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Euclid's algorithm in complex quartic fields

by

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1. Introduction. An algebraic number field K is called *euclidean* if for any integers $\alpha, \beta \neq 0$ of K there is an integer γ of K such that (N denotes norm)

(1) $$|N(\alpha-\beta\gamma)| < |N(\beta)|.$$

We also say that K has a Euclidean Algorithm (E. A.). A euclidean field has class number 1 (unique factorization). It is well-known (see, e.g., [1], [3]) that exactly 5 complex quadratic fields and 16 real quadratic fields have an E. A.

In this paper we consider the class $\mathscr{K}$ of complex quartic fields K which contain a complex quadratic subfield. We show that 51 fields of this type (30 not counting conjugate fields) have an E. A.

THEOREM. *Let K be a quadratic extension of F with relative discriminant δ ($\delta \in F$). In all of the following* 51 *cases K has a Euclidean Algorithm*:

(i) $F = Q(\sqrt{-1})$ *and* $N\delta \leqslant 52$ (14 *cases*).

(ii) $F = Q(\sqrt{-3})$ *and* $N\delta \leqslant 133$ (36 *cases*).

(iii) $F = Q(\sqrt{-7})$ *and* $N\delta \leqslant 16$ (4 *cases*).

(*The fields* $Q(\sqrt{-1}, \sqrt{-3})$, $Q(\sqrt{-1}, \sqrt{-7})$, *and* $Q(\sqrt{-3}, \sqrt{-7})$ *are each counted twice.*)

The proof generalizes that of Perron [10] for real quadratic fields. In fact for (i), (ii) the key Lemmas are special cases of results obtained by Perron [8], [9] for quite a different application. (See also [7].) For these special cases we give new proofs which are simpler than, and were obtained before learning of, the proofs in [8], [9]. Furthermore, the present paper appears to be the first to treat the E. A. in fields of this type (cf. [6], pp. 174–176).

The problem of determining all quartic fields K in the class $\mathscr{K}$ which have an E. A. is in quite a different state than the problem was for quad-

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ratic fields from 1927 to 1948. In the quadratic case general methods were developed to handle infinite classes of fields before it was shown in [5] that no real quadratic field with discriminant greater than 16384 can have an E. A. In the quartic case the problem was already in 1952 reduced to a finite number of cases [2], [4]. In fact, the bound in [2], p. 85, shows that for a totally complex quartic field K to have an E. A. it is necessary that its discriminant D_K satisfy $D_K < 24846000$. We shall deal with the problem of showing nonexistence of an E. A. (in such fields) in a subsequent paper.

2. Table of Euclidean fields. The following fields $K = F(\sqrt{\mu})$ are euclidean. Two values of μ in the same column (e. g. $1+4i$, $1-4i$) are conjugate over F and generate fields K, K' which are conjugate over Q.

(i) $F = Q(\sqrt{-1})$, $K = F(\sqrt{\mu})$.

μ	-3	i	$1+4i$ $1-4i$	$1+2i$ $1-2i$	5	$1+i$ $1-i$	$5+4i$ $5-4i$	-7	$3+2i$ $3-2i$
$N\delta$	9	16	17	20	25	32	41	49	52

(ii) $F = Q(\sqrt{-3})$, $K = F(\sqrt{\mu})$, $\varrho = \frac{1}{2}(-1+\sqrt{-3})$.

μ	$1+4\varrho$ $-3-4\varrho$	-1	$1-4\varrho$ $5+4\varrho$	5	$-3+4\varrho$ $-7-4\varrho$	$1+2\varrho$ $-1-2\varrho$	-7	$1+8\varrho$ $-7-8\varrho$
$N\delta$	13	16	21	25	37	48	49	57

μ	$5-4\varrho$ $9+4\varrho$	2	-2	$1-8\varrho$ $9+8\varrho$	$-7+4\varrho$ $-11-4\varrho$	$-3+8\varrho$ $-11-8\varrho$	$5+12\varrho$ $-7-12\varrho$
$N\delta$	61	64	64	73	93	97	109

μ	$1-2\varrho$ $3+2\varrho$	$-1+2\varrho$ $-3-2\varrho$	-11	$5-8\varrho$ $13+8\varrho$	$1+12\varrho$ $-11-12\varrho$	$9-4\varrho$ $13+4\varrho$
$N\delta$	112	112	121	129	133	133

(iii) $F = Q(\sqrt{-7})$, $K = F(\sqrt{\mu})$, $\omega = \frac{1}{2}(1+\sqrt{-7})$.

μ	$-\omega$ $\omega-1$	-3	-1
$N\delta$	8	9	16

3. Outline of proof. Let F be a complex quadratic field and let $K = F(\sqrt{\mu})$, $\mu \in F$. Then, if $x \in K$ and $N_{K/F}$ denotes the relative norm,

$$N(x) = N_{F/Q} N_{K/F}(x) = |N_{K/F}(x)|^2.$$

Therefore in condition (1) we may replace the absolute norm $N = N_{K/Q}$ by $N_{K/F}$. Since moreover the norm is multiplicative, K is euclidean if and only if for any $\xi = a/\beta \in K$ there is an integer γ of K such that

$$|N_{K/F}(\xi-\gamma)| < 1. \tag{2}$$

But if $a = a+b\sqrt{\mu} \in K$ $(a, b \in F)$, then $N_{K/F}(a) = a^2-\mu b^2$. Therefore the problem is formally the same as for quadratic fields, as in [10].

Now let $F = F_1, F_3$, or F_7, where F_m denotes $Q(\sqrt{-m})$. So $F_1 = Q(i)$, $F_3 = Q(\varrho)$, $F_7 = Q(\omega)$, where ϱ and ω are as in Section 2. Let O_K and O_F denote the ring of integers of K and of F, respectively. By unique factorization in O_F we may assume that $K = F(\sqrt{\mu})$ with μ in O_F and square free. There is a basis $\{1, \Omega\}$ for O_K as an O_F-module, where Ω depends on the residue class of μ modulo 4. The various cases can be treated at the same time by using the notation

$$\Omega = \varepsilon'+\sqrt{\mu}/\varepsilon \tag{3}$$

with appropriate $\varepsilon, \varepsilon'$ in F. In particular $\varepsilon|2$ in O_F and $\mu/\varepsilon^2 = \delta/4$, where δ is the relative discriminant of K/F. If now $\xi = a+b\sqrt{\mu} \in K$ and $\gamma = r+ +s\Omega \in O_K$ $(a, b \in F;\ r, s \in O_F)$ then

$$N_{K/F}(\xi-\gamma) = (a-\varepsilon' s-r)^2-\delta(\varepsilon b-s)^2/4. \tag{4}$$

Thus by (2) and (4) K is euclidean if and only if, given any $a, b \in F$ there exist $r, s \in O_F$ such that

$$|(a-\varepsilon' s-r)^2-\delta(\varepsilon b-s)^2/4| < 1. \tag{5}$$

The basic tool is contained in the following Lemma.

LEMMA. *Let $c(F_1) = 3/4$, $c(F_3) = \sqrt{13}/4$, $c(F_7) = 1/2$, and let $F = F_1$, F_3, or F_7. If d is any complex number with $|d| < c(F)$ then d has the following property: given any complex z_1 there is a "homologous" number z (i. e. $z-z_1 \in O_F$) such that $|z^2-d| < 1$. Moreover the constants $c(F_1)$ and $c(F_3)$ are best possible.* (The geometric proof is sketched in Section 4.)

Proof of Theorem. Following [10] we set

$$d = \delta(\varepsilon b-s)^2/4, \tag{6}$$

$$z_1 = a-\varepsilon' s, \quad z = a-\varepsilon' s-r. \tag{7}$$

Thus the inequality $|z^2-d|<1$ in the Lemma is exactly inequality (5).

Now let $a, b \in F$ be given. We see first, from consideration of the lattice O_F in the complex plane, that it is possible to choose $s \in O_F$ such that

(8) $$|\varepsilon b - s|^2 \leqslant 1/2;\ 1/3;\ 4/7.$$

(We give successively the results for F_1, F_3, F_7.) It follows from (6) and (8) that $|d| < c(F)$ so long as

$$|\delta| < 6;\ 3\sqrt{13};\ 7/2,$$

or

(9) $$N\delta < 36;\ 117;\ 49/4.$$

If (9) is satisfied then by the Lemma we can choose $r \in O_F$ (thus defining z by (7)) such that $|z^2-d|<1$ – that is, such that inequality (5) holds. Consequently for δ as in (9), K is euclidean.

This accounts for 39 of the cases listed in Section 2. The geometric arguments used to prove the Lemma immediately yield three more cases. These, plus the remaining 9 cases are obtained in Section 5.

4. Geometric proof of the Lemma. As before let $F = F_1, F_3$, or F_7. We shall say that a complex number d is *admissible* if it has the property stated in the Lemma: for any complex z_1, there is a homologous z such that $|z^2-d|<1$. Let C denote the complex plane, and $B(d)$ denotes the open disc $|z-d|<1$. $T = T(F)$ is the group of all translations of C by integers of F: $t(z) = z+a$ $(z \in C)$ for some $a \in O_F$. Thus z is homologous to z_1 if and only if $z = t(z_1)$ for some $t \in T$. By a *fundamental region* $\mathscr{R}$ we shall mean a closed region whose T-translates cover C, and having no two interior points homologous. For

$$S \subset C;\quad S^2 = \{z^2|\ z \in S\},\ \ aS = \{az|\ z \in S\},\ \ \bar{S} = \{\bar{z}|\ \ z \in S\}$$

(complex conjugate).

Now let $\mathscr{R}$ be a fundamental region and let $d \in C$. We say that d is *$\mathscr{R}$-admissible* if $\mathscr{R}^2 \subset B(d)$. Thus d is admissible if and only if d is $\mathscr{R}$-admissible for some fundamental region $\mathscr{R}$. The Lemma is proved by exhibiting a small number of fundamental regions $\mathscr{R}_j$ such that every d with $|d| < c(F)$ is $\mathscr{R}_j$-admissible for some j. We sketch the proof for $F = F_1$ and $F = F_7$.

Proof for F_1. Let $\mathscr{R}_0$ be the (closed) square with vertices $(\pm 1/2, \pm 1/2)$. Then $\mathscr{R}_0^2$ is the region $|x| \leqslant 1/4 - y^2$ (region $ABCD$ in Fig. 1). Any point $d = x+yi$ in the region

$$S_0:\quad |y| < \sqrt{(1-x^2)} - 1/2,\ |x| < \sqrt{(1-y^2)} - 1/4$$

is $\mathscr{R}_0$-admissible. (S_0 is $HAJLCM$ in Fig. 1.)

Next let $\mathscr{R}_1$ be the parallelogram with vertices $\pm(0, 1/2)$, $\pm(1, 1/2)$; $\mathscr{R}_1^2$ ($BEDGA$ in Fig. 1) is given by $x \geqslant y^2 - 1/4$, $y \geqslant 2x^2 - 1/8$. Any point d in the region S_1 (AHN in Fig. 1) is $\mathscr{R}_1$-admissible. S_1 is bounded by AN: $x = 0$; AH: $x^2 + (y+1/2)^2 = 1$; HR: $(x+1/4)^2 + y^2 = 1$, and

$$RN:\ x = t - 4t(1+16t^2)^{-1/2},\ y = 2t^2 - 1/8 + (1+16t^2)^{-1/2} \qquad (-1/4 \leqslant t \leqslant 0)$$

($R = (1/\sqrt{2} - 1/4, 1/\sqrt{2})$. RN is traced by the tip of the unit normal to the parabolic arc BE.)

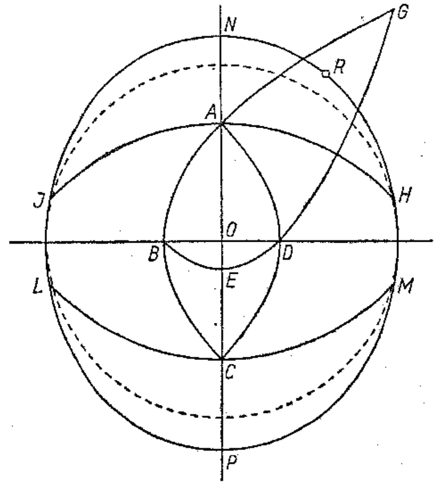

Fig. 1

Now for $j = 2, 3, 4$ let S_j be the region in the jth quadrant symmetric to S_1: $S_2 = -\bar{S}_1$, $S_3 = -S_1$, $S_4 = \bar{S}_1$. Set $\mathscr{R}_2 = i\bar{\mathscr{R}}_1$, $\mathscr{R}_3 = i\mathscr{R}_1$, $\mathscr{R}_4 = \bar{\mathscr{R}}_1$ (all fundamental regions). By symmetry any point of S_j is $\mathscr{R}_j$-admissible. Hence every point d in the region $\mathscr{S} = S_0 \cup S_1 \cup S_2 \cup S_3 \cup S_4$ ($HNJLPM$ in Fig. 1) is admissible. In particular every d with $|d| < 3/4$ (dashed circle in Fig. 1) is in $\mathscr{S}$, so the Lemma is proved for $F = F_1$. [Note that $\mathscr{S}$ includes its boundary, except for the two points $(\pm 3/4, 0)$.]

Proof for F_7. It is easily checked that every point d with $|d| < 1/2$ is $\mathscr{R}_j$-admissible for one of the following fundamental regions: $\mathscr{R}_0$ is the rectangle with vertices $(\pm 1/2, \pm\sqrt{7}/4)$. $\mathscr{R}_1$ is the parallelogram with vertices $\pm(-1/4, \sqrt{7}/4)$, $\pm(3/4, \sqrt{7}/4)$; and $\mathscr{R}_2 = \bar{\mathscr{R}}_1$.

The geometric proof for F_3 is omitted. It is considerably more complicated and delicate than the others. Moreover, the proof given above for F_1 immediately yields (see Section 5) three extra values of μ, in the table in Section 2, which are not given by the Lemma alone. On the other hand no extra values of μ are obtained from the geometric proof for F_3. Similarly, the constants $c(F_1) = 3/4$, $c(F_3) = \sqrt{13}/4$ are best possible (cf. [8] and [9], p. 135). But there is no benefit, for the present application, in obtaining the best possible value of $c(F_7)$ (which appears to be 9/16).

5. The remaining cases.

(5.1) First we consider the case $F = F_1$, $\delta = \mu \equiv 1 \bmod 4$, so $\varepsilon = 2$, $\varepsilon' = \frac{1}{2}$ in (3). Denote

$$N_1(a, b, r, s) = (r+s/2-a)^2 - \mu(s/2-b)^2. \tag{10}$$

As we saw above (cf. (2), (4), (5)), $K = F_1(\sqrt{\mu})$ is euclidean if and only if for every $a, b \in F_1 = Q(i)$ there exist $r, s \in Z[i]$ such that $|N_1(a, b, r, s)| < 1$. By the remarks at the beginning of Section 4, it is sufficient to show that for every $b \in Q(i)$ there exists $s \in Z[i]$ such that $d = \mu(s/2-b)^2$ is *admissible*. Since

$$N_1(a, b, r, s) = N_1(a, b+m, r-m, s+2m) \tag{11}$$

for $m \in Z[i]$, it is sufficient to take $b = x+yi$ in the fundamental region $|x| \leqslant \frac{1}{2}$, $|y| \leqslant \frac{1}{2}$. The further relations

$$N_1(a, b, r, s) = N_1(a+\zeta/2, b+\zeta/2, r, s+\zeta) \tag{12}$$

for $\zeta = \pm 1, \pm i$ allow us to further restrict $b = x+yi$ to the square $\mathscr{B}_1$: $|x| \leqslant \frac{1}{4}$, $|y| \leqslant \frac{1}{4}$.

There are three values of μ (with $N\delta > 36$) for which the proof of the Lemma shows immediately that $K = F_1(\sqrt{\mu})$ is euclidean: $\mu = 5+4i$, $5-4i$, -7. In each case the region $\mu\mathscr{B}_1^2$ is contained in the region $\mathscr{S}$; hence for each $b \in \mathscr{B}_1$, $d = \mu b^2$ is admissible. (Note that for larger values of $\delta = \mu \equiv 1 \bmod 4$ this simple method fails: for certain $b \in \mathscr{B}_1$, μb^2 is surely not admissible.)

(5.2) Next let $F = F_1$, $\mu \equiv \pm 1 + 2i \pmod 4$, $\delta = 2i\mu$, $\varepsilon = 1+i$, $\varepsilon' = 1/\varepsilon$ in (3). Denote

$$N_2(a, b, r, s) = \big(r+s/(1+i)-a\big)^2 - \mu\big(s/(1+i)-b\big)^2.$$

As above $K = F_1(\sqrt{\mu})$ is euclidean if and only if for every $a, b \in Q(i)$ there exist $r, s \in Z[i]$ such that $|N_2(a, b, r, s)| < 1$. By relations analogous to (11) and (12), we may restrict $b = x+yi$ to the square $\mathscr{B}_2$: $|x \pm y| \leqslant \frac{1}{2}$, with vertices $(\pm\frac{1}{2}, 0)$, $(0, \pm\frac{1}{2})$. We set $d = \mu\big(s/(1+i)-b\big)^2$. If $|b|^2 < 3/4|\mu|$, take $s = 0$; then $|d| < 3/4$, so d is admissible.

Now let $\mu = 3+2i$. We need only consider the region $\mathscr{B} = \{b \in \mathscr{B}_2, |b|^2 \geqslant 3/4\sqrt{13}\}$, which consists of four symmetric pieces (corners of $\mathscr{B}_2$). By a symmetry argument like that used in the proof of the Lemma for F_1, it is sufficient to deal with only one corner, say the one at $b = \frac{1}{2}$:

$$\mathscr{C}\colon\ b = x+yi,\ \tfrac{1}{4} < x \leqslant \tfrac{1}{2},\ |y| \leqslant \tfrac{1}{2}-x,\ |b|^2 \geqslant 3/4\sqrt{13}.$$

We may take as a fundamental region $\mathscr{R} = R_1 \cup R_2$, where R_1 is the trapezoid with vertices $(0, 0)$, $(1, 0)$, $(1, .66)$, $(0, .34)$; and $R_2 = -R_1$. It is easily checked that for each $b \in \mathscr{C}$, $d = (3+2i)b^2$ is $\mathscr{R}$-admissible — i. e., for $a \in \mathscr{R}$ and $b \in \mathscr{C}$,

$$|N_2(a, b, 0, 0)| = |a^2-(3+2i)b^2| < 1.$$

(The worst part is $a = .34i$, $b = \frac{1}{2}$, for which $|N_2(a, b, 0, 0)| = .9997$.) Thus $K = F_1(\sqrt{3+2i})$ is euclidean.

(5.3) Now we consider the case $F = F_3$, $\delta = \mu \equiv 1 \bmod 4$, with $N_1(a, b, r, s)$ as in (10).

Just as in (5.1), in order to show that $K = F_3(\sqrt{\mu})$ is euclidean, it is sufficient to show that for each $b \in F_3 = Q(\varrho)$ there exists $s \in Z[\varrho]$ such that $d = \mu(s/2-b)^2$ is admissible. The relations (11) for $m \in Z[\varrho]$, and (12) for $\zeta = \pm 1, \pm\varrho, \pm\varrho^2$ allow us to restrict b to the hexagon $\mathscr{B}_3$ centered at the origin, with vertices at $\zeta(-3)^{1/2}/6$ ($\zeta = \pm 1, \pm\varrho, \pm\varrho^2$) — i. e., at $(0, \pm\sqrt{3}/6)$, $(\pm\frac{1}{4}, \pm\sqrt{3}/12)$. By the Lemma, if $|b|^2 < \sqrt{13}/4|\mu|$ then ($s = 0$) $d = \mu b^2$ is admissible. Hence just as in (5.2) we need only consider the region $\mathscr{B} = \{b \in \mathscr{B}_3\colon |b|^2 \geqslant \sqrt{13}/4|\mu|\}$, which, for $117 < |\mu|^2 \leqslant 201$, consists of six symmetric pieces. Because of the six-fold symmetry in $Z[\varrho]$, it is sufficient to deal with just one corner, say the one at $b = (2+\varrho)/6 = (\frac{1}{4}, \sqrt{3}/12)$:

$$\mathscr{C}\colon\ b = x+yi.\ 1/8 < x \leqslant 1/4,\ 0 < y \leqslant (\tfrac{1}{2}-x)/\sqrt{3},\ |b|^2 \geqslant \sqrt{13}/4|\mu|.$$

We shall show that $K = F_3(\sqrt{\mu})$ is euclidean for $\mu = -11, 5-8\varrho$, $9-4\varrho$, $1+12\varrho$ (and the complex conjugates of the last three). In each case it remains only to exhibit a fundamental region $\mathscr{R}$ such that for each $b \in \mathscr{C}$, $d = \mu b^2$ is $\mathscr{R}$-admissible. It is possible to take $\mathscr{R} = R_1 \cup R_2$, where R_1 is a parallelogram and $R_2 = -R_1$. For each value of μ we list the vertices of R_1.

$\mu = -11$: $(-.915, \sqrt{3}/2)$, $(.085, \sqrt{3}/2)$, $(-.84, \sqrt{3}/4)$, $(.16, \sqrt{3}/4)$,

$\mu = 5-8\varrho$: $(.2674, 0)$, $(1.2674, 0)$, $(.2674, \sqrt{3}/4)$, $(1.2674, \sqrt{3}/4)$,

$\mu = 9-4\varrho$: $(0, 0)$, $(1, 0)$, $(5/4, \sqrt{3}/4)$, $(1/4, \sqrt{3}/4)$,

$\mu = 1+12\varrho$: $(0, 0)$, $(1/2, \sqrt{3}/2)$, $(1/4, 3\sqrt{3}/4)$, $(-1/4, \sqrt{3}/4)$.

Note that $\mathscr{R}$ is given in this simple form to simplify the description, but it makes it very delicate to check that the condition $|a^2-\mu b^2|<1$ is actually satisfied. Thus for $\mu = 5-8\varrho$, $a = .2674$, $b = (2+\varrho)/3$, $|a^2-\mu b^2| = .9994$. It is possible to get somewhat more comfortable upper bounds for $|a^2-\mu b^2|$, but only by using more complicated fundamental regions $\mathscr{R}$.

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Indépendance statistique d'ensembles liés à la fonction "somme des chiffres"

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0. INTRODUCTION ET PLAN DE L'ARTICLE

0.1. Introduction.

a) **Des problèmes de R. Salem, E. G. Straus et A. O. Gel'fond.** A la fin de [11] R. Salem pose (p. 62–63) une question sur le comportement à l'infini de la transformée de Fourier d'une mesure portée par la somme de deux ensembles de Cantor. E. G. Straus, plus tard, conjecture le résultat arithmétique suivant qui résoud, en partie, le problème d'analyse harmonique de Salem. Soit g_1, g_2 deux entiers $\geqslant 2$ premiers entre eux, s_{g_1}, s_{g_2} les "sommes des chiffres" en bases g_1 et g_2 (respectivement): L'ensemble des entiers n tels que $s_{g_1}(n)\leqslant A$, $s_{g_2}(n)\leqslant B$ (A, B donnés) est fini. Toujours dans le même esprit, A. O. Gel'fond qui dans [5] avait obtenu des résultats sur la somme des chiffres, pose le problème suivant ([5], p. 265) qu'il qualifie d'intéressant (¹): Démontrer que:

$$\operatorname{card}\{n\leqslant x;\ s_{g_1}(n)\equiv c_1 \pmod{m_1},\ s_{g_2}(n)\equiv c_2 \pmod{m_2}\} = \frac{x}{m_1m_2}+O(x^a) \qquad (0\leqslant a<1)$$

si $(m_1, g_1-1) = (m_2, g_2-1) = 1$.

b) Ce sont des problèmes du genre de celui de Gel'fond qui seront ici résolus. Nous les énoncerons sous forme "d'indépendance" d'ensembles relativement à la densité d des suites d'entiers.

On a par exemple le résultat d'indépendance suivant:

$$d\big(s_{g_1}(n)\equiv c_1 \pmod{m_1},\ s_{g_2}(n)\equiv c_2 \pmod{m_2}\big) = d\big(s_{g_1}(n)\equiv c_1 \pmod{m_1}\big)\cdot d\big(s_{g_2}(n)\equiv c_2 \pmod{m_2}\big) = \frac{1}{m_1m_2}$$

(¹) Dans le même article Gel'fond pose deux autres problèmes, dont l'un est résolu par M. Olivier dans [10].