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Some numerical results in the Selberg sieve method

by

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Ankeny and Onishi [1] have shown the importance of the solutions of certain differential-difference equations and parameters defined in terms of them in the Selberg lower-bound sieve method. A full account of their work will appear in Halberstam and Richert [2] and in this note we shall follow the notation of the latter.

For each $\varkappa > 0$, let $\sigma_\varkappa(u)$ denote the (continuous) solution of the differential-difference equation

$$\big(u^{-\varkappa}\sigma_\varkappa(u)\big)' = -\varkappa u^{-\varkappa-1}\sigma_\varkappa(u-2) \qquad (u \geqslant 2),$$
$$\sigma_\varkappa(u) = \frac{2^{-\varkappa}e^{-\gamma\varkappa}}{\Gamma(\varkappa+1)}u^\varkappa \qquad (0 \leqslant u < 2)$$

where γ is Euler's constant.

Further, let $\nu_\varkappa$ denote the (unique and positive) solution of the equation

$$\eta_\varkappa(x) \equiv \varkappa x^{-\varkappa}\int_x^\infty\left(\frac{1}{\sigma_\varkappa(t-1)}-1\right)t^{\varkappa-1}dt = 1. \tag{1}$$

(In the notation of Ankeny and Onishi,

$$\sigma_\varkappa(u) = \frac{e^{-\gamma\varkappa}}{\Gamma(\varkappa)}J_\varkappa(\tfrac{1}{2}u) \quad \text{and} \quad \nu_\varkappa = 2\zeta_\varkappa.)$$

Ankeny and Onishi give tables of $\zeta_\varkappa$ for $\varkappa = 1, 1.5, 2, 2.5, 3, 3.5$, and of $J_2(u)$ at intervals of 0.05 from $u = 1$ to $u = 2.5$. They apply these results to obtain Selberg's result on the prime-twin problem that there exist infinitely many integers n such that $n(n+2)$ is the product of at most 5 prime factors.

In this note we give an extension of the table of values of $\nu_\varkappa$, and correct the values of $J_2(u)$ given by Ankeny and Onishi, which appear to be in error in the range $2 < u \leqslant 2.5$. If this is so, doubt is thereby thrown on the validity of their result for the prime-twin problem. We shall show, however, that it can still be obtained. A few other applications will also be given.

The method used in the calculation of $\sigma_\varkappa$ was the predictor-corrector method well-known to numerical analysts (see, e.g. Hildebrand [3], Chapter 6). For the early parts of the range it is feasible to derive expressions for the $\sigma_\varkappa$-functions in terms of infinite series, and the results obtained by the predictor-corrector method were checked against these series in the case where the discrepancy with the table of Ankeny and Onishi occurred. In fact, for $2 < u < 3$,

$$J_2(u) = u^2\log^2 u - 4u^2\log u + (\tfrac{5}{4}\log 2 + \tfrac{7}{2} - \log^2 2)u^2 + \\ + 4u\log u - 6u - \log u + \tfrac{5}{2} - u^2\sum_{n=1}^{\infty}\frac{4}{(n+2)^2(n+1)n}\left(\frac{1}{2^{n+2}} - \frac{1}{u^{n+2}}\right).$$

For the calculation of $\nu_\varkappa$ it was found to be sufficiently accurate to split the range of the integral in (1) at some number N and to obtain upper and lower bounds for $\eta_\varkappa(x)$ for successively decreasing values of x by using Ankeny and Onishi's estimate for $\eta_\varkappa(N)$ $(= G_\varkappa(N/2)$ in their notation) together with approximation by rectangles above and below for the range from x to N.

Table 1. Values of $\sigma_2(u)$

u	$\sigma_2(u)$	u	$\sigma_2(u)$	u	$\sigma_2(u)$	u	$\sigma_2(u)$
0.1	0.0004	2.6	0.2637	5.1	0.7327	7.6	0.9485
0.2	0.0016	2.7	0.2831	5.2	0.7469	7.7	0.9523
0.3	0.0035	2.8	0.3028	5.3	0.7607	7.8	0.9559
0.4	0.0063	2.9	0.3227	5.4	0.7739	7.9	0.9592
0.5	0.0099	3.0	0.3429	5.5	0.7865	8.0	0.9624
0.6	0.0142	3.1	0.3632	5.6	0.7987	8.1	0.9652
0.7	0.0193	3.2	0.3836	5.7	0.8104	8.2	0.9679
0.8	0.0252	3.3	0.4040	5.8	0.8215	8.3	0.9705
0.9	0.0319	3.4	0.4245	5.9	0.8322	8.4	0.9728
1.0	0.0394	3.5	0.4449	6.0	0.8423	8.5	0.9750
1.1	0.0477	3.6	0.4652	6.1	0.8520	8.6	0.9770
1.2	0.0567	3.7	0.4853	6.2	0.8613	8.7	0.9788
1.3	0.0666	3.8	0.5053	6.3	0.8700	8.8	0.9806
1.4	0.0772	3.9	0.5251	6.4	0.8783	8.9	0.9822
1.5	0.0887	4.0	0.5445	6.5	0.8862	9.0	0.9836
1.6	0.1009	4.1	0.5637	6.6	0.8937	9.1	0.9850
1.7	0.1139	4.2	0.5826	6.7	0.9008	9.2	0.9863
1.8	0.1277	4.3	0.6011	6.8	0.9074	9.3	0.9874
1.9	0.1423	4.4	0.6191	6.9	0.9137	9.4	0.9885
2.0	0.1576	4.5	0.6368	7.0	0.9197	9.5	0.9895
2.1	0.1738	4.6	0.6540	7.1	0.9253	9.6	0.9904
2.2	0.1906	4.7	0.6707	7.2	0.9305	9.7	0.9912
2.3	0.2081	4.8	0.6870	7.3	0.9355	9.8	0.9920
2.4	0.2262	4.9	0.7027	7.4	0.9401	9.9	0.9927
2.5	0.2447	5.0	0.7180	7.5	0.9445	10.0	0.9934

Table 2. Values of $\nu_\varkappa$

$\varkappa$	$\nu_\varkappa$	$\nu_\varkappa/\varkappa$	$\varkappa$	$\nu_\varkappa$	$\nu_\varkappa/\varkappa$
1	2.06...	2.06...	9	21.74...	2.41...
2	4.42...	2.21...	10	24.22...	2.42...
3	6.85...	2.28...	11	26.70...	2.42...
4	9.32...	2.33...	12	29.21...	2.43...
5	11.80...	2.35...	13	31.68...	2.43...
6	14.28...	2.38...	14	34.15...	2.43...
7	16.77...	2.39...	15	36.62...	2.44...
8	19.25...	2.40...	16	39.09...	2.44...

For the applications we quote some theorems from Halberstam and Richert [2], Chapter 10.

THEOREM 1. *Let g be a natural number, and let a_i, b_i $(i = 1, \dots, g)$ be integers satisfying*

$$\prod_{i=1}^{g} a_i \prod_{1\le t<s\le g}(a_t b_s - a_s b_t) \neq 0.$$

Suppose also that $\prod_{i=1}^{g}(a_i n + b_i)$ has no fixed prime divisor. Then for any natural number r satisfying

$$r > (g+1)\log \nu_g + g - 1,$$

there are infinitely many n such that $\prod_{i=1}^{g}(a_i n + b_i)$ is the product of at most r prime factors.

THEOREM 2. *Let g be a natural number, and let a_i, b_i $(i = 1, \dots, g)$ be integers satisfying*

$$\prod_{i=1}^{g} a_i b_i \prod_{1\le t<s\le g}(a_t b_s - a_s b_t) \neq 0.$$

Let $\varrho(p)$ denote the number of solutions of

$$\prod_{i=1}^{g}(a_i n + b_i) \equiv 0 \pmod{p}$$

and suppose that

$$\varrho(p) < p \quad \textit{for all } p,$$

as well as that

$$\varrho(p) < p - 1 \quad \textit{if } p \nmid b_1 \dots b_g.$$

Then, for any natural number r satisfying

$$r > (g+\tfrac{1}{2})\log(2\nu_g) + 2g - 1 - \tfrac{1}{2}\cdot\frac{g}{\nu_g}$$

there are infinitely many primes p such that $\prod_{i=1}^{g}(a_ip+b_i)$ is the product of at most r prime factors.

The values of $\nu_\varkappa$ tabulated above lead to the conclusion that the following are admissible choices for r.

In the case of Theorem 1:

g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
r	2	6	10	15	19	24	29	34	39	45	50	55	61	66	72	78

In the case of Theorem 2:

g	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
r	3	9	14	20	27	33	40	46	53	60	67	74	81	89	96	103

In fact Halberstam and Richert show that it suffices to take r satisfying

$$r > gu - 1 + \frac{g\int\limits_u^v \frac{1}{\sigma_g\big((\alpha-1/t)v\big)}\left(1-\frac{u}{t}\right)\frac{dt}{t}}{1-\eta_g(av)} \tag{2}$$

where $\alpha = 1$ in the case of Theorem 1 and $\alpha = 1/2$ in the case of Theorem 2 and u, v are two real numbers satisfying the inequalities

$$1/\alpha < u < v, \qquad \nu_g/\alpha < v.$$

If we choose u, v suitably it is (at least in some cases) possible to obtain values for r better than those tabulated above by finding an upper bound for the quantity on the right of (2) by numerical integration. In particular, if we take $g = 2$, $\alpha = 1$, $u = 1.5$, and $v = 9$ we find we may take

$$r > 4.24,$$

i. e. $r = 5$ is an admissible choice. This yields again the result mentioned above, viz. that there are infinitely many integers n for which $n(n+2)$ is a product of at most 5 prime factors.

Further, if we take $g = 2$, $\alpha = 1/2$, $u = 2.2$, and $v = 22$ we find we may take

$$r > 6.70,$$

i. e. $r = 7$ is an admissible choice — which represents an improvement of 2 on the result tabulated above. This and two other results obtainable by the same method are summarized in the following theorem:

THEOREM. (i) *There are infinitely many primes p such that $(p+2)(p+6)$ is the product of at most 7 prime factors.*

(ii) *There are infinitely many n such that $(8n+1)(n^2+n+1)$ is the product of at most 6 prime factors.*

(iii) *There are infinitely many primes p such that $(p+2)(p^2+p+1)$ is the product of at most 9 prime factors.*

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