ACTA ARITHMETICA
XX (1972)

On the probability that n and $f(n)$ are relatively prime III

by

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It is known that if n and m are randomly chosen integers then the probability that n is prime to m is $6/\pi^2$. In the preceding papers [1], [2] of this series I considered the following problem. Let $g(p)$ be an integer valued function defined on the set of primes p, and

$$f(n) = \sum_{p|n} g(p), \quad T(x) = \sum_{\substack{n\leqslant x\\(n,f(n))=1}} 1.$$

Is it true that $T(x)\sim 6x/\pi^2$? In [1], I studied the case $g(p) = p$; this particular problem was suggested to me by Professor Erdös. It was shown that

$$T(x) = \frac{6}{\pi^2}x + O\left(\frac{x}{(\log\log\log x)^{1/4}(\log\log\log\log x)^{3/4}}\right).$$

In [2] I called g a pseudo-polynomial if for all n and k,

$$g(n+k)\equiv g(n) \pmod k$$

and proved that if g satisfied some fairly natural conditions then $T(x)\sim 6x/\pi^2$. Clearly a polynomial with integer coefficients is a pseudo-polynomial, and to justify the definition I constructed a pseudo-polynomial which is not a polynomial at all.

We are now able to give much more precise information about the problem raised by Professor Erdös and treated in [1]. We have the following

THEOREM. *Let $T(x)$ denote the number of integers $n\leqslant x$ prime to the sum of their distinct prime factors. Then*

$$T(x) = \frac{6}{\pi^2}x + \frac{x}{\log x}\sum_{k=2}^{\infty} A_k(\log x)^{\mu(k)/\varphi(k)} + O\left(\frac{x}{\log x}\exp\left(\frac{(\log_2 x)(\log_4 x)}{(\log_3 x)}\right)\right)$$

where the sum extends over squarefree k and the series

$$\sum_{k=2}^{\infty} A_k$$

is absolutely convergent; in fact we will show that for any positive ε,

$$A_k = O(k^{-3/2+\varepsilon}).$$

As usual, $\log_r x$ denotes the iterated logarithm: $\log_2 x = \log\log x$ etc. In view of the error term we may, if we wish, regard the sum as being over just those k for which $\mu(k) > 0$.

The maximum value of $\mu(k)/\varphi(k)$ is 1/2, attained when $k = 6$. At the end of the proof we give a formula for A_k which is not very helpful in the general case, but does enable us to show that

$$A_6 > 0.$$

We therefore have

COROLLARY 1.

$$T(x) - \frac{6}{\pi^2} x \sim \frac{A_6 x}{\sqrt{\log x}}.$$

COROLLARY 2. *There exists an x_0 such that for $x \geqslant x_0$,*

$$T(x) > \frac{6}{\pi^2} x.$$

I do not yet know to what extent these results generalize to the case where g is a polynomial, still less a pseudo-polynomial. One difficulty s to find a sufficiently good estimate for the sum S_5 below. If a general formula held, the corresponding sum over k would be

$$\sum_{k=2}^{\infty} \sum_{\substack{j=1 \\ (j,k)=1}}^{k} A_{k,j} (\log x)^{\tau_g(k,j)/\varphi(k)}$$

where

$$\tau_g(k,j) = \sum_{\substack{h=1 \\ (h,k)=1}}^{k} e^{2i\pi g(h)j/k}.$$

This is not in general invariant over j as in the special case $g(h) = h$ considered in the present paper, and the double sum over k and j in the final formula is inevitably less striking. To derive the corresponding corollaries would involve calculating

$$\lambda(g) = \sup_{k,j} \boldsymbol{R} \frac{\tau_g(k,j)}{\varphi(k)} \quad (k \geqslant 2,\ |\mu(k)| = 1,\ (j,k) = 1).$$

Note that $\boldsymbol{R}\tau_g(k,j) < \varphi(k)$ in every case, for one of the conditions imposed on g in Hall [2] was that for every squarefree k, there is at least one a prime to k for which $k \nmid g(a)$. A result of Hua [4] implies that for polynomial g,

$$\max_j |\tau_g(k,j)| = o(\varphi(k))$$

so that $\lambda(g)$ is attained, and is strictly less than 1. I imagine that for the polynomials g satisfying some natural conditions including the one above, there exist constants $\lambda(g)$, $A(g)$ such that

$$T(x) - \frac{6}{\pi^2} x \sim A(g) \frac{x}{(\log x)^{1-\lambda(g)}}.$$

We now give the proof of our main result.

Notation. $C_1, C_2, \ldots$ will denote positive absolute constants, independent of all parameters unless written in the form $C_j(\varepsilon)$ when there is dependence on ε. They are understood to be large enough, or in some cases small enough, to ensure the validity of every formula in which they occur.

Proof of the Theorem. We have

$$T(x) = \sum_{n \leqslant x} \sum_{q | (n, f(n))} \mu(q) = \sum_{q \leqslant x} \mu(q) \sum_{\substack{m \leqslant x/q \\ q | f(mq)}} 1$$
$$= \sum_{q \leqslant \omega} \mu(q) \sum_{\substack{m \leqslant x/q \\ f_q(m) \equiv -f(q) (\mathrm{mod}\, q)}} 1 + \theta \sum_{\omega < q \leqslant x} \sum_{\substack{m \leqslant x/q \\ q | f(mq)}} 1 = S_1 + \theta S_2$$

say, where $-1 \leqslant \theta \leqslant 1$, and $f_q(m) = f(mq) - f(q)$. We introduce the function $f_q(m)$ as it has the advantage over $f(mq)$ of being additive. We investigate S_2 first.

Treatment of S_2. We require the following lemmas.

LEMMA 1. *For $q \leqslant \sqrt{x}$ and all a,*

$$\sum_{\substack{m \leqslant x \\ f(m) \equiv a (\mathrm{mod}\, q)}} |\mu(m)| \leqslant C_1 x \left(\frac{1}{\varphi(q)} + \frac{\log q}{\log x} \right).$$

This is Lemma 1 of Hall [1]. It was stated there for a prime modulus but as the proof did not depend on this we may replace it by the general modulus q.

LEMMA 2. *For $q \leqslant 9x$ and $0 \leqslant a \leqslant \delta q$, δ fixed < 1, we have that*

$$\sum_{\substack{m \leqslant x \\ f(m) \equiv -a (\mathrm{mod}\, q)}} |\mu(m)| \leqslant C_2(\delta) \frac{x \log x}{\varphi(q)}.$$

Proof. Except in the case $m=1$, the relation $f(m) = -a \pmod q$ implies $f(m) \geqslant (1-\delta)q$, for $f(m)$ is positive. The number of prime factors of m does not exceed

$$l = \frac{C_3 \log x}{\log\log x}$$

and so m is divisible by a prime $\tilde\omega \geqslant (1-\delta)ql^{-1}$. Hence if $m \geqslant 2$ and $f(m) \equiv -a \pmod q$, m has a divisor $d = m/\tilde\omega$ satisfying

$$d \leqslant H = \frac{xl}{(1-\delta)q}, \qquad \nu(d) = \nu(m)-1.$$

Therefore

$$\sum_{\substack{m\leqslant x\\ f(m)\equiv -a \pmod q}} |\mu(m)| \leqslant 1 + \sum_{\substack{m\leqslant x\\ f(m)\equiv -a \pmod q}} |\mu(m)| \sum_{\substack{d\mid m,\, d\leqslant H\\ \nu(d)=\nu(m)-1}} 1 \leqslant 1 + \sum_{d\leqslant H} \sum_{\substack{\tilde\omega\leqslant x/d\\ \tilde\omega\equiv -a-f(d) \pmod q}} 1.$$

Note that we have dropped the conditions $\tilde\omega \nmid d$, $|\mu(d)| = 1$.

We cannot apply the Brun–Titchmarsh estimate directly to the inner sum, since the necessary condition $q < x/d$ might not hold. But provided $2l \geqslant 1$, as we may suppose, the inner sum does not exceed

$$\pi\left(\frac{2lx}{(1-\delta)d};\, q,\, -a-f(d)\right) \leqslant \frac{lx}{(1-\delta)d\varphi(q)\log\{2lx/(1-\delta)dq\}}$$

and therefore

$$\sum_{\substack{m\leqslant x\\ f(m)\equiv -a \pmod q}} |\mu(m)| \leqslant 1 + \frac{lx}{(1-\delta)\varphi(q)} \sum_{d\leqslant H} \frac{1}{d\log(2H/d)}$$

$$\leqslant \frac{lx\log\log H}{(1-\delta)\varphi(q)} \leqslant C_2(\delta)\frac{x\log x}{\varphi(q)}.$$

This completes the proof. The condition $a \leqslant \delta q$ seems rather unnatural, nevertheless it is satisfied in the application with $\delta = 5/6$.

We now deduce from Lemmas 1 and 2 estimates for similar sums over all m rather than squarefree m; this was done in the previous papers by different methods, and we adopt that of Hall [2]. We have the following extension of Lemma 3 of that paper:

LEMMA 3. *Let $Q(x, m)$ denote the number of integers $n \leqslant x$ whose squarefree kernel is m, that is, for which*

$$\prod_{p\mid n} p = m.$$

Then for non-negative r,

$$\sum_{m\leqslant x} \{Q(x, m)\}^r \leqslant x\exp\exp\{C_7 r\log(r+1)\}.$$

The method of proof is the same as before, and is due to Erdös. We have

$$\sum_{m\leqslant x} \{Q(x, m)\}^r = \sum_{k=1}^{\infty} k^r \sum_{\substack{m\leqslant x\\ Q(x,m)=k}} 1 \leqslant \sum_{k=1}^{\infty} k^r \sum_{\substack{m\leqslant x\\ Q(x,m)\geqslant k}} 1.$$

The inner sum does not exceed

$$\frac{x}{k^{r+2}} + \sum_{\substack{x/k^{r+2}\leqslant m\leqslant x\\ Q(x,m)\geqslant k}} 1$$

and we show that this last sum is $O(x/k^{r+2})$. If the squarefree kernel of n is m, and $m \geqslant x/k^{r+2}$, then n/m is a product of primes dividing m and not exceeding k^{r+2}. If the number of such primes is s, $Q(x, m)$ is the number of solutions of

$$a_1\log p_1 + a_2\log p_2 + \ldots + a_s\log p_s \leqslant \log(x/m), \qquad 0 \leqslant a_i \in Z,$$

which does not exceed the number of solutions of

$$a_1 + a_2 + \ldots + a_s \leqslant \frac{(r+2)\log k}{\log 2}.$$

We proved in [2] that this does not exceed

$$\frac{1}{s!}\left(s + \frac{(r+2)\log k}{\log 2}\right)^s \leqslant \left\{e\left(1 + \frac{(r+2)\log k}{s\log 2}\right)\right\}^s.$$

If $Q(x, m) \geqslant k$ this implies that

$$s \geqslant \frac{C_4\log k}{\log(r+2)}.$$

Now the number of m's not exceeding x with at least s prime factors not exceeding k^{r+2} is

$$\leqslant \sum_{p_i\leqslant k^{r+2}} \left[\frac{x}{p_1p_2\ldots p_s}\right] \leqslant \frac{x}{s!}\left(\sum_{p\leqslant k^{r+2}} \frac{1}{p}\right)^s \leqslant \frac{x}{s!}\left(\log(r+2) + \log\log k + C_5\right)^s$$

$$\leqslant x\left(\frac{e\{\log(r+2) + \log\log k + C_5\}}{s}\right)^s.$$

This is a decreasing function of s if

$$s \geqslant \log(r+2) + \log\log k + C_5$$

and this is true of the s above if $k \geqslant k_0(r)$. Hence for these k,

$$\sum_{\substack{x/k^{r+2}\leqslant m\leqslant x\\ Q(x,m)\geqslant k}} 1 \leqslant x\left(\frac{e\{\log(r+2) + \log\log k + C_5\}\log(r+2)}{C_4\log k}\right)^{C_4\log k/\log(r+2)}.$$

This does not exceed x/k^{r+2} if

$$C_4 \log k \geqslant \big(e\{\log(r+2)+\log\log k + C_5\}\log(r+2)\big)\exp\frac{(r+2)\log(r+2)}{C_4}.$$

That is, if

$$k \geqslant \exp\exp\big(C_6(r+2)\log(r+2)\big) = k_1(r)$$

say. (Clearly $k_1(r) \geqslant k_0(r)$.)

In any event the number of m's for which $Q(x, m) \geqslant k$ does not exceed x and so

$$\sum_{m\leqslant x}\{Q(x, m)\}^r \leqslant x \sum_{k<k_1(r)} k^r + 2x \sum_{k\geqslant k_1(r)} k^{-2} \leqslant x\exp\exp\{C_7 r\log(r+1)\}.$$

This is the result stated. We apply it as follows. We have

$$\sum_{\substack{n\leqslant x\\ f(n)\equiv -a \,(\mathrm{mod}\, q)}} 1 = \sum_{\substack{m\leqslant x\\ f(m)\equiv -a \,(\mathrm{mod}\, q)}} |\mu(m)|\,Q(x, m)$$
$$\leqslant \Big(\sum_{\substack{m\leqslant x\\ f(m)\equiv -a \,(\mathrm{mod}\, q)}} |\mu(m)|\Big)^{1-1/r}\Big(\sum_{m\leqslant x}\{Q(x, m)\}^r\Big)^{1/r}$$

for $r > 1$, by Hölder's inequality. So for $q \leqslant 9x$ and $0 \leqslant a \leqslant \delta q$,

$$(1)\qquad \sum_{\substack{n\leqslant x\\ f(n)\equiv -a \,(\mathrm{mod}\, q)}} 1 \leqslant x\left(C_2(\delta)\frac{\log x}{\varphi(q)}\right)^{1-1/r}\exp\exp\{C_7 r\log(r+1)\}$$

and for $q \leqslant \sqrt{x}$ and all a,

$$\sum_{\substack{n\leqslant x\\ f(n)\equiv -a \,(\mathrm{mod}\, q)}} 1 \leqslant x\left(\frac{1}{\{\varphi(q)\}^{1-1/r}} + \left(\frac{\log q}{\log x}\right)^{1-1/r}\right)\exp\exp\{C_7 r\log(r+1)\}.$$

These estimates give about the same information when q is approximately equal to

$$Q = \frac{\log^2 x}{\log\log x}.$$

We are now in a position to estimate the sum

$$S_2 = \sum_{\omega<q\leqslant x}\ \sum_{\substack{m\leqslant x/q\\ q|f(mq)}} 1.$$

We split this into the three parts

$$S_3 = \sum_{\omega<q\leqslant Q}\ \sum_{\substack{m\leqslant x/q\\ q|f(mq)}} 1,$$

$$S_4 = \sum_{Q<q\leqslant 3\sqrt{x}}\ \sum_{\substack{m\leqslant x/q\\ q|f(mq)}} 1,$$

and

$$S_5 = \sum_{3\sqrt{x}<q\leqslant x}\ \sum_{\substack{m\leqslant x/q\\ q|f(mq)}} 1.$$

Since $f(mq) = f(m) + \sum_{p|q,\, p\nmid m} p$, the function $f(mq) - f(m)$ takes at most $2^{\nu(q)}$ different values, the sums of subsets of the prime factors of q. Let these fall into the residue classes $a_1, a_2, \ldots, a_h \pmod q$, where $h = h(q) \leqslant \tau(q)$. If q is prime, $f(mq) - f(m) \equiv 0 \pmod q$ whether $q|m$ or not. If q has two or more distinct prime factors, one of them, say $\tilde\omega$, does not exceed $\sqrt{q}$ and so

$$0 \leqslant a_j \leqslant f(q) \leqslant \tilde\omega + q/\tilde\omega \leqslant 2 + \frac{q}{2} \leqslant \frac{5}{6}q.$$

Thus for all square free q and all a_j, $1 \leqslant j \leqslant h(q)$, we have $0 \leqslant a_j \leqslant 5q/6$. Next,

$$\sum_{\substack{m\leqslant x/q\\ q|f(mq)}} 1 \leqslant \sum_{j=1}^{h}\ \sum_{\substack{m\leqslant x/q\\ f(m)\equiv -a_j (\mathrm{mod}\, q)}} 1.$$

Therefore

$$S_3 \ll x \sum_{\omega<q\leqslant Q}\frac{\tau(q)}{q}\left\{\frac{1}{\{\varphi(q)\}^{1-1/r}} + \left(\frac{\log q}{\log(x/q)}\right)^{1-1/r}\right\}\exp\exp\{C_7 r\log(r+1)\}$$
$$\ll x\left(\frac{\log^2\omega}{\omega^{1-1/r}} + \frac{(\log Q)^{3-1/r}}{(\log x)^{1-1/r}}\right)\exp\exp\{C_7 r\log(r+1)\}$$

by partial summation. We may apply the estimate (1) above to S_4 with $\delta = 5/6$, noting that $q \leqslant 3\sqrt{x}$ gives $q \leqslant 9x/q$. We obtain

$$S_4 \ll x \sum_{Q<q\leqslant 3\sqrt{x}}\frac{\tau(q)}{q}\left(\frac{\log x}{\varphi(q)}\right)^{1-1/r}\exp\exp\{C_7 r\log(r+1)\}$$
$$\ll x\left(\frac{(\log x)(\log Q)}{Q}\right)^{1-1/r}\exp\exp\{C_7 r\log(r+1)\}.$$

Substituting the value of Q we have that

$$S_3 + S_4 \ll x\left(\frac{\log^2\omega}{\omega^{1-1/r}} + \frac{(\log\log x)^{3-1/r}}{(\log x)^{1-1/r}}\right)\exp\exp\{C_7 r\log(r+1)\}.$$

Note that $f(m) \leqslant m$ for all m, for the sum of numbers not less than 2 does not exceed their product. We know that if q has two or more prime factors then $f(q) \leqslant 5q/6$.

To deal with S_5 we observe then that if $\nu(q) \geqslant 2$ and $m \leqslant x/q \leqslant \frac{1}{3}\sqrt{x} \leqslant q/9$, we have

$$f(mq) \leqslant f(m) + f(q) \leqslant m + \frac{5}{6}q \leqslant \left(\frac{1}{9} + \frac{5}{6}\right) q \leqslant \frac{17}{18} q.$$

Therefore q cannot divide $f(mq)$ (which is not zero) in this case. Therefore

$$S_5 \leqslant \sum_{3\sqrt{x} < p \leqslant x} \sum_{\substack{m \leqslant x/p \\ p | f(mp)}} 1 \leqslant \pi(x)$$

the term $m = 1$ contributing 1 to each inner sum. Hence

$$S_2 \ll x\left(\frac{\log^2 \omega}{\omega^{1-1/r}} + \frac{(\log\log x)^{3-1/r}}{(\log x)^{1-1/r}}\right) \exp\exp\{C_7 r \log(r+1)\}.$$

Treatment of S_1. In order to study this sum we need two lemmas from the theory of functions of a complex variable, of which I believe the second is new.

LEMMA 4. *Suppose that $F(s)$ is regular and $|F(s)| \leqslant K$ for $|s-1| \leqslant 2\beta$, and that Γ is a lacet from $1-\beta$ around 1 described in the positive sense. Then for complex ϱ, $|\boldsymbol{R}\varrho| \leqslant 1$ and $\beta \leqslant 1$, we have that*

$$\left|\frac{1}{2i\pi}\int_{\Gamma} x^{s-1} F(s)(s-1)^{-\varrho}\, ds - \sum_{r=0}^{m} \frac{F^{(r)}(1)}{r!} \cdot \frac{(\log x)^{\varrho-r-1}}{\Gamma(\varrho - r)}\right| \leqslant C_9 K x^{-\beta/2} \beta^{-1},$$

where

$$m = [2\beta \log x].$$

The conditions on β and ϱ are not necessary for a result of this type but they make the proof and statement of the lemma more concise and are satisfied in the application.

Proof. By Cauchy's theorem, we have

$$F(s) = \sum_{r=0}^{m} \frac{F^{(r)}(1)}{r!}(s-1)^r + \frac{1}{2i\pi}\int_{D} \left(\frac{s-1}{\omega-1}\right)^{m+1} \frac{F(\omega)}{\omega - s}\, d\omega$$

where D is the circle $|\omega - 1| = 2\beta$. So for $|s-1| \leqslant \beta$,

$$\left|F(s) - \sum_{r=0}^{m} \frac{F^{(r)}(1)}{r!}(s-1)^r\right| \leqslant \frac{K|s-1|^{m+1}}{\beta(2\beta)^m}.$$

Let Γ be the contour comprising the arc $|s-1| = \nu$, $-\pi < \arg(s-1) < \pi$ and the lines joining $1-\beta$, $1-\nu$ in the positive and negative directions. Since $m+1-\boldsymbol{R}\varrho > -1$ the integral of $|s-1|^{m+1-\boldsymbol{R}\varrho}$ around the arc tends to zero with ν. Hence

$$\left|\frac{1}{2i\pi}\int_{\Gamma} x^{s-1}\left\{F(s)(s-1)^{-\varrho} - \sum_{r=0}^{m} \frac{F^{(r)}(1)}{r!}(s-1)^{r-\varrho}\right\} ds\right| \leqslant \frac{1}{\pi}\int_0^{\beta} \frac{K x^{-u} u^{m+1-\boldsymbol{R}\varrho}}{\beta(2\beta)^m}\, du$$

$$\leqslant \frac{K\Gamma(m+2-\boldsymbol{R}\varrho)}{\pi\beta(2\beta)^m(\log x)^{m+2-\boldsymbol{R}\varrho}} \leqslant 4\beta K\left(\frac{m+2-\boldsymbol{R}\varrho}{2\beta e \log x}\right)^{m+2-\boldsymbol{R}\varrho} (2\beta)^{-\boldsymbol{R}\varrho}$$

by Stirling's formula. Setting $m = [2\beta \log x]$ this does not exceed

$$\frac{1}{3} C_9 K x^{-2\beta} \beta^{-1}.$$

Next, we replace the contour of integration Γ by a loop integral from $-\infty$ around 1 and back. The error involved does not exceed

$$\left|\frac{1}{2i\pi}\int_{\Gamma} - \int_{(-\infty)} x^{s-1} \sum_{r=0}^{m} \frac{F^{(r)}(1)}{r!}(s-1)^{r-\varrho}\, ds\right| < \frac{1}{\pi}\int_{\beta}^{\infty} x^{-u} \sum_{r=0}^{m} \frac{|F^{(r)}(1)|}{r!} u^{r-\boldsymbol{R}\varrho}\, du.$$

The term for which $r = 0$ contributes at most

$$K x^{-\beta/2} \beta^{-\boldsymbol{R}\varrho} \log x, \quad 2K x^{-\beta/2} \Gamma(1 - \boldsymbol{R}\varrho)(\log x)^{\boldsymbol{R}\varrho - 1}$$

in the cases $\boldsymbol{R}\varrho > 0$, $\boldsymbol{R}\varrho < 0$ respectively. (In the former we remove the factor

$$\max_{u > \beta} x^{-u/2} u^{-\boldsymbol{R}\varrho} = x^{-\beta/2}\beta^{-\boldsymbol{R}\varrho}$$

from the integral and replace the lower limit of integration by zero. In the other case simply the factor $x^{-\beta/2}$.) Neither of these exceeds

$$\frac{1}{3} C_9 K x^{-\beta/2} \beta^{-1}.$$

The other terms, for which $1 < r < m$, contribute at most

$$\frac{K}{\pi}\sum_{r=1}^{m} \frac{(2\beta)^{-r} x^{-\beta/2}}{(\log x)^{r+1-\boldsymbol{R}\varrho}} \int_0^{\infty} e^{-u/2} u^{r-\boldsymbol{R}\varrho}\, du \leqslant \frac{K}{\pi} x^{-\beta/2}(2\beta)^{1-\boldsymbol{R}\varrho} \sum_{r=1}^{m} \frac{\Gamma(r+1-\boldsymbol{R}\varrho)}{(\beta \log x)^{r+1-\boldsymbol{R}\varrho}}.$$

This sum does not exceed m times its largest term, which is either the first or the last. So the expression above does not exceed

$$\frac{K}{\pi}[2\beta \log x] x^{-\beta/2}(2\beta)^{1-\boldsymbol{R}\varrho}\left\{\frac{2}{(\beta \log x)^{2-\boldsymbol{R}\varrho}} + 3\left(\frac{m+1-\boldsymbol{R}\varrho}{e\beta \log x}\right)^{m+1-\boldsymbol{R}\varrho}\right\}.$$

This is zero unless $2\beta \log x \geqslant 1$, when it does not exceed

$$\frac{1}{3} C_9 K x^{-\beta/2} \beta^{-1}.$$

Since

$$\frac{1}{2i\pi}\int_{(-\infty)} x^{s-1}\sum_{r=0}^{m}\frac{F^{(r)}(1)}{r!}(s-1)^{r-\varrho}ds = \sum_{r=0}^{m}\frac{F^{(r)}(1)(\log x)^{\varrho-r-1}}{r!\,\Gamma(\varrho-r)},$$

(see, for example, § 12.22 of Whittaker and Watson [7]) the result follows.

In the application of Lemma 4 to the present problem we need a better estimate for $|F^{(r)}(1)|$ than that given by Cauchy's coefficient formula. This is

LEMMA 5. *Let*

$$F(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}, \qquad |a_n| \leqslant (1+y)^{\nu(n)}$$

for all n; $y \in [0,1]$. *Suppose that in the region*

$$\sigma \geqslant 1-2\beta, \quad (0<\beta\leqslant \tfrac{1}{3}),\ |t|\leqslant 2,$$

$F(s)$ *is regular and* $|F(s)|\leqslant K$. *Then for all non-negative integers* r, *we have*

$$|F^{(r)}(1)| \leqslant 3e^{C_{10}y}\left(\frac{2\log K}{\beta}\right)^{r+y+1} + r!\,\beta^{-r-1}e^{(r+1)/2}.$$

Proof. For any non-negative integers r and m we have

$$F^{(r)}(1) = \sum_{j=0}^{m}(1-\sigma)^j\frac{F^{(r+j)}(\sigma)}{j!} + \frac{1}{2i\pi}\int_C\left(\frac{1-\sigma}{\omega-\sigma}\right)^{m+1}\frac{F^{(r)}(\omega)}{\omega-1}\,d\omega,$$

by Cauchy's theorem. We shall select a value of σ from the range

$$0<\sigma-1\leqslant 2(1-\beta).$$

Now

$$(-1)^kF^{(k)}(\sigma) = \sum_{n=1}^{\infty}\frac{a_n(\log n)^k}{n^\sigma}$$

whence

$$|F^{(k)}(\sigma)| \leqslant \sum_{n=2}^{\infty}\frac{(1+y)^{\nu(n)}(\log n)^k}{n^\sigma}.$$

Now

$$\sum_{n\leqslant x}(1+y)^{\nu(n)} = \sum_{n\leqslant x}\sum_{r=0}^{\nu(n)}\binom{\nu(n)}{r}y^r = \sum_{n\leqslant x}\sum_{d|n}|\mu(d)|\,y^{\nu(d)} \leqslant x\sum_{d\leqslant x}\frac{|\mu(d)|}{d}y^{\nu(d)}$$

$$\leqslant x\sum_{r=0}^{\infty}\frac{y^r}{r!}\left(\sum_{p\leqslant x}\frac{1}{p}\right)^r \leqslant xe^{C_{10}y}(\log x)^y$$

since each squarefree d with $\nu(d)=r$ occurs $r!$ times in the multinomial expansion of

$$\left(\sum_{p\leqslant x}\frac{1}{p}\right)^r.$$

It follows that

$$|F^{(k)}(\sigma)| \leqslant \sum_{n=2}^{\infty}\left(\frac{\log^k n}{n^\sigma}-\frac{\log^k(n+1)}{(n+1)^\sigma}\right)\sum_{m\leqslant x}(1+y)^{\nu(m)}$$

$$\leqslant e^{C_{10}y}\sum_{n=2}^{\infty}n(\log n)^y\int_n^{n+1}-\frac{d}{dx}\left(\frac{\log^k x}{x^\sigma}\right)dx$$

$$\leqslant e^{C_{10}y}\int_2^{\infty}x(\log x)^y\left\{\frac{\sigma\log^k x}{x^{\sigma+1}}-\frac{k\log^{k-1}x}{x^{\sigma+1}}\right\}dx$$

$$\leqslant e^{C_{10}y}\int_0^{\infty}e^{-(\sigma-1)t}\sigma t^{y+k}dt \leqslant \frac{\sigma e^{C_{10}y}\Gamma(y+k+1)}{(\sigma-1)^{y+k+1}}.$$

Now let C be a circle, centre σ and radius $\sigma-1+\beta$. If $\omega\in C$ and $|z-\omega|=\beta$ then

$$\boldsymbol{R}z \geqslant \boldsymbol{R}\omega-\beta\geqslant 1-2\beta, \qquad |\boldsymbol{I}z|\leqslant|\boldsymbol{I}\omega|+\beta\leqslant\sigma-1+2\beta.$$

If $\sigma-1+2\beta\leqslant 2$, $F(z)$ is regular and $|F(z)|\leqslant K$, so that

$$\left|\frac{F^{(r)}(\omega)}{r!}\right| = \frac{1}{2\pi}\left|\int_{|z-\omega|=\beta}\frac{F(z)}{(z-\omega)^{r+1}}dz\right| \leqslant K\beta^{-r}.$$

We deduce from these results that

$$|F^{(r)}(1)| \leqslant \frac{\sigma e^{C_{10}y}}{(\sigma-1)^{y+r+1}}\sum_{j=0}^{m}\frac{\Gamma(y+r+j+1)}{j!} + r!\,K\beta^{-r-1}\left(\frac{\sigma-1}{\sigma-1+\beta}\right)^{m+1}$$

$$\leqslant \frac{\sigma e^{C_{10}y}\Gamma(y+r+m+2)}{(\sigma-1)^{y+r+1}m!} + r!\,K\beta^{-r-1}\exp\left(\frac{-\beta(m+1)}{\sigma-1+\beta}\right).$$

Now

$$\Gamma(k+y+1) = \int_0^{\infty}e^{-(1-y)t}t^{(1-y)k}e^{-yt}t^{yk+y}dt \leqslant \{\Gamma(k+1)\}^{1-y}\{\Gamma(k+2)\}^y$$

by Hölder's inequality. Therefore

$$\frac{\Gamma(y+r+m+2)}{m!} \leqslant (r+m+2)^y(r+m+1)\ldots(m+1) \leqslant (r+m+2)^{r+1+y},$$

and

$$|F^{(r)}(1)| \leqslant \sigma e^{C_{10}y}\left(\frac{r+m+2}{\sigma-1}\right)^{r+1+y} + r!\,K\beta^{-r-1}\exp\left(\frac{-\beta(m+1)}{\sigma-1+\beta}\right).$$

By Cauchy's coefficient formula,

$$|F^{(r)}(1)| \leqslant r!\,K(2\beta)^{-r} \leqslant r!\,\beta^{-r-1}e^{(r+1)/2}$$

if $2\log K\beta \leqslant (r+1)$. So we may assume that $r+1 \leqslant [2\log K]$, moreover that $K \geqslant \beta^{-1} \geqslant 3$. We select

$$m = [2\log K]-r-1, \qquad \frac{r+m+2}{\sigma-1} = \frac{2\log K}{\beta}$$

which imply that

$$\beta < \sigma-1, \qquad \sigma-1+2\beta = \left(\frac{[2\log K]+1}{2\log K}+2\right)\beta \leqslant 7/6.$$

Hence

$$|F^{(r)}(1)| \leqslant \sigma e^{C_{10}y}\left(\frac{2\log K}{\beta}\right)^{r+1+y} + r!\,K\beta^{-r-1}\exp\left(\frac{-\beta(r+m+2)}{2(\sigma-1)}+\frac{\beta(r+1)}{2\beta}\right)$$
$$\leqslant 3e^{C_{10}y}\left(\frac{2\log K}{\beta}\right)^{r+1+y} + r!\,\beta^{-r-1}e^{(r+1)/2}.$$

This completes the proof.

LEMMA 6. *For*

$$q < \frac{C_{21}(\varepsilon)(\log x)^{8/7}}{(\log\log x)^{20/7}}, \qquad |\mu(q)| = 1,$$

and all a, and with $m = [C_{12}(\varepsilon)q^{-\varepsilon}\log x]$, $0 < \varepsilon < 3/8$, *we have*

$$\sum_{\substack{n\leqslant x\\ f_q(n)\equiv a(\mathrm{mod}\,q)}} 1 = \frac{x}{q}+\frac{x}{q}\sum_{l=1}^{q-1} e^{-2i\pi al/q}\sum_{r=0}^{m}\frac{\hat F_q^{(r)}(1;l/q)}{r!}\cdot\frac{(\log x)^{\mu(k)/\varphi(k)-r-1}}{\Gamma\big(\mu(k)/\varphi(k)-r\big)} +$$
$$+O\left(x\exp\left\{\frac{-C_{22}(\varepsilon)(\log x)^{4/7}}{(\log\log x)^{3/7}}\right\}\right)$$

where $f_q(n) = f(nq)-f(q)$, $k = q/(q,l)$ *for each* l *and*

$$\hat F_q(s;l/q) = \frac{1}{s}(s-1)^{\mu(k)/\varphi(k)}\sum_{n=1}^{\infty}\frac{1}{n^s}e^{2i\pi f_q(n)l/q}.$$

We show in the course of the proof that this function is regular in the neighbourhood of $s = 1$, moreover that

$$|\hat F_q(s;l/q)| \leqslant \exp\{C_{23}(\varepsilon)\sqrt{k}\log q\}$$

near $s = 1$. We require estimates for $|\hat F_q^{(r)}(1;l/q)|$ in the application and we use Lemma 5 which gives a considerable saving. Without it, we only know for example that

$$|\hat F_q(1;l/q)| \leqslant \exp\{C_{23}(\varepsilon)\sqrt{k}\log q\}$$

and the formula above is then only useful if $\sqrt{q}\log q \ll \log\log x$ (cf. Lemma 7 of Hall [1]).

Proof. Let

$$F_q(s,a) = \sum_{\substack{n=1\\ f_q(n)=a(\mathrm{mod}\,q)}}^{\infty} n^{-s}, \qquad \sigma > 1.$$

As $\sigma \to 1$, we have

$$F_q(\sigma,a) = O\left(\frac{1}{\sigma-1}\right)$$

uniformly in q. By Lemma 3.12 of Titchmarsh [6] we have

$$\sum_{\substack{n\leqslant x\\ f_q(n)\equiv a(\mathrm{mod}\,q)}} 1 = \frac{1}{2i\pi}\int_{c-iT}^{c+iT}\frac{x^s}{s}F_q(s,a)\,ds + O\left(\frac{x^c}{T(c-1)}\right)+O\left(\frac{x\log x}{T}\right)$$

where x is half an odd integer. Suppose that

$$F_q(s,t) = \sum_{n=1}^{\infty}\frac{1}{n^s}e^{2i\pi f_q(n)t} = \prod_{p\nmid q}\left(1+\frac{e^{2i\pi pt}}{p^s-1}\right)\prod_{p|q}\left(1-\frac{1}{p^s}\right)^{-1}.$$

Then

$$F_q(s,a) = \frac{1}{q}\sum_{l=1}^{q} e^{-2i\pi al/q}F_q(s,l/q) = \frac{1}{q}\zeta(s)+\frac{1}{q}\sum_{l=1}^{q-1}e^{-2i\pi al/q}F_q(s,l/q).$$

We choose $c = 1+1/\log x$ and deduce from the above that

$$\sum_{\substack{n\leqslant x\\ f_q(n)\equiv a(\mathrm{mod}\,q)}} 1 = \frac{1}{q}\left(x-\frac{1}{2}\right)+\frac{1}{q}\sum_{l=1}^{q-1}e^{-2i\pi al/q}W_q(x;l/q)+O\left(\frac{x\log x}{T}\right),$$

where

$$W_q(x;l/q) = \frac{1}{2i\pi}\int_{c-iT}^{c+iT}\frac{x^s}{s}F_q(s;l/q)\,ds.$$

In the first paper of this series we arrived at an integral similar to this, and all we needed there was an upper bound for $|W_q(x;l/q)|$, as we were only interested in the main term $6x/\pi^2$ in the asymptotic formula for $T(x)$. We now require an approximate formula for $W_q(x; l/q)$. First, it is clear that $F_q(s; l/q) = F_q(s; h/k)$ where h/k is in its lowest terms. Moreover,

$$F_q(s; h/k) = G(s; k, h)\prod_{p|\frac{q}{k}}\left(1+\frac{e^{2i\pi ph/k}}{p^s}\right)^{-1}\prod_{p\nmid q}\left(1+\frac{e^{2i\pi ph/k}}{(p^s-1)(p^s+e^{2i\pi ph/k})}\right)\prod_{p|q}\left(1-\frac{1}{p^s}\right)^{-1}$$

where

$$G(s; k, h) = \prod_{p\nmid k}\left(1+\frac{e^{2i\pi ph/k}}{p^s}\right)$$

and the remaining factor is regular for $\boldsymbol{R}s > 1/2$. It was shown in Hall [1] that

$$\log G(s; k, h) = \frac{\mu(k)}{\varphi(k)}\log L(s, \chi_0)+\frac{1}{\varphi(k)}\sum_{\chi\neq\chi_0}\chi(h)\tau(\chi)\log L(s,\chi)+H(s; k, h)$$

where H is regular, and bounded independently of k, for $\boldsymbol{R}s \geqslant 1/2+\delta$, $\delta > 0$. We conclude that $F_q(s; h/k)$ may be analytically continued into a simply connected region containing no zeros of L-functions (mod k), nor the point $s = 1$, and wholly included in the half-plane $\boldsymbol{R}s > 1/2$. Thus

$$F_q(s; l/q) = F_q^*(s; l/q)\exp\left\{\frac{1}{\varphi(k)}\sum_{\chi}\chi(h)\tau(\chi)\log L(s, \chi)\right\}$$

where the sum over χ runs over all characters, and

$$\tau(\chi) = \sum_{b=1}^{k}\bar{\chi}(b)e^{2i\pi b/k}, \quad |\tau(\chi)| \leqslant \sqrt{k},$$

and for $\boldsymbol{R}s \geqslant 3/4$,

$$|F_q^*(s; l/q)| \leqslant C_{11}(\varepsilon)q^{\varepsilon}.$$

Now let

$$M(k, t) = \max\{k^{\varepsilon}, (\log(|t|+3)\log\log(|t|+3))^{3/4}\}.$$

Then it is known (Prachar [5], p. 295) that $L(s, \chi) \neq 0$ for

$$\sigma \geqslant 1-\frac{C_{12}(\varepsilon)}{M(k,t)}.$$

We have replaced the $\log k$ in Prachar's definition of $M(k, t)$ by k^{ε}, to exclude the possible real Siegel zero of one of the L-functions. Since $M(q, t) \geqslant M(k, t)$ we may move the contour of integration in the formula for $W_q(x; l/q)$ to $\Gamma_1 \cup \Gamma_2$, where Γ_1 is a lacet around $s = 1$ and on Γ_2,

$$\sigma = 1-\frac{C_{12}(\varepsilon)}{M(q,t)}, \quad 0 < |t| < T.$$

The contour is completed by horizontal lines at $t = \pm T$, joining the points $\sigma\pm iT, c\pm iT$ $(\sigma = 1-C_{12}(\varepsilon)/M(q, T))$. We require bounds for $|\log L(s, \chi)|$ and these were derived in Lemma 3 of Hall [3].

It was shown that for $\chi \neq \chi_0$, the principal character,

$$|\log L(s, \chi)| \leqslant C_{13}\log k\frac{\{\log\log(|t|+3)\}^{9/4}}{\{\log(|t|+3)\}^{3/4}}+C_{14}\{\log\log(|t|+3)\}^3+ +C_{15}\log\log 3k+C_{16}(\varepsilon)$$

on and to the right of Γ_2. A better estimate was found for $|t| < 2$ but we do not need this. We remark that if we replace $C_{13}\log k$ by $C_{13}(\varepsilon)\log q$, and C_{15} by $C_{15}(\varepsilon)$, the same estimate holds for $\chi = \chi_0$ on Γ_2 itself. For observe that

$$|\log\zeta(s)| \leqslant C_{14}\{\log\log(|t|+3)\}^3$$

for $|t| > 1$; and since $|s-1| \geqslant C_{12}(\varepsilon)q^{-\varepsilon}$ we have for $|t| \leqslant 1$,

$$|\log\zeta(s)| \leqslant C_{17}(\varepsilon)\log q \leqslant C_{13}(\varepsilon)\log q\frac{\{\log\log(|t|+3)\}^{9/4}}{\{\log(|t|+3)\}^{3/4}}.$$

Also,

$$\left|\log\prod_{p|k}\left(1-\frac{1}{p^s}\right)\right| \leqslant \sum_{p|k}\frac{1}{p^{\sigma}-1} \leqslant C_{15}(\varepsilon)\log\log 3k$$

since

$$\sigma \geqslant 1-\frac{C_{12}(\varepsilon)}{q^{\varepsilon}} \geqslant 1-\frac{C_{12}(\varepsilon)}{p^{\varepsilon}} \geqslant 1-\frac{C_{12}(\varepsilon)}{\varepsilon\log p}.$$

Putting all this together we deduce that for $1 \leqslant l \leqslant q-1$ and $s\in\Gamma_2$,

$$|F_q(s; l/q)| \leqslant C_{18}(\varepsilon)\exp\{C_{19}(\varepsilon)\sqrt{k}(\log q+(\log\log T)^3)\}$$

and hence that

$$\left|\frac{1}{2i\pi}\int_{\Gamma_2}\frac{x^s}{s}F_q(s; l/q)\,ds\right| \leqslant C_{20}(\varepsilon)\left(\frac{x}{T}+x^{1-C_{12}(\varepsilon)/M(q,T)}\log T\right)\exp\{C_{19}(\varepsilon)\sqrt{k}(\log q+(\log\log T)^3)\}.$$

We set

$$\log T = \frac{(\log x)^{4/7}}{(\log\log x)^{3/7}}$$

and deduce that for

$$q \leqslant \frac{C_{21}(\varepsilon)(\log x)^{8/7}}{(\log\log x)^{20/7}},$$

$$W_q(x; l/q) = \frac{1}{2i\pi}\int_{\Gamma_1}\frac{x^s}{s}F_q(s; l/q)\,ds+O\left(x\exp\left\{\frac{-C_{22}(\varepsilon)(\log x)^{4/7}}{(\log\log x)^{3/7}}\right\}\right).$$

We now examine the integral on Γ_1. In the neighbourhood of $s=1$ we have

$$\frac{1}{s}F_q(s;l/q)=\frac{1}{s}F_q^*(s;l/q)\exp\Big\{\frac{1}{\varphi(k)}\sum_{\chi}\chi(h)\tau(\chi)\log L(s,\chi)\Big\}$$

$$=(s-1)^{-\mu(k)/\varphi(k)}\hat{F}_q(s;l/q)$$

say, where

$$|\hat{F}_q(s;l/q)|\leqslant\exp\{C_{23}(\varepsilon)\sqrt{k}\log q\}.$$

We apply Lemma 4 with

$$K=\exp\{C_{23}(\varepsilon)\sqrt{k}\log q\},\qquad 2\beta=C_{12}(\varepsilon)/M(q,0),\qquad \varrho=\mu(k)/\varphi(k).$$

We obtain

$$\left|\frac{1}{2i\pi}\int_{\Gamma_1}\frac{x^s}{s}F_q(s;l/q)\,ds-x\sum_{r=0}^{m}\frac{\hat{F}_q^{(r)}(1;l/q)}{r!}\cdot\frac{(\log x)^{\mu(k)/\varphi(k)-r-1}}{\Gamma\big(\mu(k)/\varphi(k)-r\big)}\right|$$

$$\leqslant C_9Kx^{1-\beta/2}\beta^{-1}\leqslant C_{24}(\varepsilon)q^{\varepsilon}x\exp\Big\{C_{23}(\varepsilon)\sqrt{k}\log q-\frac{C_{12}(\varepsilon)\log x}{4M(q,0)}\Big\}$$

where

$$m=\left[\frac{C_{12}(\varepsilon)\log x}{M(q,0)}\right].$$

Selecting $\varepsilon<3/8$ we deduce that

$$W_q(x;l/q)=x\sum_{r=0}^{m}\frac{\hat{F}_q^{(r)}(1;l/q)}{r!}\cdot\frac{(\log x)^{\mu(k)/\varphi(k)-r-1}}{\Gamma\big(\mu(k)/\varphi(k)-r\big)}+$$

$$+O\left(x\exp\Big\{\frac{-C_{22}(\varepsilon)(\log x)^{4/7}}{(\log\log x)^{3/7}}\Big\}\right)$$

and the result follows from this and the fact that

$$\sum_{\substack{n<x\\ f_q(n)\equiv a\,(\mathrm{mod}\,q)}}1=\frac{1}{q}\Big(x-\frac{1}{2}\Big)+\frac{1}{q}\sum_{l=1}^{q-1}e^{-2i\pi al/q}W_q(x;l/q)+O\Big(\frac{x\log x}{T}\Big).$$

The condition that x is half an odd integer is unnecessary to the result by considerations of continuity.

We have proved rather more in Lemma 6 than is needed in the present application, for the series

$$\frac{x}{\log x}\sum_{k=1}^{\infty}A_k(\log x)^{\mu(k)/\varphi(k)}$$

arises from the terms in which $r=0$ in the formula we have just proved. To deal with the other terms we first derive an upper bound for $|\hat{F}_q^{(r)}(1;l/q)|$. We have

LEMMA 7. *Let $k=q/(q,l)$ and $y=|\mu(k)/\varphi(k)|$. Then for*

$$6r\leqslant C_{26}(\varepsilon)k^{1/2}q^{2\varepsilon}$$

we have

$$|\hat{F}_q^{(r)}(1;l/q)|\leqslant C_{29}\big(C_{30}(\varepsilon)k^{1/2}q^{3\varepsilon}\big)^{r+y+1}$$

and for all non-negative integers r,

$$|\hat{F}_q^{(r)}(1;l/q)|\leqslant r!\,\big(C_{27}(\varepsilon)q^{\varepsilon}\big)^{r}\exp\{C_{23}(\varepsilon)\sqrt{k}\log q\}.$$

The second estimate is a direct consequence of Cauchy's coefficient formula. For small values of r the first estimate is better, but as r increases it practically dovetails into the second by virtue of Stirling's formula.

Proof. We have

$$\hat{F}_q(s;l/q)=G_k(s)E_q(s;h/k)$$

where

$$G_k(s)=\frac{1}{s}(s-1)^{\mu(k)/\varphi(k)}\prod_p\Big(1-\frac{\mu(k)}{\varphi(k)}p^{-s}\Big)^{-1}$$

and

$$E_q(s;h/k)=\prod_p\left(1+\frac{1}{p^s}\Big(e^{2i\pi f_q(p)h/k}-\frac{\mu(k)}{\varphi(k)}\Big)+\frac{e^{2i\pi f_q(p)h/k}}{p^s(p^s-1)}\Big(1-\frac{\mu(k)}{\varphi(k)}\Big)\right)$$

$$=\sum_{n=1}^{\infty}a_nn^{-s}$$

say. We observe that for all n,

$$|a_n|\leqslant(1+y)^{\nu(n)}$$

where $y=|\mu(k)/\varphi(k)|$. We apply Lemma 5 with

$$K=\exp\{C_{23}(\varepsilon)\sqrt{k}\log q\},\qquad 2\beta=C_{12}(\varepsilon)q^{-\varepsilon}$$

and we deduce that for integers $r\geqslant 0$,

$$|E_q^{(r)}(1;h/k)|\leqslant C_{25}\big(C_{26}(\varepsilon)\sqrt{k}q^{2\varepsilon}\big)^{r+y+1}+r!\,\big(C_{27}(\varepsilon)q^{\varepsilon}\big)^{r+1}e^{(r+1)/2}.$$

Since $G_k(s)$ is regular for $\boldsymbol{R}s>1/2$, $|t|\leqslant 1$, we have

$$|G_k^{(r)}(1)|\leqslant C_{28}3^rr!.$$

Hence

$$|\hat{F}_q^{(r)}(1;\, l/q)| \leqslant \sum_{l=0}^{r} \binom{r}{l} |G_k^{(r-l)}(1)\, E_q^{(l)}(1;\, h/k)|$$

$$\leqslant r! \sum_{l=0}^{r} C_{28} 3^{r-l} \left(\frac{C_{25}}{l!} \left(C_{26}(\varepsilon)\sqrt{k}\, q^{2\varepsilon}\right)^{l+\nu+1} + \left(C_{27}(\varepsilon)\, q^{\varepsilon}\right)^{l+1} e^{(l+1)/2} \right).$$

Provided the ratio between the terms of this sum is never less than 2, it does not exceed twice its last term. We require that

$$r \leqslant \frac{1}{6} C_{26}(\varepsilon)\sqrt{k}\, q^{2\varepsilon}, \qquad \sqrt{e}\, C_{27}(\varepsilon)\, q^{\varepsilon} \geqslant 2,$$

and then

$$|\hat{F}_q^{(r)}(1;\, l/q)| \leqslant 2C_{28} C_{25} \left(C_{26}(\varepsilon)\sqrt{k}\, q^{2\varepsilon}\right)^{r+\nu+1} + 2r!\, e^{(r+1)/2} \left(C_{27}(\varepsilon)\, q^{\varepsilon}\right)^{r+1}$$
$$\leqslant C_{29} \left(C_{30}(\varepsilon)\sqrt{k}\, q^{3\varepsilon}\right)^{r+\nu+1}$$

applying the estimate $r! \leqslant 3\left((r+1)/e\right)^{r+1}$ to the second term. But by Cauchy's coefficient formula,

$$|\hat{F}_q^{(r)}(1;\, l/q)| \leqslant K\beta^{-r} r! \leqslant r! \left(C_{27}(\varepsilon)\, q^{\varepsilon}\right)^r \exp\{C_{23}(\varepsilon)\sqrt{k} \log q\}$$

and we use this for higher values of r. Note that

$$C_{27}(\varepsilon) = 2/C_{12}(\varepsilon)$$

where $C_{12}(\varepsilon)$ is essentially the undetermined constant in Siegel's theorem. Provided it is small enough, we can make it as small as we wish, so we may assume that

$$\sqrt{e}\, C_{27}(\varepsilon)\, q^{\varepsilon} \geqslant 2.$$

In the next lemma we show that a good approximation to the sum over r in Lemma 6 may be obtained by taking its first term only. We have

LEMMA 8. *For*

$$q \leqslant \frac{C_{33}(\varepsilon)(\log x)^{8/7}}{(\log\log x)^{20/7}}, \qquad \varepsilon \leqslant 1/32,$$

and all a, *we have that*

$$\sum_{\substack{n \leqslant x \\ f_q(n) = a (\mathrm{mod}\, q)}} 1 = \frac{x}{q} + \frac{x}{q} \sum_{l=1}^{q-1} e^{-2i\pi a l/q} \hat{F}_q(1;\, l/q) \frac{(\log x)^{\mu(k)/\varphi(k)-1}}{\Gamma\left(\mu(k)/\varphi(k)\right)} + O\left(x q^{9\varepsilon} (\log x)^{-3/2}\right).$$

We are interested in the sum over $n \leqslant x/q$ in the application, and we have the following

COROLLARY. *For*

$$q \leqslant \frac{C_{34}(\varepsilon)(\log x)^{8/7}}{(\log\log x)^{20/7}}, \qquad \varepsilon \leqslant 1/32,$$

and all a, *we have that*

$$\sum_{\substack{n \leqslant x/q \\ f_q(n) = a (\mathrm{mod}\, q)}} 1 = \frac{x}{q^2} + \frac{x}{q^2} \sum_{l=1}^{q-1} e^{-2i\pi a l/q} \hat{F}_q(1;\, l/q) \frac{(\log x)^{\mu(k)/\varphi(k)-1}}{\Gamma\left(\mu(k)/\varphi(k)\right)} + O\left(x q^{9\varepsilon-1} (\log x)^{-3/2}\right).$$

Proof. By Lemma 6, for

$$q \leqslant \frac{C_{21}(\varepsilon)(\log x)^{8/7}}{(\log\log x)^{20/7}}$$

and all a, we have

$$\left| \sum_{\substack{n \leqslant x \\ f_q(n) = a (\mathrm{mod}\, q)}} 1 - \frac{x}{q} - \frac{x}{q} \sum_{l=1}^{q-1} e^{-2i\pi a l/q} \hat{F}_q(1;\, l/q) \frac{(\log x)^{\mu(k)/\varphi(k)-1}}{\Gamma\left(\mu(k)/\varphi(k)\right)} \right|$$

$$\leqslant C_{31}(\varepsilon)\, x \exp\left\{ \frac{-C_{22}(\varepsilon)(\log x)^{4/7}}{(\log\log x)^{3/7}} \right\} + \frac{x}{q} \sum_{l=1}^{q-1} \sum_{r=1}^{m} \frac{|\hat{F}_q^{(r)}(1;\, l/q)(\log x)^{\mu(k)/\varphi(k)-r-1}|}{\left|\Gamma\left(\mu(k)/\varphi(k)-r\right)\right| r!}$$

where

$$m = [C_{12}(\varepsilon)\, q^{-\varepsilon} \log x], \qquad 0 < \varepsilon < 3/8.$$

We let

$$R = \frac{1}{6} C_{26}(\varepsilon)\sqrt{k}\, q^{2\varepsilon}$$

and we deduce from Lemma 7 that the inner sum on the right does not exceed

$$\frac{1}{\varphi(k)} \sum_{1 \leqslant r \leqslant R} C_{29}(r+1)\left(C_{30}(\varepsilon)\sqrt{k}\, q^{3\varepsilon}\right)^{r+\nu+1} (\log x)^{\mu(k)/\varphi(k)-r-1} +$$
$$+ \frac{1}{\varphi(k)} \sum_{R < r \leqslant m} (r+1)! \left(C_{27}(\varepsilon)\, q^{\varepsilon}\right)^r (\log x)^{\mu(k)/\varphi(k)-r-1} \exp\left(C_{23}(\varepsilon)\sqrt{k} \log q\right)$$

since

$$\left| 1/\Gamma\left(\frac{\mu(k)}{\varphi(k)} - r\right) \right| = \left| \Gamma\left(r+1-\frac{\mu(k)}{\varphi(k)}\right) \pi^{-1} \sin\left(\frac{\mu(k)}{\varphi(k)} - r\right)\pi \right| \leqslant (r+1)!/\varphi(k).$$

The first of these sums does not exceed twice its first term, that is

$$4C_{29}\left(C_{30}(\varepsilon)\sqrt{k}\, q^{3\varepsilon}\right)^{2+\nu} (\log x)^{\mu(k)/\varphi(k)-2} / \varphi(k)$$

if the ratio between the terms is $\leqslant 1/2$, that is, if

$$4C_{30}(\varepsilon)\sqrt{k}\,q^{3\varepsilon} \leqslant \log x,$$

which we may assume to be the case. The second sum does not exceed m times its largest term, which is either the first or the last. So it is less than or equal to

$$\frac{m(\log x)^{\mu(k)/\varphi(k)}}{\varphi(k)}\left\{\frac{([R]+2)!\big(C_{27}(\varepsilon)q^{\varepsilon}\big)^{[R]+1}}{(\log x)^{[R]+2}}+\frac{(m+1)!\big(C_{27}(\varepsilon)q^{\varepsilon}\big)^{m}}{(\log x)^{m+1}}\right\}\times$$
$$\times\exp\big(C_{23}(\varepsilon)\sqrt{k}\log q\big)$$
$$\leqslant\frac{(\log x)^{\mu(k)/\varphi(k)}}{\varphi(k)}\left\{\frac{2([R]+2)^2}{\log x}\left(\frac{C_{32}(\varepsilon)\sqrt{k}\,q^{3\varepsilon}}{\log x}\right)^{[R]}+6m^2\left(\frac{2}{e}\right)^{m}\right\}\times$$
$$\times\exp\big(C_{23}(\varepsilon)\sqrt{k}\log q\big).$$

If $e\leqslant 1/32$ and $C_{23}(\varepsilon)\geqslant 48$, as we may suppose, so that $R\geqslant 8$ for all k and q, the first term here is

$$O\left(\frac{1}{\varphi(k)}(\log x)^{\mu(k)/\varphi(k)-2}\right)$$

and the second term is of smaller order. Hence

$$\left|\sum_{\substack{n\leqslant x\\ f_q(n)=a(\mathrm{mod}\,q)}} 1-\frac{x}{q}-\frac{x}{q}\sum_{l=1}^{q-1}e^{-2i\pi al/q}\hat{F}_q(1;l/q)\frac{(\log x)^{\mu(k)/\varphi(k)-1}}{\Gamma\big(\mu(k)/\varphi(k)\big)}\right|$$
$$=O\left(\frac{x}{q}\sum_{l=1}^{q-1}\frac{(\sqrt{k}q^{3\varepsilon})^{2+\nu}}{\varphi(k)}(\log x)^{\mu(k)/\varphi(k)-2}\right).$$

Since $k=q/(q,l)$ for each k dividing q other than $k=1$, there are $\varphi(k)$ values of l. The maximum value of $\mu(k)/\varphi(k)$ is $1/2$, and $(\sqrt{k})^{\nu}$ is $O(1)$. So this is

$$O\big(xq^{9\varepsilon}(\log x)^{-3/2}\big).$$

This is the result stated, and the corollary follows, using the fact that

$$\left(\log\frac{x}{q}\right)^{\mu(k)/\varphi(k)-1}=(\log x)^{\mu(k)/\varphi(k)-1}+O\Big((\log q)(\log x)^{\mu(k)/\varphi(k)-2}\Big).$$

We are now in a position to evaluate S_1. Provided

$$\omega\leqslant\frac{C_{34}(\varepsilon)(\log x)^{8/7}}{(\log\log x)^{20/7}},\qquad \varepsilon\leqslant 1/32,$$

we have by Lemma 8 that

$$S_1=\sum_{q\leqslant\omega}\mu(q)\sum_{\substack{n\leqslant x/q\\ f_q(n)=-f(q)(\mathrm{mod}\,q)}}1$$
$$=x\sum_{q\leqslant\omega}\frac{\mu(q)}{q^2}\left(1+\sum_{l=1}^{q-1}e^{2i\pi f(q)l/q}\hat{F}_q(1;l/q)\frac{(\log x)^{\mu(k)/\varphi(k)-1}}{\Gamma\big(\mu(k)/\varphi(k)\big)}\right)+O\big(x\omega^{9\varepsilon}(\log x)^{-3/2}\big).$$

Therefore

$$S_1=\frac{6}{\pi^2}x+O\left(\frac{x}{\omega}+x\omega^{9\varepsilon}(\log x)^{-3/2}\right)+x\sum_{2\leqslant k\leqslant\omega}\frac{\mu(k)}{k^2\Gamma\big(\mu(k)/\varphi(k)\big)}(\log x)^{\mu(k)/\varphi(k)-1}\times$$
$$\times\sum_{\substack{j=1\\ (j,k)=1}}^{k}\sum_{\substack{q'\leqslant\omega/k\\ (q',k)=1}}\frac{\mu(q')}{q'^2}e^{2i\pi f(q'k)j/k}\hat{F}_{q'k}(1;j/k).$$

Since

$$|\hat{F}_{q'k}(1;j/k)|=O\big((k^{1/2+3\varepsilon}q'^{3\varepsilon})^{1+\nu}\big)$$

the inner sum is

$$\sum_{\substack{q'=1\\ (q',k)=1}}^{\infty}\frac{\mu(q')}{q'^2}e^{2i\pi f(q'k)j/k}+O\big(\omega^{5\varepsilon-1}k^{3/2+\nu/2}\big)$$

and since k^{ν} is $O(1)$, the whole sum is

$$\frac{6}{\pi^2}x+O\left(\frac{x}{\omega}+x\omega^{9\varepsilon}(\log x)^{-3/2}+x\omega^{5\varepsilon-1/2}(\log x)^{-1/2}\right)+$$
$$+x\sum_{2\leqslant k\leqslant\omega}\frac{\mu(k)}{k^2\Gamma\big(\mu(k)/\varphi(k)\big)}(\log x)^{\mu(k)/\varphi(k)-1}\times$$
$$\times\sum_{\substack{j=1\\ (j,k)=1}}^{k}\sum_{\substack{q'=1\\ (q',k)=1}}^{\infty}\frac{\mu(q')}{q'^2}e^{2i\pi f(q'k)j/k}\hat{F}_{q'k}(1;j/k).$$

If we now let k range from 2 to infinity, the error introduced is

$$O\left(x\sum_{k>\omega}k^{-3/2+3\varepsilon}(\log x)^{\mu(k)/\varphi(k)-1}\right)=O\big(x\omega^{3\varepsilon-1/2}(\log x)^{-1/2}\big).$$

Hence

$$S_1=\frac{6}{\pi^2}x+\frac{x}{\log x}\sum_{k=2}^{\infty}A_k(\log x)^{\mu(k)/\varphi(k)}+$$
$$+O\left(\frac{x}{\omega}+x\omega^{9\varepsilon}(\log x)^{-3/2}+x\omega^{5\varepsilon-1/2}(\log x)^{-1/2}\right),$$

where

$$A_k = \frac{\mu(k)}{k^2\Gamma(\mu(k)/\varphi(k))} \sum_{\substack{j=1\\(j,k)=1}}^{k} \sum_{\substack{q=1\\(q,k)=1}}^{\infty} \frac{\mu(q)}{q^2} e^{2i\pi f(qk)j/k} \hat{F}_{qk}(1; j/k).$$

We set $\varepsilon = 1/90$, $\omega = C_{35}(\log x)^{9/8}$, and

$$r = C_8 \frac{\log_3 x}{\log_4 x}$$

and deduce from our results for S_1 and S_2 that

$$T(x) = \frac{6}{\pi^2}x + \frac{x}{\log x}\sum_{k=2}^{\infty} A_k(\log x)^{\mu(k)/\varphi(k)} + O\left(\frac{x}{\log x}\exp\left(\frac{C(\log_2 x)(\log_4 x)}{\log_3 x}\right)\right)$$

where C is an absolute constant and

$$\sum_{k=2}^{\infty} A_k$$

is absolutely convergent. This completes the proof.

Remarks concerning the sequence $\{A_k\}$. A_k is real, for the jth and $(k-j)$-th terms in the sum over j above are complex conjugates. From the estimate for $|\hat{F}|$ we know that

$$A_k = O(k^{-3/2+\varepsilon})$$

for all $\varepsilon > 0$. We also have the formula

$$A_k = B_k \sum_{\substack{j=1\\(j,k)=1}}^{k} e^{2i\pi f(k)j/k} \prod_{p\nmid k}\left(1+\frac{1}{p\varphi(k)}\sum_{\chi\neq\chi_0}\chi(pj)\tau(\chi) - \frac{\mu(k)e^{2i\pi pj/k}}{p^2\varphi(k)}\right)$$

where

$$B_k = \frac{\mu(k)}{k\varphi(k)\Gamma(\mu(k)/\varphi(k))}\prod_{p|k}\left(1-\frac{\mu(k)}{p\varphi(k)}\right)\operatorname*{Lt}_{s\to 1}(s-1)^{\mu(k)/\varphi(k)} \times$$
$$\times \prod_{\tilde{\omega}}\left(1-\frac{\mu(k)}{\varphi(k)}\tilde{\omega}^{-s}\right)^{-1};$$

the characters are to modulus k and

$$\tau(\chi) = \sum_{\substack{l=1\\(l,k)=1}}^{k} \bar{\chi}(l)e^{2i\pi l/k}.$$

The proof of this is straightforward but involves infinite products which would be awkward to print and I suppress it. Note that $B_k \geq 0$ for squarefree k, for

$$\frac{\mu(k)}{\varphi(k)}\Gamma\left(\frac{\mu(k)}{\varphi(k)}\right) = \Gamma\left(1+\frac{\mu(k)}{\varphi(k)}\right) > 0$$

and the remaining factors are positive. I cannot determine the sign of A_k/B_k in general, however we can show that

$$A_6 > 0.$$

For we have

$$A_6 = 2B_6\prod_{p\nmid 6}\left(1-\frac{1}{4p^2}\right)\boldsymbol{R}\left[e^{-i\pi/3}\prod_{p\nmid 6}\left(1+\frac{i\chi(p)\sqrt{3}}{2p+1}\right)\right]$$

that is, twice the real part of the term involving $j = 1$. Denoting the term in square brackets by A we may show that

$$\left|\arg A + \frac{\pi}{3} - \frac{\sqrt{3}}{2}\log L(1,\chi)\right| \leq 3/25.$$

But

$$1 > L(1,\chi) = 1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\ldots > \frac{4}{5}$$

and so

$$-\frac{\pi}{2} < -\frac{\pi}{3} - \frac{\sqrt{3}}{2}\log\frac{5}{4} - \frac{3}{25} < \arg A < -\frac{\pi}{3} + \frac{3}{25} < 0.$$

Hence A is in the fourth quadrant of the Argand diagram and A_6 is positive.

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Received on 3. 11. 1970 (121)