise

$$\delta(x) = x^{m-n} + \frac{b}{a} = x^p + \frac{d}{c} \quad \text{or} \quad \delta(x) = x^{m-p} + \frac{c}{a} = x^n + \frac{d}{b}$$

and $q(x)$ is a product of two binomials. We may assume without loss of generality that $m > n+p$. If $r < p$ then comparing the coefficients of x^m on both sides of (22) we get $\pm a = ed$, which together with (26) gives $e = \pm 1$, contrary to (21). If $r > p$ then on the right hand side of (22) occurs the term cx^{m-p+r} lacking on the left hand side. If $r = p$ then comparing the coefficients of x^m on both sides of (22) we get

$$\pm a = de + c. \tag{27}$$

If (23) holds then $e = d/c$ and since $\pm ae = d$ we get $\pm a = c$, $de = 0$ a contradiction. If (24) holds, then $\delta(x) \mid ax^m + cx^p$ gives

$$p \mid m, \quad c/a = -(-e)^{\frac{m}{p}-1}$$

and by (26) and (27) $e^2 \mp (-e)^{\frac{m}{p}-1} = 1$, which has no rational solution.

Proof of Theorem 3. In virtue of Theorem 2 $L(x^m + \varepsilon x^n + cx^p + d)$ is reducible if and only if at least one of the conditions specified in the assertion is satisfied.

Suppose that $\dfrac{x^m + \varepsilon x^n + cx^p + d}{L(x^m + \varepsilon x^n + cx^p + d)}$ is non-constant and let

$$\lambda^m + \varepsilon\lambda^n + c\lambda^p + d = 0 = \lambda^{-m} + \varepsilon\lambda^{-n} + c\lambda^{-p} + d.$$

Thus $d\lambda^{n+m} + c\lambda^{n+m-p} + \varepsilon\lambda^m + \lambda^n = 0$, $\varepsilon\lambda^m + \lambda^n + c\varepsilon\lambda^p + d = 0$, hence

$$F(\lambda) = d\lambda^{m+n} + c\lambda^{m+n-p} - c\varepsilon\lambda^p - d\varepsilon = 0.$$

By a theorem of A. Cohn [1] (p. 113), the equations $F(x) = 0$ and $x^{m+n-1}F'(x^{-1}) = 0$ have the same number of zeros inside the unit circle. We have

$$\begin{aligned}\lambda^{m+n-1}F'(\lambda^{-1}) &= \lambda^{m+n-1}\big(d(m+n)\lambda^{1-m-n} + c(m+n-p)\lambda^{1+p-m-n} - c\varepsilon p\lambda^{1-p}\big)\\ &= d(m+n) + c(m+n-p)\lambda^p - a\varepsilon p\lambda^{n+m-p}.\end{aligned}$$

Assuming $|\lambda| < 1$ we obtain

$$|d(m+n)| < |c|(n+m-p) + |c|p = |c|(m+n),$$

which is impossible.

Consequently F has no zero inside the unit circle and since F is reciprocal all zeros are on the boundary of the unit circle. It follows that the same is true for $\dfrac{x^m + \varepsilon x^n + cx^p + d}{L(x^m + \varepsilon x^n + cx^p + d)}$. However the last polynomial is monic with integer coefficients, thus by Kronecker's theorem all its zeros are roots of unity.

Therefore $K(x^m + \varepsilon x^n + cx^p + d) = L(x^m + \varepsilon x^n + cx^p + d)$ and the proof of the theorem is complete.

Proof of Corollary. In virtue of Theorem 3, $x^m + \varepsilon x^n + cx^p + d$ is reducible if and only if either one of the conditions specified in the theorem is satisfied or $x^m + \varepsilon x^n + cx^p + d$ has a proper cyclotomic factor. Now, by a theorem of Mann [7] if a root of unity λ satisfies

$$\lambda^m + \varepsilon\lambda^n + c\lambda^p + d = 0$$

then either the left hand side can be divided into two vanishing summands or $\lambda^{6(m,n,p)} = 1$. The first possibility corresponds to the first three equalities specified in the corollary, the second gives

$$\zeta^{m/\delta} + \varepsilon\zeta^{n/\delta} + c\zeta^{p/\delta} + d = 0,$$

where $\zeta^6 = 1$, $\delta = (m, n, p)$.

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