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A mean value theorem of Bombieri's type

by

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In memory of W. Sierpiński

1. Introduction. In the present paper $K = K(\sqrt{\Delta})$ denotes any fixed quadratic field of the discriminant $\Delta \ll 1$ and $\mathfrak{K}$ stands for classes of ideals in K (either in the usual or in the restricted meaning; cf. [9], § 1). Let q be any natural number > 1 and G_1 denote a group of reduced classes of residues $l \pmod q$ formed by the residues of the idealnorms $N\mathfrak{a}$ with $(\mathfrak{a}, [q]) = 1$ and $\mathfrak{a}$ belonging to the principal class $\mathfrak{K}_1$. Let $\varphi_1(q)$ be the order of G_1. Then for any class $\mathfrak{K}$ there are as many different normresidues $N\mathfrak{a} \pmod q$ with $\mathfrak{a} \in \mathfrak{K}$ and $(\mathfrak{a}, [q]) = 1$ (cf. [6], § 4), and we have

$$c\varphi(q) \leqslant \varphi_1(q) \leqslant \varphi(q) \tag{1}$$

where $c = c(\Delta)$ stands for a positive constant < 1 and $\varphi(q)$ denotes the number of reduced classes mod q (cf. [18], §§ 107, 108). The number of classes $\mathfrak{K}$ will be denoted by h.

The aim of this paper is the proof of the following

THEOREM. *Let $a = a(q, \mathfrak{K})$ be a normresidue* mod q *with $(a, q) = 1$ for the class $\mathfrak{K}$ of ideals in the quadratic field $K(\sqrt{\Delta})$ and let $\pi(x; \mathfrak{K}, q, a)$ denote the number of primes $p \leqslant x$ such that $p \equiv a \pmod q$ and such that $p = N\mathfrak{p}$ with $\mathfrak{p} \in \mathfrak{K}$. Then for any constant $A > 0$ there is a corresponding constant $B > 0$ such that*

$$\sum_{\mathfrak{K}} \sum_{q \leqslant z^{1/2}\log^{-B} z} \max_{a(q,\mathfrak{K})} \max_{x \leqslant z} \left| \pi(x; \mathfrak{K}, q, a) - \frac{1}{h\varphi_1(q)} \operatorname{Li} x \right| \ll \frac{z}{(\log z)^A} \tag{2}$$

with the constant in the notation depending merely on A and Δ.

A theorem of this type for the function $\psi(x; q, a)$ (denoting the sum of $\log p$ extended over all prime powers $p^k \leqslant x$ such that $p^k \equiv a \pmod q$) was first proved by Bombieri [1]; his proof is based on the distribution of the zeros of L-functions. Gallagher [11] gave a more direct proof based on the Pólya and Vinogradov's estimate for character sums ([16], p. 146).

As good estimate for the character sums in $K(\sqrt{\Delta})$ not being known, in the proof of (2) we shall follow the method of Bombieri.

Bombieri's theorem has been used in some additive problems as a substitute for the extended Riemann hypothesis and for the dispersion method of Linnik (cf. [4], [2]). The theorem of the present paper may be used for similar purposes if in the given problem the set of all primes is reduced to that of the primes in the sequence of idealnorms of a given class $\mathfrak{K}$ in $K(\sqrt{\Delta})$.

The result of the present paper has been announced in [10].

2. Preliminaries. In the proof of (2) we shall need the following auxiliary theorems.

(i) *Let $\chi_{\mathfrak{K}}$ denote characters of classes $\mathfrak{K}$ of ideals (in any algebraic number field) and let χ_q be a Dirichlet's character* mod q. *Then the function*

(3) $$\zeta(s, \chi_q, \chi_{\mathfrak{K}}) = \sum_{\mathfrak{a}} \frac{\chi_q(N\mathfrak{a})\chi_{\mathfrak{K}}(\mathfrak{a})}{N\mathfrak{a}^s}$$

(where $s = \sigma+it$, $\sigma > 1$, the sum is over all integer ideals $\mathfrak{a} \neq 0$) is identical to some L-function of Hecke

(4) $$\zeta(s, \chi_q, \chi_{\mathfrak{K}}) = \zeta(s, \chi') = \sum_{\mathfrak{a}} \frac{\chi'(\mathfrak{a})}{N\mathfrak{a}^s}$$

where χ' is a character mod $[q]$.

For the proof see [9], Lemma 1.

(ii) Let again $\mathfrak{K}$ be the classes of ideals in any fixed algebraic number field and let $g = g(\mathfrak{K}, q)$ denote the number of mod q incongruent idealnorms $N\mathfrak{a}$ with $(N\mathfrak{a}, q) = 1$ and $\mathfrak{a} \in \mathfrak{K}$. By what has been proved in [6], p. 260, this number actually is the same for all classes $\mathfrak{K}$. Therefore we shall write simply $g = g(q)$.

The reduced classes $l \pmod q$ for which there are idealnorms $N\mathfrak{a}$ with $N\mathfrak{a} \equiv l \pmod q$ evidently form a group $G = H_1 + H_2 + \ldots + H_r$, the H_i $(1 \leqslant i \leqslant r)$ denoting its different elements. Let $G_1 = H_1 + \ldots + H_g$ be the subgroup of G containing the normrests of the principal class of ideals. Since in the group representation

$$G = H_1G_1 + H_2G_1 + \ldots + H_rG_1$$

any two cosets H_iG_1, H_jG_1 are either identical or they have no common element (see [13], § 6), we deduce that the normrests of any class $\mathfrak{K}_i$ are those of the principal class $\mathfrak{K}_1$ multiplied by a suitable number k_i with $(k_i, q) = 1$.

Considering that the group of characters of any Abelian group is isomorphic with the group itself (see [13], § 10) we deduce that there is a group Γ of characters χ_q mod q of order g corresponding to the subgroup G_1 in the group of all reduced classes l mod q.

LEMMA 1. *Let a denote the reduced normrests* mod q *of the principal class $\mathfrak{K}_1$ of ideals. Then the reduced normrests of any other class $\mathfrak{K}_i$ are $\equiv ak_i$* (mod q) *where k_i stands for a suitable constant with $(k_i, q) = 1$ depending merely on the class $\mathfrak{K}_i$, and we have*

(5) $$\sum_{\chi_{\mathfrak{K}}} \bar{\chi}_{\mathfrak{K}}(\mathfrak{K}_i) \sum_{\chi_q \in \Gamma} \bar{\chi}_q(ak_i)\chi_q(N\mathfrak{a})\chi_{\mathfrak{K}}(\mathfrak{a}) = \begin{cases} gh & \text{if } \mathfrak{a} \in \mathfrak{K}_i,\ N\mathfrak{a} \equiv ak_i \pmod q, \\ 0 & \text{otherwise}. \end{cases}$$

Here $\chi_{\mathfrak{K}}$ runs through all the characters of the classes $\mathfrak{K}$ (h in number) and Γ is a suitable group of order g of Dirichlet's characters χ_q, isomorphic with the group of the numbers a.

Proof. By the orthogonality of the characters ([13], § 10)

$$\sum_{\chi_{\mathfrak{K}}} \bar{\chi}_{\mathfrak{K}}(\mathfrak{K}_i)\chi_{\mathfrak{K}}(\mathfrak{K}_j) = \begin{cases} h & \text{if } \mathfrak{K}_i = \mathfrak{K}_j, \\ 0 & \text{otherwise}. \end{cases}$$

Hence the sum (5) differs from zero merely for ideals $\mathfrak{a} \in \mathfrak{K}_i$ and thus $N\mathfrak{a} \equiv ak_i \pmod q$ with a fixed k_i and a running through the normresidues (g in number) of the principal class $\mathfrak{K}_1$ making the group G_1. Considering that Γ is the group of all characters of G_1 we deduce that for any $a' \in G_1$

$$\sum_{\chi_q \in \Gamma} \bar{\chi}_q(a')\chi_q(a) = \begin{cases} g & \text{if } a \equiv a' \pmod q, \\ 0 & \text{otherwise}. \end{cases}$$

Hence (5) follows. It is a correction of the erroneous formula (12) of [6].

In what follows we shall deal exclusively with Hecke's L-functions

(6) $$\zeta(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} \quad (s = \sigma+it,\ \sigma > 1)$$

with characters χ mod $\mathfrak{f}$ on the quadratic field of the discriminant Δ. Write

(7) $$d = \frac{1}{\pi}\sqrt{|\Delta| \cdot N\mathfrak{f}}.$$

The following properties (iii)–(v) are particular cases of what has been proved for Hecke's L-functions of any algebraic field.

(iii) *There is an absolute constant $c_1 > 0$ such that in the region*

$$G'\{\sigma \geqslant 1 - c_1/\log d(1+|t|) \geqslant 3/4\}$$

there is no zero of the function (6) in the case of a complex χ. For at most one real χ there may be in G' a simple zero $1-\delta$ of $\zeta(s, \chi)$; it is real and $\delta > c_2(\varepsilon)d^{-\varepsilon}$ for any $\varepsilon > 0$ and a suitable $c_2(\varepsilon) > 0$ (independent of d).

For the proof see [5] and [7] (¹).

(iv) *Let $N(T)$ denote the number of zeros of the function* (6) *in the rectangle* $(0 \leqslant \sigma \leqslant 1,\ |t-T| \leqslant 1)$ (²). *Then for appropriate absolute constant* $c_3 > 0$

$$N(T) < c_3 \log d(1+|T|). \tag{8}$$

For the proof see [5], Lemma 5.

(v) *Let $e_0 = 1$ if χ is the principal character* mod $\mathfrak{f}$, $e_0 = 0$ *otherwise and let ϱ run through the zeros of the function* (6) (³). *Then in the strip* $-1 \leqslant \sigma \leqslant 3$

$$\frac{\zeta'}{\zeta}(s,\chi) - \sum_{|s-\varrho|\leqslant 1} \frac{1}{s-\varrho} + \frac{e_0}{s-1} \ll \log d(1+|t|) \tag{9}$$

with an absolute constant in the notation.

For the proof see [5], Lemma 6.

(vi) *For any fixed natural number q and the quadratic field $K(\sqrt{\Delta})$ let $\Gamma = \Gamma_q$ be the group of Dirichlet characters χ_q as explained in Lemma* 1 *and let*

$$\tau(\chi_q) = \sum_{1\leqslant m\leqslant q} \chi_q(m) e^{2\pi i m/q}. \tag{10}$$

If $N(\alpha, T, \chi_q, \chi_{\mathfrak{K}})$ denotes the number of zeros of the function (3) *in the rectangle* $(\alpha \leqslant \sigma \leqslant 1,\ |t| \leqslant T)$, $d(q)$ *stands for the number of natural divisors of q and $\sum_{\chi_q}'$ denotes a sum over all characters of the group Γ excluding the principal character χ_q^0, then for any $\alpha \in [1/2, 1]$ we have uniformly in* $D \geqslant 2$, $M \geqslant 2$, $T \geqslant 2$

(11)
$$\sum_{\chi_{\mathfrak{K}}} \sum_{\substack{q\leqslant M \\ d(q)\leqslant D}} \frac{1}{\varphi(q)} \sideset{}{'}\sum_{\chi_q} |\tau(\chi_q)|^2 N(\alpha, T, \chi_q, \chi_{\mathfrak{K}}) \ll D^4 T (M^2 T)^{\frac{4(1-\alpha)}{3-2\alpha}} \log^{196} MT$$

with the constant in the notation depending merely on Δ.

(¹) Let $\mathfrak{F}$ be the set of functions (3) where $\chi_{\mathfrak{K}}$ runs through the characters of the classes $\mathfrak{K}$ and χ_q through those of the group Γ (cf. (ii)). Using the uniqueness theorem for Dirichlet series one can prove that no two functions of the set $\mathfrak{F}$ are identical. Hence exactly one of them has a pole at $s = 1$ and at most one has an exceptional zero, since the corresponding is true for the L-functions (4) with characters χ' mod $[q]$. See also [9], Lemma 2, footnote (⁵).

(²) Multiple zeros being counted according to their order of multiplicity.

(³) Multiple zeros being allowed for by repetition.

For the proof see [9].

If q^* is the conductor of the character χ_q, then $q^* | q$ and the sum (10) satisfies

$$|\tau(\chi_q)|^2 = \begin{cases} q^* & \text{if} \quad (q^*, q/q^*) = 1 \text{ and } q/q^* \text{ squarefree}, \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

(see [12], § 20 or [3], p. 148).

3. The function $E^*(z, q)$. In the subsequent proofs we shall use the following notation:

$$\Lambda(\mathfrak{a}) = \begin{cases} \log N\mathfrak{p} & \text{if} \quad \mathfrak{a} = \mathfrak{p}^k \ (\mathfrak{p} \text{ — prime ideal},\ k = 1, 2, \ldots), \\ 0 & \text{otherwise;} \end{cases}$$

$$\Lambda'(n) = \Lambda(\mathfrak{a}) \text{ if } n = N\mathfrak{a} \text{ and } = 0 \text{ for other } n;$$

$$\psi_1(x, \chi) = \psi_1(x, \chi_q, \chi_{\mathfrak{K}}) = \sum_{\substack{x\geqslant n = N\mathfrak{p}^k \\ k=1,2,\ldots}} \chi_q(n)\chi_{\mathfrak{K}}(\mathfrak{p}^k)(x-n)\Lambda'(n) \tag{13}$$

$$= -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \frac{\zeta'}{\zeta}(s, \chi_q, \chi_{\mathfrak{K}})\, ds \qquad (c > 1,\ x > 1)$$

(cf. [14], pp. 30–32). For any class of ideals $\mathfrak{K}_i$ and for any $a = a(q, \mathfrak{K}_i)$ with $(a, q) = 1$ and such that there is an integer ideal $\mathfrak{a} \in \mathfrak{K}_i$ with $N\mathfrak{a} \equiv a$ (mod q) let us write

$$\psi(x; \mathfrak{K}_i, q, a) = \sum_{\substack{\mathfrak{a}\in\mathfrak{K}_i \\ x\geqslant N\mathfrak{a}\equiv a \,(\mathrm{mod}\, q)}} \Lambda(\mathfrak{a}), \tag{14}$$

$$\psi_1(x; \mathfrak{K}_i, q, a) = \int_0^x \psi(u; \mathfrak{K}_i, q, a)\, du = \sum_{\substack{x\geqslant n\equiv a \,(\mathrm{mod}\, q) \\ n = N\mathfrak{p}^k,\, \mathfrak{p}^k\in\mathfrak{K}_i,\, k=1,2,\ldots}} (x-n)\Lambda'(n) \tag{15}$$

$$= -\frac{1}{2\pi i h \varphi_1(q)} \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \sum_{\chi_{\mathfrak{K}}} \bar{\chi}_{\mathfrak{K}}(\mathfrak{K}_i) \sum_{\chi_q\in\Gamma_q} \bar{\chi}_q(a) \frac{\zeta'}{\zeta}(s, \chi_q, \chi_{\mathfrak{K}})\, ds$$
$$(c > 1,\ x > 1),$$

(16)
$$E(z; \mathfrak{K}_i, q, a) = \psi_1(z; \mathfrak{K}, q, a) - z^2/2h\varphi_1(q),$$
$$E(z, q) = \max_{\mathfrak{K}} \max_{a(q,\mathfrak{K})} |E(z; \mathfrak{K}, q, a)|, \qquad E^*(z, q) = \max_{y\leqslant z} E(y, q).$$

The identity of the last two expressions in (15) is a consequence of (5).

Lemma 2. *For any $A \leqslant 1$ and $z \to \infty$*

$$E^*(z, 1) \ll z^2 (\log z)^{-A} \tag{17}$$

with the constant in the notation depending on A and Δ.

Proof. It follows from § 2, (iii) (with $d = |\Delta|^{1/2}/\pi \leqslant 1$) and (4), (8), (9) that for appropriate constants $c_4, c_5, \ldots$ (depending merely on Δ) there are no zeros of $\zeta(s, \chi_1, \chi_{\mathfrak{K}})$ in the region $\sigma \geqslant 1 - c_4/\log(3+|t|)$ and in this region $|\zeta'/\zeta(s, \chi_1, \chi_{\mathfrak{K}})| < c_5 \log^2(3+|t|) + |s-1|^{-1}$. Now we use (15) (with $q = 1$, $a = 1$, $c = 1 + 1/\log x$) and move the part of the path of integration with $|t| \leqslant t_0 = \exp(c_6\sqrt{\log x})$ to the line $\sigma = \sigma_0 = 1 - c_4/\log(3+t_0)$. In doing this we pass no other singularities of the integrand than $s = 1$ which is a simple pole of $\zeta'/\zeta(s, \chi_1, \chi_{\mathfrak{K}}^0)$ ($\chi_{\mathfrak{K}}^0$ — the principal character) with the residue -1 (cf. [15], Satz LXIII). Thus we get the estimate

$$\psi_1(x; \mathfrak{K}, 1, 1) - x^2/2h \ll x^2 \exp(-c_7\sqrt{\log x}), \tag{18}$$

whence (17) follows.

Lemma 3. *Let N and B be arbitrarily large positive constants, $z > z_0(N, B) > 1$, $x \geqslant z^{1/N}$, $\chi_q \neq \chi_q^0$ and $q \leqslant X_0 = (\log z)^N$. Then*

$$\psi_1(x, \chi_q, \chi_{\mathfrak{K}}) \ll x^2 (\log x)^{-B}$$

with the constant in the notation depending merely on N, B and Δ.

The proof begins as in the previous lemma except that now (3) is an integral function (see footnote (1)), in the region $\sigma \geqslant 1 - c_4/\log q(3+|t|)$ we have

$$|\zeta'/\zeta(s, \chi_q, \chi_{\mathfrak{K}})| < c_5 \log^2 q(3+|t|) + |s-(1-\delta)|^{-1}$$

(where $\delta \geqslant c(\varepsilon) q^{-\varepsilon}$ for any $\varepsilon > 0$) and we take $\sigma_0 = 1 - c_4/\log q(3+t_0)$, $t_0 = \exp(c_6\sqrt{\log x})$. Moving the path of integration in (13) as before we pass at most one singularity of the integrand. It is the real exceptional zero $\varrho = 1-\delta$ of $\zeta(s, \chi_q, \chi_{\mathfrak{K}})$ (if existing) and for $\varepsilon = (2N)^{-1}$ it gives a residue

$$\ll x^{2-c(\varepsilon)q^{-\varepsilon}} \ll x^2 (\log x)^{-B}.$$

The integral along the contour does not surpass the right-hand side of (18). This completes the proof.

4. Lemma 4. *Let N and A be arbitrarily large constants, $z \to \infty$, $X_0 = (\log z)^N$, $X \leqslant z^{1/2}$, $D \geqslant 2$, $M \geqslant 2$ and let Q_M denote the set of integers q such that*

$$1 < q \leqslant M, \quad d(q) \leqslant D.$$

If $\sum_{\chi_q}^*$ denotes summation over the primitive characters generated by those of the group $\Gamma = \Gamma_q$ (see § 2), then

$$\sum_{q \leqslant X} E^*(z, q) \ll z^2 \left\{ \frac{1}{(\log z)^A} + \frac{(\log z)^3}{D} \right\} + \\ + (\log z)^3 \max_{X_0 \leqslant M \leqslant X} M^{-1} \sum_{Q_M} \sum_{\chi_q}^* \max_{\chi_{\mathfrak{K}}} \max_{y \leqslant z} |\psi_1(y, \chi_q, \chi_{\mathfrak{K}})| \tag{19}$$

with the constant in the notation depending on Δ, A and N.

This is an analogous of [1], Lemma 4.

Proof. Let us write

$$\sum_{q \leqslant X} E^*(z, q) = \sum_{q \in Q_X} + \sum_{q \notin Q_X} = \Sigma_1 + \Sigma_2. \tag{20}$$

Since any prime power is the norm of at most two ideals in K, we have by (15)

$$\psi_1(x; \mathfrak{K}, q, a) \ll (\log x) \sum_{x \geqslant n \equiv a \pmod q} (x-n) \ll (\log x)\left(\frac{x^2}{q} + x + q\right)$$

whence, by (16),

$$E^*(z, q) \ll (z^2/q) \log z \quad \text{if} \quad q \leqslant X \leqslant z^{1/2}.$$

Hence, by Lemma 2,

$$\Sigma_2 \ll E^*(z, 1) + \sum_{\substack{q \leqslant X \\ d(q) > D}} \frac{z^2 \log z}{q} \tag{21}$$

$$\ll \frac{z^2}{(\log z)^A} + \frac{z^2 \log z}{D} \sum_{\substack{q \leqslant X \\ d(q) > D}} \frac{d(q)}{q} \ll \frac{z^2}{(\log z)^A} + \frac{z^2 (\log z)^3}{D}.$$

By (15) and (13)

$$\psi_1(z; \mathfrak{K}_i, q, a) = \frac{1}{h\varphi_1(q)} \sum_{\chi_{\mathfrak{K}}} \sum_{\chi_q \in \Gamma_q} \bar{\chi}_{\mathfrak{K}}(\mathfrak{K}_i) \bar{\chi}_q(a) \psi_1(z, \chi_q, \chi_{\mathfrak{K}}).$$

Hence, by (16),

$$h\varphi_1(q) E(z, q) \leqslant |\psi_1(z, \chi_0) - \tfrac{1}{2}z^2| + \sum_{\chi \neq \chi_0} |\psi_1(z, \chi)|. \tag{22}$$

Summing (18) over all classes $\mathfrak{K}$ we get

$$\psi_1(z, \chi_0) - \tfrac{1}{2}z^2 \ll z^2 (\log z)^{-A-1}. \tag{23}$$

In the same manner we can prove that

$$\sum_{\substack{\chi_{\mathfrak{K}} \text{ non princ.} \\ \chi_q = \chi_q^0}} |\psi_1(z, \chi)| \ll \frac{z^2}{(\log z)^{A+1}}. \tag{24}$$

For any χ_q let χ_q^* denote the primitive character generated by χ_q (cf. [16], IV, § 6). Then for any $\chi_q \neq \chi_q^0$ we have by (13)

$$\psi_1(z,\chi) = \sum_{\substack{n\leqslant z\\ n=N\mathfrak{p}^k,k=1,2,\ldots}} \chi_q(n)\chi_{\mathfrak{K}}(\mathfrak{p}^k)(z-n)\Lambda'(n)$$

$$= \psi_1(z;\chi_q^*,\chi_{\mathfrak{K}}) - \sum_{\substack{n\leqslant z\\ (n,q)>1\\ n=N\mathfrak{p}^k,k=1,2,\ldots}} \chi_q^*(n)\chi_{\mathfrak{K}}(\mathfrak{p}^k)(z-n)\Lambda'(n)$$

$$= \psi_1(z;\chi_q^*,\chi_{\mathfrak{K}}) + O\Big(z\sum_{\substack{p^k\leqslant z\\ p\mid q}}\log p\Big) = \psi_1(z;\chi_q^*,\chi_{\mathfrak{K}}) + O(z\log z\log q).$$

Hence, by (22), (23), (24) and (16)

$$h\varphi_1(q)E^*(z,q) \ll \frac{z^2}{(\log z)^{A+1}} + \varphi_1(q)z(\log z)^2 + \sum_{\chi_q\neq\chi_q^0}\max_{\chi_{\mathfrak{K}}}\max_{y\leqslant z}|\psi_1(y,\chi_q^*,\chi_{\mathfrak{K}})|.$$

Hence by (20), since $\varphi_1(q) \geqslant \varphi(q)$ (see (1)),

$$\Sigma_1 \ll \frac{z^2}{(\log z)^{A+1}}\sum_{q\leqslant X}\frac{1}{\varphi_1(q)} + Xz(\log z)^2 + \tag{25}$$

$$+\sum_{q\in Q_X}\frac{1}{\varphi_1(q)}\sum_{\chi_q\neq\chi_q^0}\max_{\chi_{\mathfrak{K}}}\max_{y\leqslant z}|\psi_1(y,\chi_q^*,\chi_{\mathfrak{K}})|$$

$$\ll \frac{z^2}{(\log z)^{A}} + \sum_{q\in Q_X}\frac{1}{\varphi_1(q)}\sum_{\chi_q\neq\chi_q^0}\max_{\chi_{\mathfrak{K}}}\max_{y\leqslant z}|\psi_1(y,\chi_q^*,\chi_{\mathfrak{K}})|.$$

Let the conductor of χ_q be q^* $(q^*\mid q)$. Then

$$\sum_{q\in Q_X}\frac{1}{\varphi_1(q)}\sum_{\chi_q\neq\chi_q^0}\max_{\chi_{\mathfrak{K}}}\max_{y\leqslant z}|\psi_1(y,\chi_q^*,\chi_{\mathfrak{K}})|$$

$$\ll \sum_{q^*\in Q_X}\sum_{\chi_{q^*}}{}^{*}\max_{\chi_{\mathfrak{K}}}\max_{y\leqslant z}|\psi_1(y,\chi_q,\chi_{\mathfrak{K}})|\sum_{\substack{q\in Q_X\\ q=q^*r}}{}^{\prime}\frac{1}{\varphi(q)}.$$

Since $\varphi(q^*r) \geqslant \varphi(q^*)\varphi(r) \gg q^*\varphi(r)/\log X$ (see [16], I, Satz 5.1), this does not exceed

$$(\log z)^2\sum_{q^*\in Q_X}(q^*)^{-1}\sum_{\chi_{q^*}}{}^{*}\max_{\chi_{\mathfrak{K}}}\max_{y\leqslant z}|\psi_1(y,\chi_{q^*},\chi_{\mathfrak{K}})|. \tag{26}$$

Let $\chi_q\neq\chi_q^0$, $q\leqslant X_0 = (\log z)^N$. Then by Lemma 3 (with $B = N+A+3$)

$$\sum_{q\in Q_{X^0}} q^{-1}\sum_{\chi_q}{}^{*}\max_{\chi_{\mathfrak{K}}}\max_{y\leqslant z}|\psi_1(y,\chi_q,\chi_{\mathfrak{K}})| \ll \frac{z^2}{(\log z)^{A+2}}.$$

Splitting the remaining sum (over $q\in[X_0,X]$) into $\ll\log X$ parts $(U,2U)$ we deduce that

$$\sum_{\substack{q\in Q_X\\ q\geqslant X_0}} q^{-1}\sum_{\chi_q}{}^{*}\max_{\chi_{\mathfrak{K}}}\max_{y\leqslant z}|\psi_1(y,\chi_q,\chi_{\mathfrak{K}})|$$

$$\ll (\log z)\max_{X_0\leqslant M\leqslant X} M^{-1}\sum_{Q_M}\sum_{\chi_q}{}^{*}\max_{\chi_{\mathfrak{K}}}\max_{y\leqslant z}|\psi_1(y,\chi_q,\chi_{\mathfrak{K}})|.$$

Hence, by (25), (26)

$$\Sigma_1 \ll \frac{z^2}{(\log z)^{A}} + (\log z)^3\max_{X_0\leqslant M\leqslant X} M^{-1}\sum_{Q_M}\sum_{\chi_q}{}^{*}\max_{\chi_{\mathfrak{K}}}\max_{y\leqslant z}|\psi_1(y,\chi_q,\chi_{\mathfrak{K}})|,$$

whence (19) follows by (20) and (21).

5. Proof of the estimate (28).

LEMMA 5. *Let $\varrho = \beta+i\gamma$ run through the zeros of $\zeta(s,\chi_q,\chi_{\mathfrak{K}})$ $(\chi_q\neq\chi_q^0)$*[3] *in $c_0<\sigma<1$ (where c_0 stands for any positive constant $\leqslant 1/8$) and let $\psi_1(z,\chi_q,\chi_{\mathfrak{K}})$ be the function (13). Then for $q\leqslant z^{1/2}$ and $T\in[2,z^{1/2}]$*

$$\psi_1(z,\chi_q,\chi_{\mathfrak{K}}) \ll \sum_{\substack{\beta\geqslant 1/2\\ |\gamma|\leqslant T}}\frac{z^{1+\beta}}{|\varrho(\varrho+1)|} + \frac{z^2(\log z)^2}{T} \tag{27}$$

with the constant in the notation depending merely on A.

Proof. Using the properties (8) and (9) of the function $\zeta(s,\chi) = \zeta(s,\chi_q,\chi_{\mathfrak{K}})$ we can prove (cf. [17], Anhang III, Lemma IV) that for any $T>2$ there is a line l_1 $(t = t_0\in[T-1,T],\ c_0\leqslant\sigma\leqslant 1+c_0)$ such that the distance between l_1 and any zero ϱ of $\zeta(s,\chi)$ is $\gg 1/\log qT$ and thus for all $s\in l_1$ we have $\zeta'/\zeta(s,\chi)\ll\log^2 qT$. And there is a line l_3 $(t = -t_0'\in[-T,-T+1],\ c_0\leqslant\sigma\leqslant 1+c_0)$ with the same property. This property possesses also some broken line $l_2\in(c_0\leqslant\sigma\leqslant 2c_0,\ |t|\leqslant T)$ satisfying the following condition: For any integer $n\in[-T,T]$ the part of l_2 with $t\in[n,n+1]$ is of the length <2.

We replace the part $-t_0'\leqslant t\leqslant t_0$ of the path of integration in (13) (where we use $c = 1+1/\log z$) by a contour along l_1, l_2, l_3. The contour integral is evidently

$$\ll\frac{z^2\log^2 z}{T} + z^{1+1/4}\log^2 z$$

and the sum of residues at the poles ϱ being passed over does not exceed

$$\ll\sum_{|\gamma|\leqslant T}\frac{z^{1+\beta}}{|\varrho(\varrho+1)|} = \sum_{\substack{|\gamma|\leqslant T\\ \beta\geqslant 1/2}}\frac{z^{1+\beta}}{|\varrho(\varrho+1)|} + O(z^{3/2}\log^2 z) = \sum_{\substack{|\gamma|\leqslant T\\ \beta\geqslant 1/2}} + O\Big(\frac{z^2\log^2 z}{T}\Big).$$

Hence (27) follows.

LEMMA 6. *Let $E^*(z, q)$ be the function* (16). *Then for any constant $A > 0$ there is a corresponding constant $B = 5A+213$ such that*

$$\sum_{q\leqslant X} E^*(z, q) \ll z^2(\log z)^{-A} \quad \text{if} \quad X \leqslant z^{1/2}(\log z)^{-B}. \tag{28}$$

(The constant in the notation depends on A and Δ.)

Proof. Splitting the sum of (27) into parts $|\gamma| \leqslant 1$ and $2^{m-1} \leqslant |\gamma| < 2^m$ $(1 \leqslant m \ll \log z)$ we get

$$\sum_{\substack{|\gamma|\leqslant T\\ \beta\geqslant 1/2}} \frac{z^{1+\beta}}{|\varrho(\varrho+1)|} \ll \sum_{2^{m-1}\leqslant T} 2^{-2m} \sum_{|\gamma|\leqslant 2^m} z^{1+\beta}. \tag{29}$$

By partial summation (cf. [14], p. 18, Theorem A)

$$\sum_{\substack{|\gamma|\leqslant 2^m\\ \beta\geqslant 1/2}} z^{1+\beta} = z \sum_{|\gamma|\leqslant 2^m} \Big\{ z^{1/2} N(\tfrac{1}{2}, 2^m, \chi_q, \chi_{\mathfrak{K}}) + (\log z) \int_{1/2}^{1} N(\alpha, 2^m, \chi_q, \chi_{\mathfrak{K}}) z^\alpha d\alpha \Big\}.$$

Hence, by (27) and (29),

$$\max_{y\leqslant z} |\psi_1(y, \chi_q, \chi_{\mathfrak{K}})| \ll \frac{z^2(\log z)^2}{T} + $$
$$+ (\log z) \sum_{2^{m-1}\leqslant T} 2^{-2m} \Big\{ z^{3/2} N(\tfrac{1}{2}, 2^m, \chi_q, \chi_{\mathfrak{K}}) + \int_{1/2}^{1} N(\alpha, 2^m, \chi_q, \chi_{\mathfrak{K}}) z^{1+\alpha} d\alpha \Big\}$$

and thus (cf. Lemma 4)

$$M^{-1} \sum_{Q_M} \sideset{}{^*}\sum_{\chi_q} \max_{\chi_{\mathfrak{K}}} \max_{y\leqslant z} |\psi_1(y, \chi_q, \chi_{\mathfrak{K}})| \tag{30}$$

$$\ll M\frac{z^2(\log z)^2}{T} + \frac{\log z}{M} \sum_{2^{m-1}\leqslant T} 2^{-2m} \Big\{ z^{3/2} \sum_{Q_M} \sideset{}{^*}\sum_{\chi_q} \max_{\chi_{\mathfrak{K}}} N(\tfrac{1}{2}, 2^m, \chi_q, \chi_{\mathfrak{K}}) +$$
$$+ \int_{1/2}^{1} \sum_{Q_M} \sideset{}{^*}\sum_{\chi_q} \max_{\chi_{\mathfrak{K}}} N(\alpha, 2^m, \chi_q, \chi_{\mathfrak{K}}) z^{1+\alpha} d\alpha \Big\}$$

$$\ll M\frac{z^2(\log z)^2}{T} + \frac{\log z}{M} \sum_{2^{m-1}\leqslant T} 2^{-2m} \max_{\alpha} \Big\{ \sum_{Q_M} \sideset{}{^*}\sum_{\chi_q} \max_{\chi_{\mathfrak{K}}} N(\alpha, 2^m, \chi_q, \chi_{\mathfrak{K}}) z^{1+\alpha} \Big\}$$

$$\ll M\frac{z^2(\log z)^2}{T} + \frac{(\log z)^2}{M} \max_{2<T'\leqslant T} (T')^{-2} \max_{\alpha} \Big\{ \sum_{Q_M} \sideset{}{^*}\sum_{\chi_q} \max_{\chi_{\mathfrak{K}}} N(\alpha, T', \chi_q, \chi_{\mathfrak{K}}) z^{1+\alpha} \Big\}.$$

Since $q^{-1}|\tau(\chi_q)|^2 = 1$ for a primitive χ_q and $\geqslant 0$ otherwise (cf. (12)), by (11)

$$\sum_{Q_M} \sideset{}{^*}\sum_{\chi_q} N(\alpha, T', \chi_q, \chi_{\mathfrak{K}}) \ll D^4 T' (M^2 T')^{\frac{4(1-\alpha)}{3-2\alpha}} \log^{196} MT.$$

Hence, by (30) and (19)

$$\sum_{q\leqslant X} E^*(z, q) \ll z^2 \Big(\frac{1}{(\log z)^A} + \frac{(\log z)^3}{D} \Big) +$$
$$+ (\log z)^3 \max_{X_0\leqslant M\leqslant X} \Big\{ M \frac{z^2\log^2 z}{T} + M^{-1}(\log z)^2 z D^4 (\log z)^{196} \max_{\alpha} z^\alpha M^{\frac{8(1-\alpha)}{3-2\alpha}} \Big\}.$$

The restrictions imposed on M, X and T are $M \leqslant X \leqslant z^{1/2}$, $T \leqslant z^{1/2}$ (cf. Lemmas 4 and 5). Further on we use $D = (\log z)^{A+3}$, $T = M(\log z)^{A+5}$, $X \leqslant z^{1/2}(\log z)^{-A-5}$ and get

$$\sum_{q\leqslant X} E^*(z, q) \ll \frac{z^2}{(\log z)^A} + z(\log z)^{4A+213} \max_{1/2\leqslant\alpha\leqslant 1} z^\alpha M^{\frac{8(1-\alpha)}{3-2\alpha}-1} \tag{31}$$
$$\ll \frac{z^2}{(\log z)^A} + z(\log z)^{4A+213} \max_{1/2\leqslant\alpha\leqslant 1} z^\alpha M^{3/2-2\alpha^2}$$

(cf. [1], p. 214). Taking $X = z^{1/2}(\log z)^{-B}$, $M \geqslant X_0 = (\log z)^N$ we have (cf. [1], p. 215)

$$z^\alpha M^{3/2-2\alpha^2} \ll z(\log z)^{-N/2} + z(\log z)^{-B} \quad \text{for all } \alpha \in [1/2, 1].$$

Now (28) follows from (31) if we take $N/2 = B = 5A+213$.

6. Proof of the estimate (37).

LEMMA 7. *Let C be any constant > 4, $A = 4C$, $B = 5A+213$, $z \geqslant z_0(C)$ and let*

$$y/2 < q \leqslant y \quad \text{where} \quad 2 < y < z^{1/2}(\log z)^{-B}.$$

Then for almost all $q \in (y/2, y]$ (with exception of $\ll y(\log z)^{-C}$ integers of this interval)

$$\psi_1(x; \mathfrak{K}, q, a) - \frac{x^2/2}{h\varphi_1(q)} \ll \frac{x^2}{\varphi_1(q)(\log x)^C} \quad \text{if} \quad x \in [z(\log z)^{-C}, z] \tag{32}$$

and a is a normresidue mod q *with $(a, q) = 1$ for the class $\mathfrak{K}$. The constant in the notation depends on C and Δ.*

Proof. Having fixed z and y let us call $q \in (y/2, y]$ a *normal* integer if (cf. (16))

$$E^*(z, q) < \frac{z^2}{\varphi_1(q)(\log z)^{3C}}. \tag{33}$$

Let N_y denote the number of the exceptional $q \in (y/2, y]$ not satisfying (33). By (28)

$$N_y \frac{z^2}{\varphi_1(q)(\log z)^{3C}} \ll \frac{z^2}{(\log z)^A},$$

whence (since $\varphi_1(q) < y$) $N_y < y(\log z)^{-C}$. For any of the normal $q \in (y/2, y]$ we have by (16)

$$\left|\psi_1(x; \mathfrak{K}, q, a) - \frac{x^2/2}{h\varphi_1(q)}\right| < \frac{z^2}{\varphi_1(q)(\log z)^{3C}} \ll \frac{x^2}{\varphi_1(q)(\log z)^C} \ll \frac{x^2}{\varphi_1(q)(\log x)^C}$$

(since $z \leqslant x\log^C z$), the desired result.

LEMMA 8. *Suppose*

$$c_n \geqslant 0, \quad f(x) = \sum_{n \leqslant x} c_n, \quad f_1(x) = \int_0^x f(u)\,du,$$

and

$$f_1(x) = ax^2 + O\left(\frac{ax^2}{(\log x)^A}\right) \quad \text{if} \quad x \in [x_1, x_2],\ x_2 > 4x_1 > 4$$

(A *stands for a positive constant*). *Then*

$$f(x) = 2ax + O\left(\frac{ax}{(\log x)^{A/2}}\right) \tag{34}$$

if

$$x \in [x_3, x_4] = [x_1 + x_1\log^{-A/2} x_1,\ x_2 - x_2\log^{-A/2} x_2].$$

Proof (cf. [14], p. 35, Theorem C). Since $f(x)$ does not decrease, for any $x_0 \in (x_1, x_4)$ and any $x \in (x_0, x_2)$ we have

$$f(x_0) \leqslant \frac{1}{x - x_0}\int_{x_0}^{x} f(u)\,du = \frac{f_1(x) - f_1(x_0)}{x - x_0}$$
$$= \frac{ax^2 - ax_0^2 + O(ax^2\log^{-A} x)}{x - x_0} = a(x + x_0) + O\left(\frac{ax^2}{(x - x_0)\log^A x}\right),$$

whence (taking $x = x_0 + x_0\log^{-A/2} x_0$)

$$f(x_0) \leqslant 2ax_0 + O(ax_0\log^{-A/2} x_0). \tag{35}$$

By a similar argument

$$f(x_0) \geqslant \frac{\log^{A/2} x_0}{x_0}\int_{x_0 - x_0/\log^{A/2} x_0}^{x_0} f(u)\,du = 2ax_0 + O(ax_0\log^{-A/2} x_0)$$

$$\text{if} \quad x_0 \in (x_3, x_2).$$

Hence by (35) the lemma follows.

LEMMA 9. *Let* $\psi(x; \mathfrak{K}, q, a)$ *be the function* (14) *and suppose that* (32) *holds. Then*

$$\psi(x; \mathfrak{K}, q, a) - \frac{x}{h\varphi_1(q)} \ll \frac{x}{\varphi_1(q)(\log z)^{C/2}} \quad \text{if} \quad x \in [z(\log z)^{-C}, z]. \tag{36}$$

Proof. (36) follows from (32) and (34) in the first instance for the interval $[x_3, x_4]$ (where $x_3 = z(\log z)^{-C} + z(\log z)^{-3C/2}$, $x_4 = z - z(\log z)^{-C/2}$) and thus ultimately for $[z(\log z)^{-C}, z]$, since changing x by $\ll x(\log x)^{-C/2}$ the variation of the left-hand side of (36) does not exceed the term on its right-hand side. (In the proof consider that $\psi(x+y; \mathfrak{K}, q, a) - \psi(x; \mathfrak{K}, q, a) \leqslant 2\sum \log p$ where the sum is over all prime powers $p^k \equiv a \pmod q$ of the interval $(x, x+y)$ and $y = x(\log x)^{-C/2}$. By [8], Lemma 1, the sum satisfies $\ll y/\varphi(q)$.)

LEMMA 10. *Let a be a normresidue* mod q *for the class* $\mathfrak{K}$, $(a, q) = 1$, *and let* $\pi(x; \mathfrak{K}, q, a)$ *denote the number of primes* $p \leqslant x$ *such that* $p \equiv a$ (mod q) *and* $p = N\mathfrak{p}$ *with* $\mathfrak{p} \in \mathfrak{K}$. *If* (32) *holds, then*

$$\pi(x; \mathfrak{K}, q, a) - \frac{1}{h\varphi_1(q)}\operatorname{Li} x \ll \frac{x}{\varphi_1(q)(\log x)^{C/2}} \quad \left(\frac{z}{(\log x)^{C/2}} \leqslant x \leqslant z\right) \tag{37}$$

with the constant in the notation depending on C and Δ.

Proof. Let us write

$$\vartheta(x; \mathfrak{K}, q, a) = \sum_{\substack{p \leqslant x \\ p \equiv a \,(\mathrm{mod}\, q) \\ p = N\mathfrak{p},\, \mathfrak{p} \in \mathfrak{K}}} \log p = \psi(x; \mathfrak{K}, q, a) - O\Big(\sum_{\substack{p^k \leqslant x,\, k = 2,3,\ldots \\ p^k \equiv a \,(\mathrm{mod}\, q)}} \log p\Big) \tag{38}$$

where p denotes primes and $\mathfrak{p}$ prime ideals (cf. § 3). Considering that the number of the solutions $u \in (0, q)$ of the congruence $u^2 \equiv a \pmod q$ does not exceed $2.2^{\nu(q)} \ll q^{\varepsilon}$ (where $\nu(q)$ denote the number of all different prime divisors of q and ε stands for any positive constant $< 1/4$) we deduce that the number of solutions $u \in (0, x^{1/2}]$ of the same congruence satisfies

$$\ll \left(1 + \frac{x^{1/2}}{q}\right)q^{\varepsilon} \ll \frac{x}{q(\log x)^{C+2}}$$

(since $q \leqslant z^{1/2}(\log z)^{-B}$, $x \geqslant z/\log^{C/2} z$). The number of solutions $u \in (0, x^{1/k}]$ of the congruence $u^k \equiv a \pmod q$ for $k \in [3, \log x/\log 2]$ satisfies the same estimate. Hence the remainder term of (38) does not surpass the right-hand side of (36), whence

$$\vartheta(x; \mathfrak{K}, q, a) = \frac{x}{h\varphi_1(q)} + O\left(\frac{x}{\varphi_1(q)\log^{C/2} x}\right) \quad \left(\frac{z}{(\log z)^C} \leqslant x \leqslant z\right). \tag{39}$$

Let us write

$$x_1 = z(\log z)^{-C}, \quad \vartheta(x; \mathfrak{K}, q, a) = S(x) + r(x)$$

where

$$S(x) = \frac{x}{h\varphi_1(q)},\qquad r(x) \ll \frac{x}{\varphi_1(q)(\log x)^{C/2}}.$$

Then

$$(40)\qquad \pi(x;\mathfrak{R},q,a) = \sum_{x_1\leqslant n\leqslant x}\frac{S(n)-S(n-1)}{\log n} + $$
$$+\sum_{x_1\leqslant n\leqslant x}\frac{r(n)-r(n-1)}{\log n} + \pi(x_1-1/2;\mathfrak{R},q,a).$$

The absolute value of the last sum does not exceed

$$\sum_{x_1\leqslant n\leqslant x-1}|r(n)|\left(\frac{1}{\log n}-\frac{1}{\log(n+1)}\right)+$$
$$+\left|\frac{r([x])}{\log[x]}\right|+\left|\frac{r([x_1])}{\log[x_1]}\right| \ll \frac{x}{\varphi_1(q)(\log x)^{C/2}}.$$

The last term in (40) does not exceed the number of primes $p\leqslant x_1$, $p\equiv a \pmod q$ which number is $\ll x_1/\varphi(q)\log x$, by the theorem of Brun–Titchmarsh (cf. [16], II, Satz 4.9). Hence, by (40) and (39),

$$\pi(x;\mathfrak{R},q,a) = \frac{1}{h\varphi_1(q)}\sum_{x_1\leqslant n\leqslant x}\frac{1}{\log n} + O\left(\frac{1}{\varphi_1(q)\log^{C/2}x}\right)$$
$$= \frac{1}{h\varphi_1(q)}\mathrm{Li}\,x + O\left(\frac{x}{\varphi_1(q)\log^{C/2}x}\right),$$

the desired result.

7. Proof of the theorem. Let q run through the normal integers $\epsilon(y/2,y]$ (cf. § 6). Then by (37) for any $\mathfrak{R}$ and any normresidue $a=a(\mathfrak{R})$ mod q with $(a,q)=1$ we have

$$\pi(x;\mathfrak{R},q,a)-\frac{1}{h\varphi_1(q)}\mathrm{Li}\,x \ll \frac{z}{\varphi_1(q)\log^{C/2}z}\qquad \left(\frac{z}{(\log z)^{C/2}}\leqslant x\leqslant z\right)$$

whence

$$(41)\qquad \sum_{\substack{y/2<q\leqslant y\\ q\,\mathrm{norm.}}}\max_{a(\mathfrak{R},q)}\left|\pi(x;\mathfrak{R},q,a)-\frac{1}{h\varphi_1(q)}\mathrm{Li}\,x\right| \ll \frac{z\log\log z}{(\log z)^{C/2}},$$

since $\varphi_1(q)\gg q/\log\log q$. The sum over the exceptional numbers $q\epsilon(y/2,y]$ not exceeding

$$\frac{y}{(\log z)^C}\cdot\frac{z}{\varphi_1(q)\log z} \ll \frac{z\log\log z}{(\log z)^{C+1}},$$

the restriction 'q norm.' in (41) may be dropped. And we may drop also the restriction $x\geqslant x_0 = z/\log^{C/2}z$, since by the theorem of Brun–Titchmarsh for $x\leqslant x_0$ the left-hand side of (41) does not exceed

$$\ll y\,\frac{x_0}{\varphi_1(q)\log x} \ll \frac{z\log\log z}{(\log z)^{C/2}\log z}.$$

Summing over all the intervals $(y/2,y]$, $(y/4,y/2]$, ... ($\ll\log z$ in number) we obtain the inequality

$$\sum_{q\leqslant z^{1/2}(\log z)^{-B}}\max_{a(\mathfrak{R},q)}\max_{x\leqslant z}\left|\pi(x;\mathfrak{R},q,a)-\frac{1}{h\varphi_1(q)}\mathrm{Li}\,x\right| \ll \frac{z}{(\log z)^{C/2-2}}.$$

This implies (2).

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