

Ω -theorems for the Riemann zeta-function

by

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Abstract. For $s = \sigma + it$ the familiar formula

$$(0.1) \quad \zeta^k(s) = \sum_{j \geq 1} \frac{d_k(j)}{j^s}$$

is valid for $\sigma > 1$ and integer $k \geq 1$. Here Ω results for $\zeta(c + it)$, $\frac{1}{2} \leq c \leq 1$ will be obtained in an elementary way from (0.1) by choosing an appropriate integer n , $n = n(k, c)$. The size of the single term $d_k(n)$ will give Ω results that are slightly better than the existing results of Littlewood for $c = 1$ and Titchmarsh for $\frac{1}{2} \leq c < 1$. Their results are based on the use of $\sum d_k^2(j)/j^s$, $\sigma > 1$, which involves Legendre functions. The method used here also applies to $1/\zeta(1 + it)$ as will be shown.

1. Let γ be the Euler constant.

THEOREM 1. *There exist arbitrarily large t for which*

$$|\zeta(1 + it)| \geq e^\gamma \log \log t + O(1).$$

This improves slightly the result of Littlewood ([2]; [5], Theorem 8.9A).

THEOREM 2. *Let $\frac{1}{2} \leq c < 1$. Then there exists a positive constant B independent of c such that for sufficiently large T*

$$\max_{1 \leq |t| \leq T} |\zeta(c + it)| \geq e^{B(\log T)^{1-c}/\log \log T}.$$

This improves slightly the result of Titchmarsh ([4]; [5], Theorem 8.12). The following improves slightly the result of Littlewood ([3]; [5], Theorem 8.9B).

THEOREM 3. *There exist arbitrarily large values of t for which*

$$\frac{1}{|\zeta(1 + it)|} \geq \frac{6e^\gamma}{\pi^2} (\log \log t - \log \log \log t) + O(1).$$

* Supported in part by the National Science Foundation, NSF GP 22928.

The proofs of Theorems 1 and 2 are entirely elementary except for the use of the fact that the integral of an analytic function around a rectangle is zero. For Theorem 3 we use a weak upper bound for $1/\zeta(s)$ for $\sigma \geq 1$ such as $O(\log^7 t)$, ([5], (3.6.5)).

The simple idea involved in choosing n appropriately is revealed in the Remark following the statement of Lemma 2.1. There are actually many choices of n that will work. However I was unable to find a large enough number of them which when combined substantially improve the Theorems.

Professor P. Turán has pointed out to me that under R. H., applying the inequality of Hadamard–Borel–Carathéodory to $\log \zeta(s)$, leads to Ω -results for $1/\zeta(s)$, $\frac{1}{2} < c < 1$, of the same kind as in Theorem 2 since otherwise Theorem 2 would be contradicted.

2. In what follows p will denote a prime number and k an integer exceeding 2. As usual let $[x]$ denote the integer part of x .

LEMMA 2.1. Let c satisfy $\frac{1}{2} \leq c \leq 1$. Let the integer $m (= m_p)$ be defined by

$$(2.1) \quad m = \left[\frac{k}{p^c - 1} \right].$$

Then there exists a constant A_1 such that

$$\left| \log \frac{d_k(p^m)}{p^{mc}} - k \log \frac{p^c}{p^c - 1} + \frac{1}{2} \log m \right| \leq A_1.$$

Remark. For the moment let the integer m merely be positive. Then from consideration of the factor $(1 - 1/p^s)^{-k}$ in the product representation of $\zeta^k(s)$ it follows easily that

$$(2.2) \quad \frac{d_k(p^m)}{p^{mc}} = \frac{(k-1+m)!}{(k-1)! m! p^{mc}}.$$

Clearly this increases with m so long as

$$\frac{k-1-m}{mp^c} \geq 1$$

or so long as $k-1 \geq m(p^c-1)$. This motivates the choice of m in (2.1) as the way to maximize $d_k(p^m)/p^{mc}$. Since $d_k(n)/n^c$ is a multiplicative function, the above maximization leads easily to the choice of n which will essentially maximize $d_k(n)/n^c$.

Proof of Lemma 2.1. Let $p^c \leq k$ so that $m \geq 1$ and let

$$R = \log \frac{d_k(p^m)}{p^{mc}}$$

with m as in (2.1). Since $\log j! = (j + \frac{1}{2}) \log j - j + O(1)$ for $j \geq 1$ it follows from (2.2) that

$$(2.3) \quad R = (k - \frac{1}{2} + m) \log(k-1+m) - (k - \frac{1}{2}) \log(k-1) - (m + \frac{1}{2}) \log m - cm \log p + O(1) \\ = I + H - \frac{1}{2} \log m + O(1)$$

where

$$I = \left(k - \frac{1}{2}\right) \log \left(1 + \frac{m}{k-1}\right), \quad H = m \log \frac{k-1+m}{mp^c}.$$

From the definition of m

$$(2.4) \quad m = \frac{k}{p^c - 1} - \alpha, \quad 0 \leq \alpha < 1$$

and so

$$I = \left(k - \frac{1}{2}\right) \log \left(1 + \frac{k}{k-1} \cdot \frac{1}{p^c - 1} - \frac{\alpha}{k-1}\right) = \left(k - \frac{1}{2}\right) \log \left(1 + \frac{1}{p^c - 1} + O\left(\frac{1}{k}\right)\right) \\ = \left(k - \frac{1}{2}\right) \log \left(\frac{p^c}{p^c - 1}\right) + O(1) = k \log \frac{p^c}{p^c - 1} + O(1).$$

Using (2.4) again

$$H = m \log \left(1 + \frac{k-1-mp^c+m}{mp^c}\right) = m \log \left(1 + \frac{\alpha(p^c-1)-1}{mp^c}\right).$$

Because $m \geq 1$, $0 \leq \alpha < 1$, $p^c \geq 2^{1/2}$

$$-\frac{1}{m2^{1/2}} \leq \frac{\alpha(p^c-1)-1}{mp^c} \leq \frac{1}{m}$$

and so $H = O(1)$. Hence from (2.3) follows

$$R = k \log \frac{p^c}{p^c - 1} - \frac{1}{2} \log m + O(1)$$

which proves the lemma.

LEMMA 2.2. (The case $c = 1$). Let $m (= m_p)$ be defined as in 2.1. Let

$$(2.5) \quad n = \prod_{p \leq k} p^m.$$

Then as k increases

$$(2.6) \quad \left(\frac{d_k(n)}{n}\right)^{1/k} = e^\gamma \log k + O(1).$$

Moreover

$$(2.7) \quad \log n = k \log k + O(k).$$

Proof. Let $\pi(x)$ represent the number of primes not exceeding x . Then $\pi(x) = O(x/\log x)$. By Lemma 2.1 with $c = 1$

$$(2.8) \quad \log \frac{d_k(n)}{n} = -k \sum_{p \leq k} \log \left(1 - \frac{1}{p}\right) - \frac{1}{2} \sum \log \left[\frac{k}{p-1}\right] + O(\pi(k)).$$

For $p \leq k$

$$0 \leq \log \left[\frac{k}{p-1}\right] \leq \log \frac{k}{p-1} \leq \log \frac{k}{p} + 1.$$

Let

$$I = \sum \log \left[\frac{k}{p-1}\right] \leq \sum \log \frac{k}{p} + \pi(k).$$

Then with the sums for $p \leq k$

$$(2.9) \quad \sum \log \frac{k}{p} = \sum_p \int_p^k \frac{dv}{v} = \int_2^k \frac{dv}{v} \sum_{p \leq v} 1 = \int_2^k \frac{\pi(v)}{v} dv = O\left(\frac{k}{\log k}\right).$$

Hence in (2.8)

$$(2.10) \quad \frac{1}{k} \log \frac{d_k(n)}{n} = - \sum \log \left(1 - \frac{1}{p}\right) + O\left(\frac{1}{\log k}\right).$$

It is an elementary result ([1], (22.7.4)) that

$$\sum_{p < x} \frac{1}{p} = \log \log x + B_1 + O\left(\frac{1}{\log x}\right)$$

where B_1 is a constant and that ([1], (22.8.1))

$$\sum_{p < \infty} \left\{ \log \left(1 - \frac{1}{p}\right) + \frac{1}{p} \right\} = B_1 - \gamma$$

and so

$$\sum_{p \leq x} \left\{ \log \left(1 - \frac{1}{p}\right) + \frac{1}{p} \right\} = B_1 - \gamma + O\left(\frac{1}{x}\right).$$

Hence

$$(2.11) \quad \sum_{p \leq x} \log \left(1 - \frac{1}{p}\right) = -\log \log x - \gamma + O\left(\frac{1}{\log x}\right).$$

Setting $x = k$ above, (2.10) becomes

$$\frac{1}{k} \log \frac{d_k(n)}{n} = \log \log k + \gamma + O\left(\frac{1}{\log k}\right)$$

which proves (2.6).

From (2.5) with the sum for $p \leq k$

$$\log n = \sum \left[\frac{k}{p-1}\right] \log p.$$

Since $\sum \log p = O(k)$,

$$\log n = k \sum \frac{\log p}{p-1} + O(k).$$

Since $\sum \log p/p(p-1) = O(1)$,

$$\log n = k \sum \frac{\log p}{p} + O(k) = k \log k + O(k)$$

by a well known elementary result which proves (2.7).

3. Here the proof of Theorem 1 will be given. It will be useful to recall the elementary formula ([5], (2.1.4))

$$\zeta(s) = \frac{s}{s-1} - s \int_1^\infty \frac{x - [x]}{x^{s+1}} dx$$

valid for $\sigma > 0$. For $\sigma \geq \frac{1}{2}$ and t large

$$(3.1) \quad |\zeta(s)| \leq 2 + |s| \int_1^\infty \frac{dx}{x^{\sigma+1}} < 3t, \quad \sigma \geq \frac{1}{2}.$$

Proof of Theorem 1. With n as in (2.5), $a > 0$ and $b > 0$, let

$$(3.2) \quad J = \frac{\pi^{-1/2}}{ib} \int_{1+a-i\infty}^{1+a+i\infty} \zeta^k(s) n^s e^{(s-1-a)^2/b^2} ds.$$

Then by (0.1)

$$J = \sum \frac{d_k(j)}{j^{1+a}} n^{1+a} \frac{\pi^{-1/2}}{b} \int_{-\infty}^\infty e^{it \log n/j - t^2/b^2} dt = \sum \frac{d_k(j)}{j^{1+a}} n^{1+a} \exp\left(-\frac{b^2}{4} \log^2 \frac{n}{j}\right)$$

by a familiar integral formula. Since $d_k(j) > 0$, it follows that

$$(3.3) \quad J > d_k(n).$$

Divide J in (3.2) into three parts J_1, J_2 and J_3 with respective ranges of integration $(1+a-i\infty, 1+a-i)$, $(1+a-i, 1+a+i)$ and $(1+a+i, 1+a+i\infty)$. Since the only singularity of $\zeta(s)$ is the pole at $s = 1$, for some A ,

$$(3.4) \quad |\zeta(s)| \leq \frac{1}{|s-1|} + A, \quad |s-1| \leq 2,$$

and so it follows easily that

$$|J_2| \leq \frac{\pi^{-1/2}}{b} \int_{-1}^1 \left(\frac{1}{a} + A \right)^k n^{1+a} dt.$$

This is minimized essentially by choosing $a = k/\log n$. For this choice

$$(3.5) \quad |J_2| \leq \frac{2ne^k}{b} \left(\frac{\log n}{k} + A \right)^k.$$

Since by (2.7) $\log n = k \log k + O(k)$

$$|J_2| \leq \frac{2n}{b} e^k (\log k)^k \left(1 + O\left(\frac{1}{\log k} \right) \right)^k.$$

If $b = e^k$ it follows from (2.6) that for large k

$$(3.6) \quad |J_2| \leq 4d_k(n) e^{-\gamma k}.$$

To appraise J_3 we deform the integral to a sum of two, J_{31} and J_{32} , with the help of (3.1), where J_{31} is for $s = \sigma + i$, $1 < \sigma < 1 + a$, and J_{32} is for $s = 1 + it$, $1 < t$. On J_{31} , $|\zeta(\sigma + it)| \leq 1 + A$ by (3.4). Hence

$$|J_{31}| \leq \frac{n^{1+a}(1+A)^k}{b} \frac{e^{(a^2-1)t^2}}{\pi^{1/2}} \int_1^{1+a} d\sigma.$$

Since a is small for large k

$$(3.7) \quad |J_{31}| \leq \frac{n^{1+a}(1+A)^k}{b} = n(1+A)^k.$$

For J_{32} we divide the range of integration into two parts $1 \leq t \leq T$ and $T < t$. Hence since b is large

$$(3.8) \quad |J_{32}| \leq \frac{n}{b} \int_1^T |\zeta(1+it)|^k e^{-t^2/b^2} dt + J'$$

where using (3.1)

$$J' = \frac{3^k n}{b} \int_T^\infty t^k e^{-t^2/b^2} dt.$$

Let $T = 2kb$. Since

$$\frac{d}{dt} \left(k \log t - \frac{t^2}{b^2} \right) = \frac{k}{t} - \frac{2t}{b^2} \leq -\frac{1}{b}$$

for $t \geq 2kb = T$ it follows that

$$(3.9) \quad J' \leq T^k e^{-T^2/b^2} \frac{3^k n}{b} \int_T^\infty e^{-(t-T)/b} dt = T^k e^{-4k^2} 3^k n.$$

Since $\log T = k + \log 2k$

$$J' \leq n \exp\{k(k + 6 \log 6k) - 4k^2\} \leq n$$

for large k . Thus if $M_T = \max |\zeta(1+it)|$ for $1 \leq |t| \leq T$ then

$$|J_{32}| \leq n M_T^k \frac{1}{b} \int_0^\infty e^{-t^2/b^2} dt + n \leq n M_T^k + n.$$

Using (3.7)

$$|J_3| \leq n M_T^k + n(1+A)^k + n \leq n M_T^k + e^{-\gamma k} d_k(n).$$

Clearly $|J_1|$ has the same dominant. Using this and (3.6) in (3.3)

$$(1e - 6^{-\gamma k}) \frac{d_k(n)}{n} \leq 2M_T^k.$$

By (2.6)

$$(1 - 6e^{-\gamma k})^{1/k} (e^\gamma \log k + O(1)) \leq 2^{1/k} M_T$$

or

$$e^\gamma \log k + O(1) \leq M_T.$$

Since $\log T = k + \log 2k$,

$$\log \log T = \log k + O\left(\frac{\log k}{k} \right)$$

and so

$$e^\gamma \log \log T + O(1) \leq M_T$$

which proves Theorem 1.

4. For Theorem 2 the following lemma is required.

LEMMA 4.1. For $\frac{1}{2} \leq c < 1$ there is a constant $A_2 > 2$ such that if

$$(4.1) \quad y = \left(\frac{k}{A_2} \right)^{1/c},$$

if $m (= m_p)$ is given by (2.1) and

$$n = \prod_{p \leq y} p^m,$$

then there exists a constant $A_3 > 0$ such that

$$(4.2) \quad \frac{1}{k} \log \frac{d_k(n)}{n^c} \geq A_3 \frac{(\log n)^{1-c}}{\log \log n}$$

for large k . Moreover $\log n = O(k^3)$ and

$$(4.3) \quad \log n \geq \frac{k^{1/c}}{4A_2}.$$



Proof of Lemma 4.1. By Lemma 2.1

$$(4.4) \quad \log \frac{d_k(n)}{n^c} \geq \sum_{p \leq y} H(p, c, k)$$

where

$$(4.5) \quad \begin{aligned} H &= k \log \frac{p^c}{p^c-1} - \frac{1}{2} \log \frac{k}{p^c-1} - A_1 \\ &= \left(k - \frac{1}{2}\right) \log \frac{p^c}{p^c-1} - \frac{1}{2} \log \frac{k}{p^c} - A_1 \\ &\geq \left(k - \frac{1}{2}\right) \frac{1}{p^c} - \frac{1}{2} \log \frac{k}{p^c} - A_1 \geq \frac{k}{p^c} - \frac{1}{2} \log \frac{k}{p^c} - \frac{A_2}{4} \end{aligned}$$

where $A_2 = 4(A_1 + \frac{1}{2}) > 2$.

It will now be shown that

$$(4.6) \quad x - \frac{1}{2} \log x - \frac{A_2}{4} \geq \frac{1}{2} x, \quad x \geq A_2.$$

Indeed $x - \log x - \frac{1}{2} A_2$ is an increasing function of x for $x \geq 1$ and it is positive for $x = A_2 > 2$ since $\frac{1}{2} v - \log v$ is increasing for $v \geq 2$ and is positive for $v = 2$. By (4.6) it follows from (4.5) that

$$H \geq \frac{1}{2} \frac{k}{p^c} \quad \text{for} \quad \frac{k}{p^c} \geq A_2.$$

Hence, if \sum is for $p \leq y$, (4.4) implies

$$(4.7) \quad \frac{1}{k} \log \frac{d_k(n)}{n^c} \geq \frac{1}{2} \sum \frac{1}{p^c}.$$

From the definition of n

$$(4.8) \quad \begin{aligned} \log n &= \sum m \log p \leq k \sum \frac{\log p}{p^c-1} < k \sum \frac{\log p}{p^c} + 4k \sum \frac{\log p}{p^{2c}} \\ &\leq k \sum \frac{\log p}{p^c} + 4k \sum \frac{\log p}{p} \leq k \log y \sum \frac{1}{p^c} + O(k \log y). \end{aligned}$$

This implies that

$$\sum \frac{1}{p^c} \geq \frac{\log n}{k \log y} + O(1)$$

and so by (4.7)

$$(4.9) \quad \frac{1}{k} \log \frac{d_k(n)}{n^c} \geq \frac{\log n}{2k \log y} + O(1) = \frac{c \log n}{2k(\log k - \log A_2)} + O(1).$$

Since $p^c \leq k/A_2$

$$m = \left[\frac{k}{p^c-1} \right] \geq \frac{k}{p^c-1} - 1 \geq \frac{k-p^c}{p^c-1} \geq \frac{k(1-1/A_2)}{p^c-1} \geq \frac{k}{2p^c}$$

and so

$$\log n = \sum m \log p \geq \frac{k}{2} \sum \frac{\log p}{p^c} \geq \frac{k}{2y^c} \sum \log p \geq \frac{ky}{4y^c}$$

for large k , and hence for large y . Using (4.1) this implies that

$$\log n \geq \frac{k^{1/c}}{4A_2^{1/c-1}} \geq \frac{k^{1/c}}{4A_2}$$

and so

$$k \leq (4A_2 \log n)^c$$

which proves (4.3) and in (4.9) implies that for large k

$$\frac{1}{k} \log \frac{d_k(n)}{n^c} \geq \frac{A_3 \log n}{(\log n)^c \log \log n}$$

for some A_3 which is (4.2).

From the top line of (4.8)

$$\log n \leq k \sum \frac{\log p}{p^c-1} \leq \frac{k}{2^{1/2}-1} \sum \log p = kO(y) = O(k^3)$$

by (4.1).

Proof of Theorem 2. Here n is defined as in Lemma 4.1. Then with that n and J given by (3.2) there follows exactly as in (3.3)

$$(4.10) \quad J > d_k(n).$$

With $a = k/\log n$ and J divided into J_1, J_2 and J_3 as below (3.3) there follows (3.5) which since $\log n = O(k^3)$ gives for large k

$$J_2 = O\left(\frac{n}{b} e^{3k \log k}\right).$$

Let $b = n$. By (4.3)

$$\log k \leq \log \log n$$

for large k . Hence

$$(4.11) \quad J_2 = O(e^{3k \log \log n}).$$

Here, where $c < 1$, J_1 and J_3 are indented to the line $\sigma = c$ instead of $\sigma = 1$. Then as in the first part of (3.7)

$$(4.12) \quad |J_{31}| \leq \frac{n^{1+a}(1+A)^k}{b} \leq e^k(1+A)^k$$



since here $b = n$ and as before $a = k/\log n$. In place of (3.8)

$$(4.13) \quad |J_{32}| \leq \frac{n^c}{b} \int_1^T |\zeta(c+it)|^k e^{-t^2/b^2} dt + J'$$

where now

$$J' = \frac{3^k n^c}{b} \int_T^\infty t^k e^{-t^2/b^2} dt.$$

Let $T = k^3 b$ so that much as before

$$J' \leq 3^k n^c T^k e^{-T^2/b^2} \frac{1}{b} \int_T^\infty e^{-(t-T)b} = n^c \exp\{-k^6 + k(3 \log k + \log 3 + \log n)\}$$

and since $\log n = O(k^3)$

$$J' = O(n^c e^{-k^6/2}).$$

Hence if $M_T = \max |\zeta(c+it)|$ for $1 \leq t \leq T$ then

$$|J_3| \leq n^c M_T^k + n^c O(e^{-k^6/2}) + e^k (1+A)^k.$$

With (4.11) and (4.10)

$$\frac{d_k(n)}{n^c} \leq 2M_T^k + O(e^{3k \log \log n}).$$

Since $(\alpha + \beta)^{1/k} \leq \alpha^{1/k} + \beta^{1/k}$ for $\alpha, \beta \geq 0$,

$$\left(\frac{d_k(n)}{n^c}\right)^{1/k} \leq 2^{1/k} M_T + O((\log n)^3).$$

By (4.2) for any fixed c , $\frac{1}{2} \leq c < 1$ and sufficiently large k , and hence n , this implies

$$(4.14) \quad M_T \geq \exp\left(\frac{A_3}{2} \frac{(\log n)^{1-c}}{\log \log n}\right).$$

Since $T = k^3 n$ it follows that $\log n = \log T - 3 \log k$ and that

$$\log \log n = \log \log T + O\left(\frac{\log k}{\log T}\right).$$

Hence

$$\frac{(\log n)^{1-c}}{\log \log n} = \frac{(\log T)^{1-c}}{\log \log T} \left(1 - O\left(\frac{\log k}{\log T}\right)\right).$$

Since $\log T > \log n \geq k^{1/c}/4A_2$

$$\frac{\log k}{\log T} = O\left(\frac{\log k}{k^{1/c}}\right)$$

and so for large k

$$\frac{(\log n)^{1-c}}{\log \log n} \geq \frac{1}{2} \frac{(\log T)^{1-c}}{\log \log T}$$

which in (4.14) proves Theorem 2.

5. For $\sigma > 1$

$$(5.1) \quad \left(\frac{1}{\zeta(s)}\right)^k = \sum \frac{a_k(j)}{j^s}$$

where from $(1-1/p^s)^k$ follows easily

$$(5.2) \quad a_k(p^r) = (-1)^r \frac{k!}{r!(k-r)!}.$$

Maximizing $|a_k(p^r)/p^r|$ much as in the Remark preceding Lemma 1.1 leads to choosing $r = m$ where

$$(5.3) \quad m = \left[\frac{k}{p+1} \right].$$

LEMMA 5.1. With $m (= m_p)$ as above let

$$(5.4) \quad n = \prod_{p < k} p^m.$$

Then

$$(5.5) \quad \frac{1}{k} \log \frac{|a_k(n)|}{n} = \sum_{p < k} \log \left(\frac{p+1}{p}\right) + O\left(\frac{1}{\log k}\right).$$

Moreover

$$(5.6) \quad \log n = k \log k + O(k).$$

Proof of Lemma 5.1. Since $a_k(n)/n$ is multiplicative it follows from (5.2) with $r = m (= m_p)$ that, if \sum is for $p < k$, then

$$\begin{aligned} R = \log \frac{|a_k(n)|}{n} &= \sum \{\log k! - \log m! - \log(k-m)! - m \log p\} \\ &= \sum \left\{ \left(k + \frac{1}{2}\right) \log k - \left(m + \frac{1}{2}\right) \log m - \left(k-m + \frac{1}{2}\right) \times \right. \\ &\quad \left. \times \log(k-m) - m \log p \right\} + O\left(\frac{k}{\log k}\right) \\ &= \sum \left\{ \left(k + \frac{1}{2}\right) \log \frac{k}{k-m} + m \log \frac{k-m}{mp} \right\} + O\left(\frac{k}{\log k}\right) \end{aligned}$$

where $\frac{1}{2} \sum \log m$ was appraised as in (2.9).



From (5.3), $m = k/(p+1) - a$ where $0 \leq a < 1$. Hence

$$\begin{aligned} \left(k + \frac{1}{2}\right) \log \frac{k}{k-m} &= \left(k + \frac{1}{2}\right) \log \frac{p+1}{p+a(p+1)/k} \\ &= \left(k + \frac{1}{2}\right) \log \frac{p+1}{p} - \left(k + \frac{1}{2}\right) \log \frac{p+a(p+1)/k}{p} \\ &= \left(k + \frac{1}{2}\right) \log \frac{p+1}{p} + O(1) \end{aligned}$$

and

$$m \log \frac{k-m}{mp} = m \log \left(1 + \frac{k-m(p+1)}{mp}\right) = m \log \left(1 + \frac{a(p+1)}{mp}\right) = O(1).$$

Therefore

$$\begin{aligned} R &= \sum \left(k + \frac{1}{2}\right) \log \frac{p+1}{p} + O\left(\frac{k}{\log k}\right) \\ &= k \sum \log \frac{p+1}{p} + \frac{1}{2} \sum \frac{1}{p} + O\left(\frac{k}{\log k}\right) \end{aligned}$$

which easily yields (5.5). The proof of (5.6) is the same as for (2.7).

Proof of Theorem 3. With n as in (5.4) let

$$(5.7) \quad J = \frac{\pi^{-1/2}}{bi} \int_{-i\infty+2}^{i\infty+2} (\zeta(s))^{-k} n^s e^{(s-2)^2/b^2} ds.$$

By (5.1)

$$J = \sum \frac{a_k(j)}{j^2} n^2 \exp\left(-\frac{b^2}{4} \log^2 \frac{n}{j}\right).$$

Hence

$$(5.8) \quad |J| \geq |a_k(n)| - I$$

where, using $|\log n/j| > 1/2n, j \neq n$,

$$(5.9) \quad I = \sum \frac{|a_k(j)|}{j^2} n^2 e^{-b^2/16n^2}.$$

Since by (5.2)

$$|a_k(j)| \leq d_k(j), \quad \sum \frac{|a_k(j)|}{j^2} \leq 2 \sum \frac{d_k(j)}{j^2} \log j + 1.$$

But

$$\sum \frac{d_k(j)}{j^2} \log j = \frac{1}{2\pi i} \int_{-i\infty+3/2}^{i\infty+3/2} \frac{\zeta^k(s)}{(s-2)^2} ds = O(e^{2k})$$

since $\zeta(3/2) < e^2$. Hence by (5.9)

$$I = O(e^{2k}) n^2 e^{-b^2/16n^2}.$$

Let $b = 8nk$. Then

$$I = O(e^{2k}) n^2 e^{-4k^2}.$$

Using (5.6) it follows easily that

$$I = O(e^{-k^2}).$$

Hence

$$(5.10) \quad |J| \geq |a_k(n)| + O(e^{-k^2}).$$

Next using ([5], (3.6.5))

$$\frac{1}{|\zeta(s)|} = O((\log t)^\sigma), \quad \sigma \geq 1$$

for $|t| \geq 1$ in (5.7) and moving the contour to $\sigma = 1$ gives

$$|J| \leq \frac{n}{b} \int_{-\infty}^{\infty} |\zeta(1+it)|^{-k} e^{-t^2/b^2} dt$$

for large k , and hence large b . Let $T = k^2 b$ and let

$$J_1 = \frac{n}{b} \int_0^T |\zeta(1+it)|^{-k} e^{-t^2/b^2} dt$$

and

$$J_2 = \frac{n}{b} \int_T^{\infty} |\zeta(1+it)|^{-k} e^{-t^2/b^2} dt.$$

Then

$$(5.11) \quad |J| \leq 2J_1 + 2J_2.$$

Clearly

$$(5.12) \quad J_1 \leq n \max_{|t| \leq T} |\zeta(1+it)|^{-k}$$

and since $(\log t)^2/t \rightarrow 0$ as $t \rightarrow \infty$,

$$J_2 \leq \frac{n}{b} \int_T^{\infty} t^k e^{-t^2/b^2} dt \leq T^k e^{-T^2/b^2} \frac{n}{b} \int_T^{\infty} e^{-(t-T)/b} dt \leq n e^{k \log T - k^4}$$

much as in (3.9). Evidently

$$\log T = \log k^2 + \log b = \log 8k^3 + \log n = O(k \log k)$$

by (5.6). Hence

$$(5.13) \quad J_2 = O(n e^{-k^4/2}).$$

Using (5.10), (5.11), (5.12) and (5.13)

$$2 \max_{|t| \leq T} |\zeta(1+it)|^{-k} + O(e^{-k^2}) \geq \left| \frac{a_k(n)}{n} \right|$$

and so by (5.5)

$$\begin{aligned} 2^{1/k} \max_{|t| \leq T} |\zeta(1+it)|^{-1} + O(e^{-k}) &\geq \prod_{p < k} \left(1 + \frac{1}{p}\right) \left(1 + O\left(\frac{1}{\log k}\right)\right) \\ &= \prod_{p < k} \left(\frac{1}{1-1/p}\right) \prod_{p < k} \left(1 - \frac{1}{p^2}\right) \left(1 + O\left(\frac{1}{\log k}\right)\right) \\ &= \frac{e^{\gamma \log k}}{\zeta(2)} + O(1) \end{aligned}$$

as in (2.11). Since $T = k^2 b = 8k^3 n$,

$$\log T = 3 \log k + \log 8 + k \log k + O(k) = k \log k + O(k).$$

Hence

$$\log \log T = \log k + \log \log k + O\left(\frac{1}{\log k}\right)$$

and so for large k , $\log \log T > \log k$ and so

$$\log k \geq \log \log T - \log \log \log T + O\left(\frac{1}{O(1)}\right).$$

Thus

$$\max_{|t| \leq T} \frac{1}{|\zeta(1+it)|} \geq \frac{e^{\gamma} 6}{\pi^2} (\log \log T - \log \log \log T) + O(1)$$

which completes the proof.

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