$\boldsymbol{R}\mathcal{O}$ is a semi-simple algebra over $\boldsymbol{R}$. By Wedderburn's structure theorems, $\boldsymbol{R}\mathcal{O} = \bigoplus_1^k T_i$ where each T_i is $\boldsymbol{R}$-isomorphic to some $M_{n_i\times n_i}(B_i)$, B_i a finite dimensional division algebra over $\boldsymbol{R}$. Choose an $\boldsymbol{R}$-basis B for $\boldsymbol{R}\mathcal{O}$ by selecting an $\boldsymbol{R}$-basis for each T_i. Then for $x \in \boldsymbol{R}\mathcal{O}$, say $x = \bigoplus_1^k x_i$, $x_i \in T_i$, we have $x \in L(\mathcal{O})$ if and only if $\prod_{i=1}^k \|x_i\|_i = \pm 1$ where $\|x_i\|_i$ is the regular norm of x_i in T_i.

We claim that if $U(\mathcal{O})$ is finite, then k must be 1. For if $k > 1$, say $\dim_{\boldsymbol{R}} T_1 = k_1$ and $\dim_{\boldsymbol{R}} T_2 = k_2$, let $x = 2 \oplus 2^{-k_1/k_2} \oplus 1 \oplus \ldots \oplus 1$. Then x, and thus all of its integral powers, are in $L(\mathcal{O})$. But $\{x^s \mid s \in \boldsymbol{Z}\}$ is unbounded, which is a contridiction. Hence if $U(\mathcal{O})$ is finite, $\boldsymbol{R}\mathcal{O} \underset{\boldsymbol{R}}{\cong} M_{n\times n}(C)$ where C is a finite dimensional division algebra over $\boldsymbol{R}$. We will show that in this case n must be 1.

Let $\psi\colon \boldsymbol{R}\mathcal{O} \to M_{n\times n}(C)$ be an $\boldsymbol{R}$-isomorphism and let $b_1, \ldots, b_s$ be an $\boldsymbol{R}$-basis for C. Then $\{e_{ij}b_q \mid 1 \leqslant i, j \leqslant n;\ 1 \leqslant q \leqslant s\}$ ordered lexicographically is an ordered $\boldsymbol{R}$-basis for $M_{n\times n}(C)$. Here the e_{ij} denote the usual matrix units. Since ψ is a $\boldsymbol{R}$-isomorphism, $B = \{\psi^{-1}(e_{ij}b_q)\}$ is an $\boldsymbol{R}$-basis for $\boldsymbol{R}\mathcal{O}$. Let $\psi(x) = I_n + e_{1n}$. Then $\hat{\varphi}_B(x)$ is of the form

$$\begin{bmatrix} 1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & . & . & . & . & . & . & . & 0 & 1 \\ 0 & . & . & . & . & . & . & . & 0 & 1 \end{bmatrix}.$$

Now $\|x\| = 1$, i.e. $x \in L(\mathcal{O})$, but $\{x^q \mid q \in \boldsymbol{Z}\}$ is unbounded if $n > 1$.

Thus we have that if $U(\mathcal{O})$ is finite, $\boldsymbol{R}\mathcal{O}$ is a finite dimensional division algebra over $\boldsymbol{R}$. Hence, since $\boldsymbol{R}\mathcal{O} \cong \boldsymbol{R} \otimes_{\boldsymbol{Q}} D$, D must be $\boldsymbol{Q}$-isomorphic to either $\boldsymbol{Q}$, an imaginary quadratic extension of $\boldsymbol{Q}$ or a positive definite quaternion algebra over $\boldsymbol{Q}$. This condition is clearly also sufficient to assure that $U(\mathcal{O})$ is finite.

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Solvability of a Diophantine inequality in algebraic number fields

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1. Introduction. Let K be a totally real algebraic number field of finite degree h over the field $\boldsymbol{Q}$ of rational numbers and $\overline{K} = K \otimes_{\boldsymbol{Q}} \boldsymbol{R}$ the tensor product of K with the field $\boldsymbol{R}$ of real numbers. Any element a in $\overline{K}$ is represented as

$$a = \begin{pmatrix} a^{(1)} & & 0 \\ & \ddots & \\ 0 & & a^{(h)} \end{pmatrix}$$

where $a^{(1)}, \ldots, a^{(h)}$ are the 'conjugates' of a. Put

$$\|a\| = \max_{1\leqslant k\leqslant h} |a^{(k)}|. \tag{1}$$

Let $m \geqslant 2$ be a rational integer and

$$f(x_1, \ldots, x_s) = \sum_{r=1}^s a_r x_r^m \tag{2}$$

be a polynomial with coefficients a_r in $\overline{K}^*$, the group of non-singular elements of $\overline{K}$. We say that $f(x_1, \ldots, x_s)$ is *totally indefinite*, if, for every k, $1 \leqslant k \leqslant h$,

$$f^{(k)}(x_1, \ldots, x_s) = \sum_{r=1}^s a_r^{(k)} x_r^m = 0$$

has a real solution with all $x_1, \ldots, x_s$ *not* equal to zero.

Let $\mathfrak{O}$ denote the ring of integers of K. The object of this paper is to prove the following

THEOREM. *Let $f(x_1, \ldots, x_s)$ be a totally indefinite polynomial over $\overline{K}^*$ given by* (2). *Let*

$$f \neq \lambda\varphi(x_1, \ldots, x_s)$$

where $\lambda \in \overline{K}^*$ and $\varphi(x_1, \ldots, x_s)$ is a polynomial with coefficients in K. Let $mh \geqslant 4$ and

$$s \geqslant \max\left(2^m + 2,\ h2^{m-1}(m-1) + h^2 + h\right). \tag{3}$$

Then given any $\varepsilon > 0$, there exist integers $x_1, \ldots, x_s$ in $\mathfrak{O}$ not all zero, such that

$$\|f(x_1, \ldots, x_s)\| < \varepsilon.$$

We make the following remarks.

1. If $f = \lambda\varphi$ then L. G. Peck [5] has proved that if

$$s \geqslant s_0 = 1 + \mathrm{Max}\left(4m^{2h+3}, (2^{m-1}+h)mh\right)$$

then the equation $f(x_1, \ldots, x_s) = 0$ is solvable in integers $x_1, \ldots, x_s$ in $\mathfrak{O}$ not all zero so that we could have stated the theorem without the condition $f \neq \lambda\varphi$ by taking $s \geqslant s_0$ and satisfying (3).

2. In the cases $mh < 4$, namely $h = 1, m = 2, 3$, the theorem is still true because of the results of Davenport and Heilbronn [1]; however it would not follow from our methods given in this paper.

3. If $m = 2$, $h = 2$, then, in view of Hasse's theorem, it follows from the theorem above that

$$\|f(x_1, \ldots, x_s)\| < \varepsilon, \quad s \geqslant 10$$

and $f(x_1, \ldots, x_s) = \sum_{r=1}^{s} a_r x_r^2$, $a_r \in \bar{K}^*$. It appears that this must be true for $s \geqslant 5$, and in case $f \neq \lambda\varphi$ even with $s \geqslant 3$. However we cannot prove these.

4. In case $h = 1$, Davenport and Roth [3] have a more precise value for large m.

The theorem can also be proved for fields K not necessarily totally real. The proof requires only trivial changes and these are pointed out at the end. An important problem is whether s can be found *independent* of the degree h of K as has been shown in a very special case recently by us [7].

2. Notation. K is a totally real algebraic number field of degree h and $\mathfrak{O}$ is its ring of integers, $\omega_1, \ldots, \omega_h$ is a fixed basis of integers of K and $\varrho_1, \ldots, \varrho_h$ its complementary basis so that $\varrho_1, \ldots, \varrho_h$ is a basis of the ideal ϑ^{-1}, where ϑ is the different of K. $N\vartheta = |d|$ where d is the discriminant of K. If α and β are two elements of $\bar{K}$ we say that $\alpha \succ \beta$ if $\alpha - \beta$ is totally positive, that is $\alpha^{(k)} - \beta^{(k)} > 0$ for all k. For $\alpha \in \bar{K}$, $\sigma(\alpha)$ and $N\alpha$ denote the *trace* and the *norm* of α.

$\mathscr{R}$ is the subset of $\alpha \in \bar{K}$ with $\alpha = \sum_{k=1}^{h} x_k \varrho_k$, $0 \leqslant x_k < 1$, $k = 1, \ldots, h$ and $\mathscr{B}_0$ is the subset of $\bar{K}$ of $\beta = \sum_{k=1}^{h} y_k \omega_k$ with $-1 \leqslant y_k < 1$, $k = 1, \ldots, h$. If $P > 0$ is a rational integer, $P\mathscr{B}_0$ denotes the set of $\beta = \sum_k y_k \omega_k$ with $-P \leqslant y_k < P$, $k = 1, \ldots, h$.

For a real number x, $[x]$ is the largest integer not exceeding x and $\langle x \rangle$ is the distance of x from the nearest integer. If f and g are two numbers or functions $f \ll g$ means $|f| \leqslant c\,|g|$ for some unspecified constant $c > 0$ depending on K and m. We also use the usual O and o symbols of Landau.

3. The method. For $\alpha \in \bar{K}$, define the exponential sum

$$S(\alpha) = S(\alpha, P) = \sum_{x \in P\mathscr{B}_0 \cap \mathfrak{O}} e^{2\pi i \sigma(\alpha x^m)} \tag{4}$$

so that x runs through all integers $x = \sum_k y_k \omega_k$ in K with $-P \leqslant y_k < P$, $k = 1, \ldots, h$. For $\beta \in \bar{K}$, put

$$L(\beta) = \prod_{k=1}^{h} \left(\frac{\sin \pi\beta^{(k)}}{\pi\beta^{(k)}}\right)^2 \tag{5}$$

where, if $\beta^{(k)} = 0$ for some k, the corresponding factor is to be replaced by 1. It is easy to see that if $y = (y^{(1)}, \ldots, y^{(h)})$ is in $\bar{K}$ and

$$\{dy\} = dy^{(1)} \ldots dy^{(h)} \tag{6}$$

then

$$\int_{-\infty}^{\infty} \cdots \int L(y)\, e^{2\pi i \sigma(\theta y)} \{dy\} = \begin{cases} 0 & \text{if} \quad \|\theta\| > 1, \\ \prod_{k=1}^{h} (1 - |\theta^{(k)}|) & \text{if} \quad \|\theta\| \leqslant 1 \end{cases} \tag{7}$$

where $\theta = (\theta^{(1)}, \ldots, \theta^{(h)}) \in \bar{K}$.

It is clearly enough to prove the theorem with $\varepsilon = 1$; for, then to obtain the theorem we have only to take $\varepsilon^{-1} f(x_1, \ldots, x_s)$ instead of $f(x_1, \ldots, x_s)$. In order to prove this we shall *assume* that for every $x_1, \ldots, x_s$ in $\mathfrak{O}$ not all zero

$$\|f(x_1, \ldots, x_s)\| \geqslant 1, \tag{8}$$

and thereby obtain a contradiction.

For $\alpha \in \bar{K}$, put

$$T(\alpha) = \prod_{j=1}^{s} S(a_j \alpha).$$

If $\alpha = \sum_{k=1}^{h} \alpha_k \varrho_k$, then

$$\int_{-\infty}^{\infty} \cdots \int T(\alpha) L(\alpha)\, d\alpha_1 \ldots d\alpha_h = |d| \int_{-\infty}^{\infty} \cdots \int T(\alpha) L(\alpha)\, d\alpha^{(1)} \ldots d\alpha^{(h)}.$$

On the other hand, the right hand integral reduces to

$$|d| \sum_{\substack{x_1, \ldots, x_s \in P\mathscr{B}_0 \cap \mathfrak{O} \\ \|\sum_j a_j x_j^m\| \leqslant 1}} N\left(1 - \Big|\sum_{j=1}^{s} a_j x_j^m\Big|\right).$$

However with the assumption (8) this reduces to $|d|$. Thus

$$(9)\qquad \int_{-\infty}^{\infty}\!\!\ldots\int T(\alpha)L(\alpha)\,d\alpha_1\ldots d\alpha_h = |d|.$$

Since $f \neq \lambda\varphi$, $\lambda \epsilon \overline{K}^*$, let us assume, without loss in generality that $a_1 a_2^{-1}$ is *not* in K. Put

$$(10)\qquad c_1 = \mathrm{Max}(\|a_1^{-1}\|, \|a_2^{-1}\|).$$

We divide the whole space $-\infty < \alpha^{(k)} < \infty$, $k = 1, \ldots, h$, into three mutually non-overlapping subsets E_1, E_2, E_3 defined by

$$(11)\qquad \begin{aligned} E_1 &= \{\alpha \epsilon \overline{K} \mid \|\alpha\| \leqslant c_1 P^{1-m-\delta}\}, \\ E_2 &= \{\alpha \epsilon \overline{K} \mid c_1 P^{1-m-\delta} < \|\alpha\| \leqslant P^{\delta/4h^2}\}, \\ E_3 &= \{\alpha \epsilon \overline{K} \mid \|\alpha\| > P^{\delta/4h^2}\} \end{aligned}$$

where $0 < \delta < 1$ to be fixed precisely later. Let us put

$$(12)\qquad J_i = \int_{E_i}\!\!\ldots\int T(\alpha)L(\alpha)\,d\alpha_1\ldots d\alpha_h, \quad i = 1, 2, 3.$$

We then show that J_1 has an estimate from below involving P (Lemma 3). For J_2 and J_3 we shall obtain upper estimates involving P which is strictly of lower order than that of the lower estimate for J_1 (Lemmas 13 and 4). Since by (8)

$$J_1 + J_2 + J_3 = |d|,$$

for P tending to infinity this would lead to a contradiction. This would mean that our assumption (8) is false. Therefore our theorem would be proved.

4. A lower estimate for J_1. For $P > 0$, define $Y(P) = \{x \epsilon \overline{K} \mid \|x\| < P\}$ and for $\theta \epsilon \overline{K}$, put

$$I(\theta) = \int_{Y(P)}\!\!\ldots\int e^{2\pi i\sigma(\theta x^m)} \prod_{k=1}^{h} dx^{(h)}.$$

Let γ be any number in K. Then $(\gamma)\vartheta = \mathfrak{b}\cdot\mathfrak{a}_\gamma^{-1}$ for two integral ideals $\mathfrak{a}_\gamma$ and $\mathfrak{b}$ of K which are coprime. Put $\mathfrak{a}_\gamma = \mathfrak{O}$ if $\gamma = 0$. Put

$$G(\gamma) = N\mathfrak{a}_\gamma^{-1} \sum_{\mu (\mathrm{mod}\, \mathfrak{a}_\gamma)} e^{2\pi i\sigma(\mu^m\gamma)}$$

μ running over a complete system of representatives of residue classes of $\mathfrak{O}$ mod $\mathfrak{a}_\gamma$. We have now the following

LEMMA 1 (Siegel [10], p. 128). *Let $\alpha \epsilon \overline{K}$ and $\gamma \epsilon K$ with $N\mathfrak{a}_\gamma \leqslant P^{1-\delta}$ and $\|\alpha-\gamma\| \leqslant c_1/N\mathfrak{a}_\gamma \cdot P^{1-m-\delta}$, $0 < \delta < 1$. Then*

$$(13)\qquad S(\alpha) = \frac{G(\gamma)}{\sqrt{|d|}} I(\alpha-\gamma) + O(P^{h-\delta}).$$

Moreover

$$(14)\qquad I(\alpha-\gamma) = O\big(P^h N(\mathrm{Min}(1, P^{-1}|\alpha-\gamma|^{-1/m}))\big).$$

LEMMA 2.

$$(15)\qquad \Big|\int_{E_1}\!\!\ldots\int \prod_{j=1}^{s} I(a_j\alpha)L(\alpha)\,d\alpha^{(1)}\ldots d\alpha^{(h)}\Big| \gg (NP)^{s-m}.$$

Proof. To prove this inequality, it is clearly enough to show that for some $\varrho > 0$ we have

$$(16)\qquad \int_{\|\alpha\|\geqslant c_1 P^{1-m-\delta}}\!\!\ldots\int \prod_{j=1}^{s} I(a_j\alpha)L(\alpha)\,d\alpha^{(1)}\ldots d\alpha^{(h)} \ll (NP)^{s-m-\varrho}$$

and

$$(17)\qquad \int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} \prod_{j=1}^{s} I(a_j\alpha)L(\alpha)\,d\alpha^{(1)}\ldots d\alpha^{(h)} \gg (NP)^{s-m}.$$

If we take (14) with $\gamma = 0$ we get an upper bound for the left side of (16) as

$$(NP)^s \int_{\|\alpha\|\geqslant P^{1-m-\delta}} \prod_{j=1}^{s} N\big(\mathrm{Min}(1, P^{-1}|a_j\alpha|^{-1/m})\big)L(\alpha)\,d\alpha^{(1)}\ldots d\alpha^{(h)}$$

$$\ll (NP)^s \int_{\|\alpha\|\geqslant P^{1-m-\delta}} N\big(\mathrm{Min}(1, |P^m\alpha|^{-s/m})\big)L(\alpha)\,d\alpha^{(1)}\ldots d\alpha^{(h)}$$

$$\ll (NP)^s \sum_{k=1}^{h} \int_{|\alpha^{(k)}|=P^{1-m-\delta}} |\alpha^{(k)}|^{-s/m}P^{-s}\,d\alpha^{(k)}\Big(\int_0^{P^{-m}} dx + \int_{P^{-m}}^{|\alpha^{(k)}|} P^{-s}x^{-s/m}\,dx\Big)^{h-1}$$

which gives after simplification

$$(18)\qquad \ll (NP)^{s(1-1/h)} \int_{P^{1-m-\delta}}^{\infty} t^{-s/m}(P^{-m}+t^{1-s/m}P^{-s})^{h-1}\,dt.$$

Now

$$1+P^{m-s}t^{1-s/m} < 1+P^{m-s+(1-s/m)(1-m-\delta)} < 1+P^{m-s+(s-m)\left(1-\frac{1-\delta}{m}\right)}$$
$$< 1+P^{(m-s)(1-\delta/h)} < 2$$

since $0 < \delta < 1$ and $s > m$. Therefore (18) is

$$\ll (NP)^{(s-m)(1-1/h)} \int_{P^{1-m-\delta}}^{\infty} t^{-s/m}\,dt \ll (NP)^{(s-m)(1-1/h)} P^{(m+\delta-1)(s-m)/m} = (NP)^{s-m-\varrho}$$

where $\varrho = (s-m)(1-\delta)/hm > 0$. Thus (16) is proved.

We now prove (17). In view of (7), the left hand side of (17) is just

$$F = \int_{\|\sum_{k=1}^{s} a_k\eta_k^m\| \leqslant 1, \|\eta_k\| \leqslant P} N\Big(1-\Big|\sum_{k=1}^{s} a_k\eta_k^m\Big|\Big) \prod_{\substack{1\leqslant l\leqslant h\\ 1\leqslant k\leqslant s}} d\eta_k^{(l)}.$$

This integral clearly exceeds the integral F_1 with the same integrand but extended over the domain

$$\Big\| \sum_{k=1}^{s} a_k \eta_k^m \Big\| \leqslant 1, \quad 0 < \eta_k^{(l)} < P,\ 1 \leqslant k \leqslant s;\ 1 \leqslant l \leqslant h.$$

In the integral F_1, we make a change of variable $\eta_k^{(l)}$ to $(\eta_k^{(l)})^m$ and then obtain

$$F \geqslant \int\limits_{\substack{0<\eta_k^{(l)}<P^m \\ 1\leqslant k\leqslant s, 1\leqslant l\leqslant h}} V(\eta_1, \ldots, \eta_s) \prod_{j=1}^{s} (N\eta_j)^{-(1-1/m)} \prod_{k,l} d\eta_k^{(l)}, \tag{19}$$

where

$$V(\eta_1, \ldots, \eta_s) = \begin{cases} N\Big(1 - \Big|\sum_{k=1}^{s} a_k \eta_k\Big|\Big) & \text{if } \Big\|\sum_k a_k \eta_k\Big\| \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Observe now that in the domain of integration in (19) there exists a subdomain $\mathscr{C}$ of volume $\gg (NP)^{(s-1)m}$ on which $\|\sum_k a_k \eta_k\| \leqslant 1/2$ and on which the integrand is $\gg (NP)^{-s(m-1)}$. For, we know that, since f is totally indefinite, there exists for every j, $1 \leqslant j \leqslant h$ at least one pair a_{p_j}, a_{q_j} such that $a_{p_j}^{(j)} > 0$ and $a_{q_j}^{(j)} < 0$. For each index j, we fix positive real numbers $c_{1j}, \ldots, c_{sj}$ such that $\sum_{k=1}^{s} a_k^{(j)} c_{kj}^m = 0$ with $0 < c_{kj} \leqslant 1/2$. This is clearly possible. If $b > 0$ is sufficiently small, the domain $\mathscr{C}$ defined by

$$\begin{gathered} |P^{-m}\eta_k^{(j)} - c_{k_j}^m| < b, \quad k \neq p_j,\ 1 \leqslant k \leqslant s, \\ \Big|\sum_{k=1}^{s} a_k^{(j)} \eta_k^{(j)}\Big| < b \end{gathered} \tag{20}$$

is contained in the domain of integration in (19). Its volume is $\gg (NP)^{(s-1)m}$. Furthermore the integrand is on this set $\gg (NP)^{-s(m-1)}$. Therefore

$$F_1 \gg \int\limits_{\mathscr{C}} V(\eta_1, \ldots, \eta_s) \prod_{j=1}^{s} (N\eta_j)^{-(1-1/m)} \prod_{k,l} d\eta_k^{(l)} \gg (NP)^{(s-1)m - s(m-1)} = (NP)^{s-m}.$$

This proves (17) and consequently Lemma 2.

Lemma 3. *For $s \geqslant \operatorname{Max}(m^2+1, h\delta^{-1})$ we have $|J_1| \gg (NP)^{s-m}$.*

Proof. From Lemma 1, we obtain

$$\begin{aligned} T(a) &= \prod_{j=1}^{s} S(a_j a) \\ &= \prod_{j=1}^{s} I\big(a_j(a-\gamma)\big) + O\Big((NP)^{s-\delta/h} N\big(\min(1, |P^m(a-\gamma)|^{-(s-1)/m})\big)\Big) + \\ &\qquad + O\big((NP)^{s(1-\delta/h)}\big). \end{aligned} \tag{21}$$

On the other hand, it is known from Siegel ([9], p. 335) that

$$\int_0^1 N\Big(\min\Big(1, \Big|P^m \sum_{k=1}^{h} x_k \omega_k\Big|^{-a/m}\Big)\Big)\, dx_1 \ldots dx_h = O(NP)^{-m}. \tag{22}$$

Using (21) and (22) we get, taking $\gamma = 0$ and $s \geqslant \operatorname{Max}(m^2+1, h/\delta)$

$$\begin{aligned} \Big| J_1 - &\int\limits_{\|a\| \leqslant c_1 P^{1-m-\delta}} \prod_{j=1}^{s} I(a_j a) L(a)\, da_1 \ldots da_h \Big| \\ &\ll (NP)^{s-\delta/h} \int\limits_{\|a\| \leqslant c_1 P^{1-m-\delta}} N\big(\min(1, |P^m a|^{-(s-1)/m})\big) L(a)\, da_1 \ldots da_h + \\ &\qquad + (NP)^{s-s\delta/h} \int\limits_{\|a\| \leqslant c_1 P^{1-m-\delta}} L(a)\, da_1 \ldots da_h. \end{aligned}$$

Since $L(\beta) \ll 1$ for $\beta \in \overline{K}$ and $L(\gamma) \ll (N\gamma)^{-2}$ for $\gamma \in \overline{K}^*$, we get, for the above, the estimate

$$\ll (NP)^{s-\delta/h-m} + (NP)^{s(1-\delta/h)-m+1-\delta} \ll (NP)^{s-m-\delta/h}, \quad \text{if } s \geqslant h/\delta. \tag{23}$$

Lemma 2 now combined with (23) proves Lemma 3.

5. An upper bound for J_3. By definition, J_3 is given by

$$J_3 = \int\limits_{\|a\| > P^{\delta_1}} T(a) L(a)\, da_1 \ldots da_h$$

where $\delta_1 = \delta/4h^2$. We now prove

Lemma 4. *For $s \geqslant 2^m$ and P sufficiently large,*

$$J_3 = o\big((NP)^{s-m}\big).$$

Proof. Using Hölder's inequality, we get

$$J_3 \leqslant \prod_{j=1}^{s} \Big\{ \int\limits_{\|a\| > P^{\delta_1}} |S(a_j a)|\ L(a)\, da \Big\}^{1/s};$$

where $da = da_1 \ldots da_h$.

It is therefore enough to find an upper estimate for

$$\int\limits_{\|a\| > P^{\delta_1}} |S(a_j a)|^s L(a)\, da,$$

which is the same as finding an estimate for

$$\int\limits_{\|a\| > P^{\delta_1}} |S(a)|^s L(a_j^{-1} a)\, da.$$

From Körner [4], Satz 5, we obtain for $s \geqslant 2^m$

$$\int\limits_{\mathfrak{R}} \Big| S\Big(\sum_{k=1}^{h} a_k \varrho_k\Big)\Big|^s da \ll (NP)^{s-m} (\log P)^{c_3} \tag{24}$$

for some $c_3 > 0$ depending only on K and m. The domain $\|\alpha\| > P^{\delta_1}$ can be covered, without gaps and overlapping, by cubes

$$F(g_1, \ldots, g_h) = \Big\{\alpha = \sum_i \alpha_i \varrho_i \in \bar{K} \mid \; g_i \leqslant \alpha_i < g_{i+1}, \; i = 1, \ldots, h\Big\}$$

$g_1, \ldots, g_h$ being rational integers. Each cube is contained in a box of type $\gamma + c_4 \mathscr{R}$ where $\gamma = (g_1, \ldots, g_h) \in \bar{K}$ and c_4 is a constant independent of γ. Since $S(\alpha+\lambda) = S(\alpha)$ for $\lambda \in \vartheta^{-1}$, we have

$$(25) \qquad \int\limits_{F(g_1,\ldots,g_h)} \Big| S\Big(\sum_k \alpha_k \varrho_k\Big)\Big|^s L(\alpha)\, d\alpha \leqslant \sup_{\alpha \in F(g_1,\ldots,g_h)} L(\alpha) \int\limits_{\gamma + c_4\mathscr{R}} |S(\alpha)|^s\, d\alpha$$
$$\leqslant \sup_{\alpha \in F(g_1,\ldots,g_h)} L(\alpha)(NP)^{s-m}(\log P)^{c_3}.$$

For the cubes covering the domain $\|\alpha\| > P^{\delta_1}$ we can ensure that at least one of the numbers $g_i \gg P^{\delta_1}$. From the properties of $L(\alpha)$ we get

$$J_3 \ll \sum_{\beta=[P^{\delta_1}]}^{\infty} \beta^{-2} \Big(\sum_{\beta=1}^{\infty} \beta^{-2}\Big)^{h-1} (NP)^{s-m} (\log P)^{c_3} \ll P^{-\delta_1} (NP)^{s-m+\delta_1/2h}$$

for large P. Thus if P is large so that

$$(26) \qquad (\log P)^{c_3} \leqslant P^{\delta/8h^2}$$

we have

$$J_3 \ll (NP)^{s-m-\delta/8h^3}, \qquad s \geqslant 2^m.$$

This proves Lemma 4.

6. Estimation of J_2. Our object is to first obtain an estimation of $S(\alpha)$. This is given in Lemma 10. It is dependent on certain lemmas which are generalizations of lemmas of C. P. Ramanujam and Davenport. Since the proofs can be obtained by suitable modifications of those of Ramanujam and Davenport, we state them without proof.

We first begin by defining 'major arcs' after Ramanujam [8].

Let $\gamma \in K$ and $\vartheta(\gamma) = \mathfrak{b}\mathfrak{a}_\gamma^{-1}$ where $\mathfrak{a}_\gamma$ and $\mathfrak{b}$ are coprime integral ideals. The 'major arc' B_γ corresponding to γ with $N\mathfrak{a}_\gamma \leqslant P^{1-\delta}$ is defined as the set of $x = \sum_{k=1}^{h} x_k \varrho_k$ in $\mathscr{R}$ such that if $\gamma = \sum_{k=1}^{h} \gamma_k \varrho_k$, then

$$(27) \qquad |x_k - \gamma_k| < N\mathfrak{a}_\gamma^{-1} P^{1-m-\delta}.$$

Put $\mathfrak{m} = \mathscr{R} - \bigcup_\gamma B_\gamma$ where γ runs over elements of K with $N\mathfrak{a}_\gamma \leqslant P^{1-\delta}$. Denote by $\mathfrak{m}_\theta$ the set of α in $\mathfrak{m}$ such that there does not exist $\lambda \neq 0$, $\lambda \in \mathfrak{O} \cap P^{(m-1)\theta+\delta}\mathscr{B}_0$ and a $\mu \in \vartheta^{-1}$ such that

$$(28) \qquad \lambda\alpha - \mu = \sum_{k=1}^{h} \varepsilon_k \varrho_k, \qquad |\varepsilon_k| < P^{-m+(m-1)\theta+\delta}$$

where $0 < \theta < 1$. We then have

LEMMA 5. *If* $\theta = (1-(h+1)\delta)/2h(m-1)$, *then* $\mathfrak{m}_\theta = \mathfrak{m}$.

This lemma is analogous to the statement on p. 701 in [8] and is proved in a similar way.

By applying Weyl's lemma $(m-1)$ times, we have from Siegel [10],

LEMMA 6. *For* $\alpha \in \bar{K}$,

$$|S(\alpha)|^{2^{m-1}}$$
$$= O\big((NP)^{2^{m-1}-m}\big) \sum_{\lambda_1,\ldots,\lambda_{m-1} \in P\mathscr{B}_0} \prod_{k=1}^{h} \mathrm{Min}\big(P, \langle m!\, L_k(\alpha, \lambda_1, \ldots, \lambda_{m-1})\rangle^{-1}\big)$$

where $\alpha\lambda_1 \ldots \lambda_{m-1} = \sum_{k=1}^{h} \varrho_k L_k(\alpha; \lambda_1, \ldots, \lambda_{m-1})$ *and* $\lambda_1, \ldots, \lambda_{m-1}$ *are integers in* K.

LEMMA 7. *Suppose that* $|S(\alpha)| > (NP)^{1-\varrho}$ *with* $0 < \varrho < 1$. *Let* N *denote the number of integer points* $\lambda_1, \ldots, \lambda_{m-1}$ *in* $P\mathscr{B}_0$ *such that*

$$\langle m!\, L_k(\alpha; \lambda_1, \ldots, \lambda_{m-1})\rangle < P^{-1}, \qquad 1 \leqslant k \leqslant h,$$

then

$$N \gg (NP)^{m-1-2^{m-1}\varrho} (\log P)^{-h}.$$

The proof of this is similar to Lemma 1.2 in [8]. It is to be noted that the proof of Lemma 7 uses Lemma 6.

The next lemma is a generalisation of Lemma 7.

LEMMA 8. *Let* $|S(\alpha)| > (NP)^{1-\varrho}$ *with* $0 < \varrho < 1$ *and let* N_{m-1} *denote the number of integer points* $X_1, \ldots, X_{m-1}$ *satisfying*

$$X_1, \ldots, X_{m-1} \in P^\theta \mathscr{B}_0, \qquad \langle m!\, L_k(\alpha, X_1, \ldots, X_{m-1})\rangle < P^{-m+(m-1)\theta}$$

for $0 < \theta < 1$. *Then*

$$N_{m-1} \gg (NP)^{(m-1)\theta - 2^{m-1}\varrho} (\log P)^{-h}.$$

Proof. From the definition, we have

$$m!\, L_p(\alpha; \lambda_1, \ldots, \lambda_{m-1}) = m!\, \sigma(\alpha\lambda_1 \ldots \lambda_{m-1}\omega_p).$$

This equals

$$m! \sum_{q=1}^{h} \lambda_{iq}\, \sigma(\alpha\lambda_1 \ldots \lambda_{i-1}\lambda_{i+1} \ldots \omega_p\omega_q)$$

where $\lambda_i = \sum \lambda_{iq}\omega_q$. Fix $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{m-1}$. Then the coefficient of λ_{iq} in $m!\, L_p(\alpha; \lambda_1, \ldots, \lambda_{m-1})$ is $m!\, \sigma(\alpha\lambda_1 \ldots \lambda_{i-1}\lambda_{i+1} \ldots \lambda_{m-1}\, \omega_p\omega_q)$. This is also the coefficient of λ_{ip} in $m!\, L_q(\alpha; \lambda_1, \ldots, \lambda_{m-1})$. Therefore fixing $\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_{m-1}$ in $P\mathscr{B}_0$ and regarding the h linear forms $m!\, L_p(\alpha; \lambda_1, \ldots, \lambda_{m-1})$ as symmetric linear forms in λ_{ip} we can apply Lemma 3.3 of Davenport [2] (see also [1], Lemma 28).

Let N_i $(0 \leqslant i \leqslant m-1)$ denote the number of integral points $X_1, \ldots, X_{m-1}$ such that

$$(29)\qquad \begin{gathered} X_1, \ldots, X_i \in P^\theta \mathscr{B}_0, \\ X_{i+1}, \ldots, X_{m-1} \in P\mathscr{B}_0, \\ \langle m!\, L_p(a; X_1, \ldots, X_{m-1})\rangle < P^{i\theta-(i+1)} \end{gathered}$$

where if $i = 0$ the first set is empty and if $i = m-1$ the second set is. We now assume as induction hypothesis that for $0, 1, 2, \ldots, i$,

$$(30)\qquad N_i \gg (NP)^{m-1-2^{m-1}\varrho+i(\theta-1)}(\log P)^{-h}.$$

Clearly for $i = 0$, this is Lemma 7 with $N_0 = N$.

Choose now, in Davenport's notation,

$$(31)\qquad \begin{aligned} Z_1 &= P^{\frac{i+2}{2}(\theta-1)}, \\ Z_2 &= P^{\frac{i}{2}(\theta-1)}, \\ A &= P^{\frac{2+i}{2}-\frac{i\theta}{2}}. \end{aligned}$$

Fix $X_1, \ldots, X_i$ in $P^\theta\mathscr{B}_0$ and $X_{i+1}, \ldots, X_{m-1}$ in $P\mathscr{B}_0$. Applying Davenport's lemma, we get

$$N_{i+1} \gg N_i\left(\frac{Z_1}{Z_2}\right)^h.$$

Combining this with (30), we get

$$N_{i+1} \gg (NP)^{(i+1)(\theta-1)-2^{m-1}\varrho+m-1}(\log P)^{-h}.$$

This shows that (30) is true for $i+1$ and so for $i = m-1$. This proves the lemma.

We deduce at once

LEMMA 9. *For $a \in \mathfrak{m}_\theta$ and large P, $|S(a)| < (NP)^{1-\theta/2^{m-1}}$.*

It is proved in Lemma 5 that for $\theta = (1-(h+1)\delta)/2h(m-1)$, $\mathfrak{m} = \mathfrak{m}_\theta$. Let us now choose

$$0 < \delta = 1/(2^{m-1}(m-1)+h+1) < 1.$$

Then $\theta/2^{m-1} = (1-(h+1)\delta)/2^m h(m-1) = \delta/2h$. On the other hand, from Lemma 9 we have for $a \in \mathfrak{m}_\theta$ and large P, $|S(a)| < (NP)^{1-\theta/2^{m-1}}$. We therefore obtain

LEMMA 10. *For $a \in \mathfrak{m}_\theta$ and $\delta = 1/((h+1)+2^{m-1}(m-1))$, we have*

$$|S(a)| < (NP)^{1-\delta/2h}.$$

We now generalize Dirichlet's approximation theorem to algebraic number fields as follows.

LEMMA 11. *Let ξ be an element of $\overline{K}$ but not of K. There exist infinitely many pairs of integers $\mu_i, \nu_i \neq 0$ in $\mathfrak{O}$ such that*

(i) *the g.c.d of μ_i, ν_i belongs to a fixed finite set of integral ideals in K;*

(ii) *ν_i and $\|\nu_i\|$ are of the same order of magnitude;*

(iii) *$\|\nu_i\|$ tends to infinity with i*

and

(iv) $\|\xi-\mu_i/\nu_i\| \ll \|\nu_i\|^{-(h+1)/h}$.

Proof. It is well-known [6] that if $x_1, \ldots, x_h$ are real numbers of which one at least is irrational there exist infinitely many h-tuples $p_i^{(1)}, \ldots, p_i^{(h)}$ and integers q_i, $i = 1, 2, \ldots$ such that

$$(32)\qquad |x_j - p_i^{(j)}/q_i| < q_i^{-(1+1/h)}, \quad i = 1, 2, \ldots;\ 1 \leqslant j \leqslant h.$$

If $\xi = \sum_i x_i\omega_i$ $(x_i \in \boldsymbol{R})$ is an element of $\overline{K}$ but not of K, then at least one x_i is irrational and so putting $a_i = \sum_{j=1}^h p_i^{(j)}\omega_j$, we get

$$(33)\qquad |\xi - a_i/q_i| \ll q_i^{-(1+1/h)}$$

absorbing $\sup_{1\leqslant i\leqslant h}|\omega_i|$ in the constant on the right side of (33). Similar inequalities exist for the other conjugates and thus

$$(34)\qquad \|\xi - a_i/q_i\| \ll q_i^{-(1+1/h)}.$$

Let $\mathfrak{a}_i = a_i\mathfrak{O}+q_i\mathfrak{O}$. Then $a_i\mathfrak{a}_i^{-1}$ and $q_i\mathfrak{a}_i^{-1}$ are integral ideals which are coprime. They are not necessarily principal ideals. In each ideal class of K choose an integral ideal, say of minimum norm and denote by $\boldsymbol{F}$ the totality of these finitely many integral ideals. Multiplying $a_i\mathfrak{a}_i^{-1}, q_i\mathfrak{a}_i^{-1}$, if necessary, by some ideal in $\boldsymbol{F}$ we get

$$\frac{a_i}{q_i} = \frac{\mu_i}{\nu_i}$$

where μ_i, ν_i are in $\mathfrak{O}$ having at most one of the ideals in $\boldsymbol{F}$ as greatest common divisor. We can multiply μ_i and ν_i, if necessary, by a unit of $\mathfrak{O}$ so that ν_i and all its conjugates have the same order $|N\nu_i|^{1/h}$.

In view of the linear independence of $\omega_1, \ldots, \omega_h$, it is clear that the number of h-tuples $(p_i^{(1)}/q_i, \ldots, p_i^{(h)}/q_i)$ and so the μ_i/ν_i are infinite in number. That $\|\nu_i\| \to \infty$ follows trivially from above.

In the sequel, we assume that P tends to infinity through a sequence of natural numbers of the form $[\|\nu_i\|^{1+1/h}]$ with $\xi = a_1 a_2^{-1}$.

We shall now prove the following fundamental lemma.

LEMMA 12. *Let*

1) *$a_1 a_2^{-1}$ be in $\overline{K}$ but not in K,*

2) $0 < \delta = 1/(2^{m-1}(m-1)+h+1) < 1$,

3) $(\log P)^{c_3} \leqslant P^{\delta/8h^2}$.

Then for $\alpha \in E_2$, $\operatorname{Min}(|S(a_1\alpha)|, |S(a_2\alpha)|) \ll (NP)^{1-\delta/2h}$.

Proof. Put $a_i\alpha = \sum_{j=1}^{h} c_j^{(i)} \varrho_i$ where $c_j^{(i)}$ are real numbers and $i = 1, 2$. Let $\mu_i = \sum_{j=1}^{h} b_j^{(i)} \varrho_j$ where $b_j^{(i)}$ are real numbers such that $0 \leq b_j^{(i)} < 1$, $j = 1, \ldots, h$; $i = 1, 2$ and $c_j^{(i)} - b_j^{(i)}$ are rational integers. Then $\mu_i \in \mathscr{R}$ and further

$$S(a_i\alpha) = S(\mu_i), \quad i = 1, 2.$$

Suppose $\mu \in \overline{K}$ is in a major arc B_γ for some γ with $(\gamma)\vartheta = \mathfrak{b}\mathfrak{a}_\gamma^{-1}$, $(\mathfrak{a}_\gamma, \mathfrak{b}) = 1$ and $N\mathfrak{a}_\gamma \leq P^{1-\delta}$. Then $\|\mu - \gamma\| \ll N\mathfrak{a}_\gamma^{-1} P^{1-m-\delta}$. Lemma 1 then gives

$$|S(\mu)| \ll NP \cdot N\mathfrak{a}_\gamma^{-1} \Big| \sum_{x (\operatorname{mod} \mathfrak{a}_\gamma)} e^{2\pi i \sigma(\gamma x^m)} \Big| + O\big((NP)^{1-\delta/h}\big).$$

The generalized 'Gauss sum' has the estimate (see [10])

$$N\mathfrak{a}_\gamma^{-1} \Big| \sum_{x (\operatorname{mod} \mathfrak{a}_\gamma)} e^{2\pi i \sigma(\gamma x^m)} \Big| \ll (N\mathfrak{a}_\gamma)^{-1/m}.$$

We therefore obtain, if say μ_1 (or μ_2) $\in B_\gamma$, then for $i = 1$ (or 2),

$$|S(a_i\alpha)| \ll NP(N\mathfrak{a}_\gamma)^{-1/m} + O\big((NP)^{1-\delta/h}\big). \tag{35}$$

On the other hand if μ_1 does not lie in any B_γ with $N\mathfrak{a}_\gamma \leq P^{1-\delta}$, then by the definition of $\mathfrak{m}$, $\mu_1 \in \mathfrak{m}$. By Lemma 5, for small θ, $\mu_1 \in \mathfrak{m} = \mathfrak{m}_0$. Using Lemma 10 we get, with the δ in that lemma

$$|S(a_1\alpha)| = |S(\mu_1)| \ll (NP)^{1-\delta/2h}. \tag{36}$$

Thus if either μ_1 or μ_2 is in $\mathfrak{m}$ or in some B_γ with $P^{m\delta/h} \leq N\mathfrak{a}_\gamma \leq P^{1-\delta}$ (δ as in Lemma 10), then (35) and (36) hold and

$$\operatorname{Min}(|S(a_1\alpha)|, |S(a_2\alpha)|) \ll (NP)^{1-\delta/2h}. \tag{37}$$

In these cases, Lemma 12 is proved.

In order to complete the proof, we have to show that if μ_1 and μ_2 lie in B_{γ_1} and B_{γ_2} respectively and

$$N\mathfrak{a}_{\gamma_i} \leq P^{1-\delta}, \quad N\mathfrak{a}_{\gamma_i} \leq P^{m\delta/h}, \quad i = 1, 2,$$

then (37) holds. What we are going to show is that this case does not even arise.

Let as before $a_j\alpha = \sum_{i=1}^{h} c_{i,j}\varrho_i$, $j = 1, 2$ and $\mu_j = \sum_{i=1}^{h} b_{i,j}\varrho_i$ so that $a_j\alpha - \mu_j = \sum_i (c_{i,j} - b_{i,j})\varrho_i$ is in ϑ^{-1}. Let $\gamma_j = \sum_{i=1}^{h} e_{i,j}\varrho_i$ with $(\gamma_j)\vartheta = \mathfrak{b}_j \mathfrak{a}_{\gamma_j}^{-1}$, $(\mathfrak{a}_{\gamma_j}, \mathfrak{b}_j) = 1$. Since $a_j\alpha - \mu_j \in \vartheta^{-1}$ we see that $(\mu_j - a_j\alpha - \gamma_j)\vartheta$ has the same denominator as $(\gamma_j)\vartheta$. If we put

$$\gamma_j' = -\mu_j + a_j\alpha + \gamma_j = \sum_{i=1}^{h} g_{i,j}\varrho_i$$

then

$$|c_{i,j} - g_{i,j}| \leq N\mathfrak{a}_{\gamma_j}^{-1} P^{1-m-\delta}. \tag{38}$$

We write $\gamma_j' = t_j u_j^{-1}$ as in Lemma 11 with t_j, u_j integers of K having at most a common divisor which is in the finite set F. Then $N\mathfrak{a}_{\gamma_j}$ is the same as $|Nu_j|$ except for a positive constant depending on K alone.

In (38), $g_{i,j}$ are rational numbers with denominator not exceeding $N\mathfrak{a}_{\gamma_j}$. If for a fixed j we have $g_{i,j} \neq 0$ for every i, then (38) gives

$$c_{i,j} = g_{i,j}\big(1 + O(P^{1-m-\delta})\big) \tag{39}$$

and therefore

$$a_j\alpha = t_j u_j^{-1}\big(1 + O(P^{1-m-\delta})\big) \tag{40}$$

where the constants implied by O depend only on K. (40) is to be understood as holding for all conjugates of the left and the right sides.

If however $g_{i,j} = 0$ for some i, then (39) is not true for that i. However for a given j, all $g_{i,j}$ cannot be zero since $\alpha \in E_2$ and so

$$\|\alpha\| > c_1 P^{1-m-\delta}, \quad c_1 = \operatorname{Max}(\|a_1^{-1}\|, \|a_2^{-1}\|).$$

(40) is thus eventually seen to be true after changing the constant in the O-term for these i if necessary and by multiplying by ϱ_k and summing over i from 1 to h.

From (42), we get

$$a_1 a_2^{-1} = \frac{t_1 u_2}{t_2 u_1}\big(1 + O(P^{1-m-\delta})\big).$$

We now use the fact that $a_1 a_2^{-1} \in \overline{K}$ and not in K. By Lemma 11, there exist infinitely many $c_0 d_0^{-1}$ in K with $|Nd_0| = O(\|d_0\|^h) \to \infty$ such that

$$|a_1 a_2^{-1} - c_0 d_0^{-1}| \ll \|d_0\|^{-(1+1/h)}.$$

Further P is of the same order as $\|d_0\|^{(h+1)/h}$. We maintain that the ideals $(c_0 d_0^{-1})$ and $(t_1 u_2 / t_2 u_1)$ are distinct, for, if they are the same, then for some unit $\eta \in \mathfrak{O}$,

$$c_0 d_0^{-1} = \eta \frac{t_1 u_2}{t_2 u_1}. \tag{41}$$

Then

$$\frac{(c_0 t_2 u_1)}{(c_0, d_0)} = \frac{(d_0 t_1 u_2)}{(c_0, d_0)}$$

since $c_0 \mathfrak{O}/(c_0, d_0)$ and $d_0 \mathfrak{O}/(c_0, d_0)$ are coprime, we have

(42) $$(t_2 u_1) = \frac{d_0 \mathfrak{O}}{(c_0, d_0)} \cdot \mathfrak{a}$$

where $\mathfrak{a}$ is some integral ideal. Since $\alpha \epsilon E_2$,

$$\|a_j \alpha\| < P^{\delta/4h^2}$$

with δ given by Lemma 10. Furthermore

$$|N u_j| \leqslant N \mathfrak{a}_{\gamma_j} \ll P^{m\delta/h}$$

and

(43) $$|N t_j| \ll |N(a_j \alpha)| |N u_j| \ll P^{(\delta/4h)+m\delta}.$$

Taking norms in (42) we see that, using (43)

(44) $$|N t_2 u_1| = O(P^{(\delta/h(2m+1/4)}).$$

On the other hand since $N\mathfrak{a} \geqslant 1$, (42) gives on taking norms

$$|N(t_2 u_1)| \geqslant \frac{|N(d_0)|}{N(c_0, d_0)} \cdot N\mathfrak{a} \gg P^{h^2/h+1}$$

since $N(c_0, d_0)$ is bounded independently of P and d_0 and its conjugates are of the order of magnitude of $P^{h/h+1}$. With δ given by Lemma 10, (42), (44) and (45) lead to a contradiction for $mh \geqslant 4$. Therefore (41) is untenable. In particular

$$\tau = \frac{c_0}{d_0} - \frac{t_1 u_2}{t_2 u_1} \neq 0.$$

We therefore get

(45) $$0 < |\tau| \ll \|d_0\|^{-(h+1)/h} + P^{-(m-1+\delta)} \ll P^{-1} \ll \|d_0\|^{-(h+1)/h}$$

by our choice of a special sequence for P, when $\|d_0\| \to \infty$.

(45) is true for all conjugates of τ and so

(46) $$0 < |N\tau| \ll \|d_0\|^{-(h+1)}.$$

But then $\tau = (c_0 t_2 u_1 - d_0 t_1 u_2)/d_0 t_2 u_1$ and so

(47) $$|N\tau| \gg \frac{|N(c_0 u_1 t_2 - d_0 u_2 t_1)|}{|N d_0| \cdot |N t_2| \cdot |N u_1|} \gg \frac{1}{(\|d_0\|^h |N t_2| \, |N u_1|)}.$$

From (46) and (47),

$$\frac{1}{\|d_0\|^{h+1}} \gg \frac{1}{\|d_0\|^h} \cdot \frac{1}{P^{(2m+1/4)\delta/h}}$$

and so

(48) $$\frac{1}{\|d_0\|} \gg \frac{1}{P^{\delta/h(2m+1/4)}} \gg \frac{1}{\|d_0\|^{(8m+1)(h+1)\delta/4h^2}}.$$

With δ as in Lemma 10, this is impossible for large $\|d\|$ since $mh \geqslant 4$. We thus arrive at a contradiction to μ_1 and μ_2 being in B_{γ_1} and B_{γ_2} as mentioned.

Lemma 12 is thus completely proved.

We now prove

LEMMA 13. *Under the hypothesis of the Theorem and with δ defined as in Lemma 10, we have*

$$J_2 = O\big((NP)^{s-m-\delta/h}\big).$$

Proof. Using Lemma 12 and the trivial estimate NP for $S(a_k \alpha)$, we get

$$J_2 \ll (NP)^{1-\delta/2h} \Big\{ \int\limits_{\substack{E_2 \\ |S(a_1\alpha)| \leqslant |S(a_2\alpha)|}} \prod_{j=2}^{s} |S(a_j \alpha)| L(\alpha)\, d\alpha + \\ + \int\limits_{\substack{E_2 \\ |S(a_2\alpha)| \leqslant |S(a_1\alpha)|}} \prod_{j\neq 2}^{s} |S(a_j \alpha)| L(\alpha)\, d\alpha \Big\}.$$

By an application of Hölder's inequality and by using (24) and by arguments similar to those in Lemma 4, we get

$$J_2 \ll (NP)^{1-\delta/2h} NP (NP)^{s-2-m+\delta/8h^3} (NP)^{\delta/4h}$$

provided P satisfies (26) and $s \geqslant 2^m + 2$. Thus

$$J_2 \ll (NP)^{s-m-\delta/8h}.$$

As indicated in § 3, Lemmas 3, 4 and 13 prove the theorem.

We make some remarks concerning the extension of the theorem to fields K which are not necessarily totally real.

Let $K = K^{(1)}, K^{(2)}, \ldots, K^{(r_1)}$ be the real conjugates of K and $K^{(r_1+1)} = \bar{K}^{(r_1+r_2+1)}, \ldots, K^{(r_1+r_2)} = \bar{K}^{(r_1+2r_2)}$ be the complex conjugates of K, so that $h = r_1 + 2r_2$. Instead of the kernel in (5), we may now take

(49) $$L(\beta) = \prod_{i=1}^{r_1} \left(\frac{\sin \pi \beta^{(i)}}{\pi \beta^{(i)}}\right)^2 \prod_{j=r_1+1}^{r_1+r_2} \left(\frac{J_1(4\pi|\beta^{(j)}|)}{\sqrt{2}|\beta^{(j)}|}\right)^2$$

where J_1 denotes the Bessel function of order 1 and we have the usual convention that for $\beta^{(j)} = 0$, the corresponding factor is to be taken as 1. (It is to be remarked that what is important is that the factors in the kernel be rapidly decreasing functions which are Fourier transforms of bounded C^∞ functions on $\boldsymbol{R}^1$ (respectively $\boldsymbol{R}^2$) not vanishing at the origin and having support in the unit interval (respectively the unit disc). Observe that for $\beta \in \overline{K}$, $\gamma \in \overline{K}^*$,

$$L(\beta) \ll 1, \quad L(\gamma) \ll \prod_{i=1}^{r_1} |\gamma^{(i)}|^{-2} \prod_{j=r_1+1}^{r_1+r_2} |\gamma^{(j)}|^{-3}.$$

For $\alpha \in \overline{K}$, if $\{d\alpha\}$ denotes the volume element $\prod_{i=1}^{r_1} d\alpha^{(i)} \prod_{j=r_1+1}^{r_1+r_2} d(\operatorname{Re}\alpha^{(j)})\, d(\operatorname{Im}\alpha^{(j)})$, then instead of (7), we have

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} L(y) e^{2\pi i \sigma(\theta y)} \{dy\} = \begin{cases} 0, & \text{if } \|\theta\| > 1 \\ \prod_{i=1}^{r_1} (1 - |\theta^{(i)}|) \prod_{j=r_1+1}^{r_1+r_2} \varphi(|\theta^{(j)}|), & \text{if } \|\theta\| \leqslant 1 \end{cases}$$

where, for complex z,

$$\varphi(z) = \begin{cases} 4\sin^{-1}(\sqrt{1-|z|^2}) - |z|\sqrt{1-|z|^2} & \text{for } 0 \leqslant |z| \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of our theorem may be checked to go through in the general case.

Finally we remark that using the same arguments as in [3], we can also establish the validity of the inequality $\|f(x_1, \ldots, x_s) + b\| < \varepsilon$ for any $b \in \overline{K}$, with integers $x_1, \ldots, x_s$ in $\mathfrak{O}$ not all zero, under the additional restriction that $m > (h+1)\big(1 - 2^{m-1}(m-1) + h + 1\big)^{-1}$.

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