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A problem of Schur and its generalizations

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§ 1. A problem of Schur. A set S of integers is said to be *sum-free* if $a, b \in S$ implies $a+b \notin S$ (a and b not necessarily distinct).

A well known theorem of I. Schur [11] states that if the integers $1, 2, \ldots, [n!e]$ are partitioned in any manner into classes, then at least one of the classes is not sum-free. Accordingly, we define $f(n)$ to be the largest positive integer such that the integers $1, 2, \ldots, f(n)$ can be partitioned in some manner into n sum-free classes.

It is easy to verify that $f(1) = 1$, $f(2) = 4$ and $f(3) = 13$. In 1961 L. D. Baumert [3] with the aid of a high speed computer, showed that $f(4) = 44$. The value of $f(n)$ for $n > 4$ is not known and it appears very difficult to determine $f(n)$, even for $n = 5$.

In [11] Schur proved that

$$f(n+1) \geqslant 3f(n)+1 \tag{1.1}$$

and as a result of this

$$f(n) \geqslant \frac{3^n-1}{2}. \tag{1.2}$$

Defining $g(m)$ to be the smallest number of sum-free classes into which the integers $1, 2, \ldots, m$ can be partitioned, H. L. Abbott and L. Moser [2] showed that for all positive integers p and q

$$f\big(pq+g(pf(q))\big) \geqslant \big(2f(q)+1\big)^p-1. \tag{1.3}$$

From this they deduce that for some absolute constant c and all n sufficiently large

$$f(n) > 89^{n/4-c\log n}, \tag{1.4}$$

which improves Schur's lower bound. On the other hand Schur's theorem states

$$f(n) \leqslant [n!e]-1. \tag{1.5}$$

In this paper we obtain a lower bound for $f(n)$ which is better than that given by (1.4). However, instead of studying $f(n)$ directly we consider some generalizations.

§ 2. A generalization of Schur's problem. As was observed by R. Rado [8] the problem of Schur is a special case of a more general problem. Consider the following equation in l unknowns $x_1, x_2, \ldots, x_l$:

$$\sum_{i=1}^{l} a_i x_i = 0, \tag{2.1}$$

where $a_1, a_2, \ldots, a_l$ are non-zero integers. Rado called equation (2.1) *n-fold regular* if there exists a non-negative integer $f(n)$, which we take to be minimal, such that if the integers $1, 2, \ldots, f(n)+1$ are partitioned in any manner into n classes, then at least one of the classes contains a solution to (2.1). Equation (2.1) is said to be *regular* if it is n-fold regular for every positive integer n.

One of the main results which Rado establishes is the following criterion giving necessary and sufficient conditions for an equation to be regular: Equation (2.1) is regular if and only if some subset of the coefficients has zero sum. Thus the equation $x_1+x_2-x_3 = 0$ is regular and it is easy to see that the problem of Schur consists of finding bounds for $f(n)$ for the equation $x_1+x_2-x_3 = 0$. The problem of estimating lower bounds for $f(n)$ for a number of regular equations was considered by H. Salié [10] and Abbott [1].

Write (2.1) in the form

$$\sum_{i=1}^{t} a_i x_i = \sum_{i=t+1}^{l} a_i x_i \tag{2.2}$$

where $a_1, a_2, \ldots, a_l$ are positive integers. Suppose (2.2) is regular. Henceforth we assume that

$$A = \sum_{i=1}^{t} a_i > \sum_{i=t+1}^{l} a_i = B,$$

since otherwise $f(n) = 0$ for all n.

THEOREM 2.1. *Let m be a positive integer. Let M and N be integers satisfying*

$$(A-1)f(m) \leqslant M < N \tag{2.3}$$

and

$$Af(m)+1 \leqslant N \leqslant \left\{\frac{A}{A-1}(M+1)\right\}, \tag{2.4}$$

where

$$\{x\} = \begin{cases} [x] & \text{if } x \text{ is not an integer,} \\ x-1 & \text{if } x \text{ is an integer.} \end{cases}$$

Let $h(M, N)$ be the least number of sets into which the integers $1, 2, \ldots, M$ can be partitioned, no set containing a solution of any of the equations

$$\sum_{i=1}^{t} a_i x_i = \sum_{i=t+1}^{l} a_i x_i + \mu N \tag{2.5}$$

where

$$\mu = -B+1, -B+2, \ldots, A-1 \quad \text{if} \quad N < \frac{A}{A-1} M$$

and

$$\mu = -B+1, -B+2, \ldots, A-2 \text{ if } \frac{A}{A-1} M \leqslant N \leqslant \left\{\frac{A}{A-1}(M+1)\right\}.$$

Let $h(m) = \min h(M, N)$ where the minimum is taken over all pairs M, N satisfying (2.3) and (2.4). Then for all positive integers n

$$f(n+h(m)) \geqslant N_1 f(n) + M_1$$

where N_1 and M_1 satisfy (2.3) and (2.4) and $h(M_1, N_1) = h(m)$.

Proof. Partition the integers $1, 2, \ldots, M_1$ into $h(m)$ classes $C_1, C_2, \ldots, C_{h(m)}$ satisfying the conditions given in defining $h(m)$. Let

$$A' = \{bN_1+c \mid b = 0, 1, \ldots, f(n), c = 1, 2, \ldots, M_1\}.$$

Partition A' into $h(m)$ classes $C_1', C_2', \ldots, C_{h(m)}'$ by placing bN_1+c in C_i' if $c \in C_i$. Partition the integers $1, 2, \ldots, f(n)$ into n classes $D_1, D_2, \ldots, D_n$ none of which contains a solution of (2.2). Let

$$B' = \{bN_1-c \mid b = 1, 2, \ldots, f(n), c = 0, 1, \ldots, N_1-M_1-1\}.$$

Partition B' into n classes $C_{h(m)+1}', C_{h(m)+2}', \ldots, C_{h(m)+n}'$ by placing bN_1-c in class $C_{h(m)+i}'$ if $b \in D_i$. It is easy to see that

$$A' \cap B' = \emptyset \quad \text{and} \quad A' \cup B' = \{1, 2, \ldots, N_1 f(n)+M_1\}$$

and thus we have partitioned the integers $1, 2, \ldots, N_1 f(n)+M_1$ into $h(m)+n$ classes $C_1', C_2', \ldots, C_{h(m)+n}'$ and it remains only to show that none of these classes contains a solution to equation (2.2).

Consider first the classes $C_1', C_2', \ldots, C_{h(m)}'$. If any one of these classes contains a solution to equation (2.2) it is of the form

$$\sum_{i=1}^{t} a_i(b_i N_1 + c_i) = \sum_{i=t+1}^{l} a_i(b_i N_1 + c_i)$$

where $0 \leqslant b_i \leqslant f(n)$ and $1 \leqslant c_i \leqslant M_1$. Hence we must have

$$\sum_{i=1}^{t} a_i c_i \equiv \sum_{i=t+1}^{l} a_i c_i \pmod{N_1}.$$

But $0 < \sum_{i=1}^{t} a_i c_i \leqslant AM_1$ and $0 < \sum_{i=t+1}^{l} a_i c_i \leqslant BM_1$. Therefore, if $M_1 \leqslant N_1 < \frac{A}{A-1} M_1$, we have $AM_1 \leqslant AN_1$ and $BM_1 \leqslant BN_1$. Then by the definition of $h(m)$ we have a contradiction. On the other hand, if $\frac{A}{A-1} M_1 \leqslant N_1 \leqslant \left\{\frac{A}{A-1}(M_1+1)\right\}$ then $AM_1 \leqslant (A-1)N_1$ and $BM_1 < BN_1$ and again by the definition of $h(m)$ we have a contradiction. Therefore none of the classes $C'_1, C'_2, \ldots, C'_{h(m)}$ contains a solution to equation (2.2).

Now consider the classes $C'_{h(m)+1}, C'_{h(m)+2}, \ldots, C'_{h(m)+n}$. If any one of these classes contains a solution to equation (2.2) it is of the form

$$\sum_{i=1}^{t} a_i(b_i N_1 - c_i) = \sum_{i=t+1}^{l} a_i(b_i N_1 - c_i)$$

where $1 \leqslant b_i \leqslant f(n)$ and $0 \leqslant c_i \leqslant N_1 - M_1 - 1$. By construction we must have either

$$\sum_{i=1}^{t} a_i b_i N_1 \leqslant \sum_{i=t+1}^{l} a_i b_i N_1 + N_1$$

or

$$\sum_{i=1}^{t} a_i b_i N_1 + N_1 \leqslant \sum_{i=t+1}^{l} a_i b_i N_1 .$$

In the first case we must have that

$$A(N_1 - M_1 - 1) \geqslant \sum_{i=1}^{t} a_i c_i \geqslant \sum_{i=t+1}^{l} a_i c_i + N_1 \geqslant N_1$$

which is false if $N_1 \leqslant \left\{\frac{A}{A-1}(M_1+1)\right\}$. In the second case we must have that

$$N_1 \leqslant N_1 + \sum_{i=1}^{t} a_i c_i \leqslant \sum_{i=t+1}^{l} a_i c_i \leqslant B(N_1 - M_1 - 1) < A(N_1 - M_1 - 1)$$

which again is false if $N_1 \leqslant \left\{\frac{A}{A-1}(M_1+1)\right\}$. Therefore none of the classes $C'_{h(m)+1}, C'_{h(m)+2}, \ldots, C'_{h(m)+n}$ contains a solution to equation (2.2) and the proof of the theorem is complete.

Consider the regular equation

$$2x_1 + x_2 = 2x_3. \tag{2.6}$$

Salié [10] proved that for equation (2.6) $f(n) \geqslant 2^n - 1$ and Abbott [1] improved this result to

$$f(n) > 40^{n/5 - c\log n} \tag{2.7}$$

for some constant c and n sufficiently large. Applying Theorem 2.1 to equation (2.6) with $m = 2$, $M_1 = 9$ and $N_1 = 12$ we have that $h(2) = 3$ as may be seen by the following partitioning of the integers $1, 2, \ldots, 9$:

$$C_1 = \{1, 6, 7\}, \quad C_2 = \{2, 5, 8\}, \quad C_3 = \{3, 4, 9\}.$$

Therefore Theorem 2.1 implies that for equation (2.6)

$$f(n+3) \geqslant 12f(n) + 9 \tag{2.8}$$

and consequently

$$f(n) > c12^{n/3} \tag{2.9}$$

for some constant c which improves (2.7) considerably.

Salié [10] also proved that $f(n) \geqslant 2^n - 1$ for the regular equation $x_1 + x_2 + x_3 = 2x_4$ and Abbott [1] improved this to $f(n) > 10^{n/3 - c\log n}$ for some constant c and n sufficiently large. Theorem 2.1 may be used to improve this result to

$$f(n) > c10^{n/3}$$

for some constant c.

Clearly estimates for $f(n)$ for many regular equations can be found in this manner. However the difficulty in determining $h(m)$ may be as difficult in general as determining $f(n)$ itself.

Let $f_k(n)$ be defined as follows: $f_k(n)$ is the largest positive integer such that the integers $1, 2, \ldots, f_k(n)$ can be partitioned into n classes, no class containing a solution to the following system, (S), of $\binom{k-1}{2}$ equations in $\binom{k}{2}$ unknowns:

$$x_{ij} + x_{j,j+1} = x_{i,j+1}, \qquad 1 \leqslant i < j \leqslant k-1.$$

We will call such classes (S)-free. It is easy to see that $f_3(n) = f(n)$, where $f(n)$ is the Schur function for sum free sets. That $f_k(n)$ exists for $k > 3$ follows from the results of R. Rado [8]. Now define $g_k(m)$ as follows: If $f_k(n-1) < m \leqslant f_k(n)$, then $g_k(m) = n$; i.e. $g_k(m)$ is the smallest number of (S)-free classes into which the integers $1, 2, \ldots, m$ can be partitioned. E. R. Williams [13] has shown for all positive integers p and q that

$$f_k\big(pq + g_k(pf_k(q))\big) \geqslant (2f_k(q)+1)^p - 1. \tag{2.10}$$

This was proven analogously to the work of Abbott and Moser and reduces, in the case $k = 3$, to their result (1.3).

The following theorem may be deduced by arguments similar to those used to prove Theorem 2.1. We omit the details.

THEOREM 2.2. *For all positive integers n and m*

$$f_k(n+m) \geqslant \big(2f_k(m)+1\big)f_k(n)+f_k(m).$$

COROLLARY 2.1. *For all positive integers m and n*

$$f(n+m) \geqslant \big(2f(m)+1\big)f(n)+f(m).$$

Proof. Let $k = 3$ in Theorem 2.2.

COROLLARY 2.2. *For $n \geqslant 4$, and for some absolute constant c,*

$$f(n) \geqslant c\,89^{n/4}.$$

Proof. By Corollary 2.1 we have $f(n+4) \geqslant 89f(n)+44$, and this implies the result with $c = 44/89$.

It is clear that the lower bound for $f(n)$ given by Corollary 2.2 is better than that given by (1.4).

COROLLARY 2.3. *For $n \geqslant 1$ and for some constant c_k, dependent only on k,*

$$f_k(n) \geqslant c_k(2k-3)^n.$$

Proof. Since $f_k(1) = k-2$, the result follows from Theorem 2.2 with $c_k = (k-2)/(2k-3)$.

§ 3. Some applications to Ramsey's Theorem. In 1930, F. P. Ramsey [9] published a combinatorial theorem which may be formulated as follows:

RAMSEY'S THEOREM. *Let n, k and r be positive integers with $k \geqslant r$. Then there exists a least positive integer $R_n(k, r)$ such that if $s \geqslant R_n(k, r)$, S is a set of s elements, and the collection of subsets of S with r elements is partitioned in an arbitrary manner into n classes, then there is some subset K of S with k elements such that the subsets of K with r elements all belong to the same class.*

In this section we shall be concerned only with the case $r = 2$. We may then reformulate Ramsey's Theorem in this special case as follows:

If G is a complete graph on $R \geqslant R_n(k, 2)$ vertices and if each edge of G is colored in any one of n colors, then there results a complete subgraph of G on k vertices, all of whose edges have the same color, i.e. a complete monochromatic k-gon.

Many studies have been done on $R_n(k, 2)$ but the problem of evaluating this function appears very difficult even for small values of n and k. Erdös [4] and Abbott [1] have shown that

$$R_2(k, 2) > ck2^{k/2} \tag{3.1}$$

for some constant c. The argument used by Erdös to prove (3.1) can be used to prove

$$\binom{R_n(k, 2)}{k} \geqslant n^{\binom{k}{2}-1}. \tag{3.2}$$

This gives a lower bound of approximately $kn^{k/2}$. On the other hand R. E. Greenwood and A. M. Gleason [6] have shown

$$R_n(k, 2) \leqslant \frac{(nk-n)!}{\big((k-1)!\big)^n}$$

and in the particular case $k = 3$ that

$$R_n(3, 2) \leqslant [n!\,e]+1. \tag{3.3}$$

In this section we shall be concerned with estimating a lower bound for $R_n(k, 2)$ for some small values of k. In this direction the best previous results are those of Guy R. Giraud [5]. Giraud has shown for $n \geqslant 2$

$$R_n(4, 2) \geqslant \frac{33}{2}5^{n-2}+\frac{3}{2}$$

and

$$R_n(5, 2) \geqslant \frac{73}{2}7^{n-2}+\frac{3}{2}.$$

Here we shall improve these results.

Let $f_k(n)$ and the system (S) be defined as in Section 2. Partition the integers $1, 2, \ldots, f_k(n)$ into n (S)-free classes $C_1, C_2, \ldots, C_n$. Let G be a graph with vertices $P_0, P_1, \ldots, P_{f_k(n)}$. Color the edge (P_i, P_j) color c_r if $|i-j| \in C_r$. Suppose that $P_{i_1}, P_{i_2}, \ldots, P_{i_k}$, where $i_1 > i_2 > \ldots > i_k$, are the vertices of a monochromatic k-gon of color c_r. Then $i_t - i_s \in C_r$ for $1 \leqslant t < s \leqslant k$. But then

$$(i_t-i_s)+(i_s-i_{s+1}) = (i_t-i_{s+1}), \qquad 1 \leqslant t < s \leqslant k-1$$

is a solution to the system (S) in C_r, a contradiction. Therefore we have

$$R_n(k, 2) \geqslant f_k(n)+2. \tag{3.4}$$

Equation (3.4) together with the result of Greenwood and Gleason (3.3) imply Schur's result (1.5).

THEOREM 3.1. *For $n \geq 1$, $k \geq 2$ and for some constant c_k, dependent only on k*

$$R_n(k, 2) \geq c_k(2k-3)^n.$$

Proof. This is an immediate consequence of equation (3.4) and Corollary 2.3.

Theorem 3.1 as opposed to the inequalities (3.1) and (3.2) is effective when k is small and n is large. The theorem could perhaps be improved substantially if some new estimates for $f_k(n)$ could be found for $n > 1$. Although one might conjecture that $f_k(n)$ grows like $R_n(k, 2)$ we cannot obtain any useful estimates of even $f_k(2)$. However in certain special cases we can improve the lower bound given by Theorem 3.1.

THEOREM 3.2. *For $n \geq 4$ and for some constant c*

$$R_n(3, 2) \geq c89^{n/4}.$$

Proof. This is an immediate consequence of (3.4) and Corollary 2.2.

THEOREM 3.3. *For $n \geq 2$ and some constant c*

$$R_n(4, 2) \geq c33^{n/2}.$$

Proof. Partition the integers $1, 2, \ldots, 16$ into the following sets:

$$C_1 = \{1, 2, 4, 8, 9, 13, 15, 16\},$$
$$C_2 = \{3, 5, 6, 7, 10, 11, 12, 14\},$$

where C_1 consists of the quadratic residues of 17 and C_2 the non residues.

From this partitioning it follows that $f_4(2) \geq 16$, since it is a routine matter to verify that C_1 and C_2 are (S)-free. The result now follows from (3.4) and Theorem 2.1.

In a similar manner it can be shown that $f_5(2) \geq 37$ and consequently $R_n(5, 2) > c75^{n/2}$ for some constant c. However in [1] Abbott has shown for integers a and $b \geq 2$

$$R_n(ab-a-b+2, 2) \geq \big(R_n(a, 2)-1\big)\big(R_n(b, 2)-1\big)+1. \tag{3.5}$$

Taking $a = b = 3$ in (3.5) we have

$$R_n(5, 2) \geq \big(R_n(3, 2)-1\big)^2+1$$

and in view of Theorem 3.2 we have

THEOREM 3.4. *For $n \geq 2$ and some constant c*

$$R_n(5, 2) > c89^{n/2}.$$

§ 4. A problem of Turán. Schur's theorem can be generalized in other directions. One such generalization is the following question raised by P. Turán [12]: If n and m are positive integers, denote by $f(m, n)$ the largest possible integer such that the integers $m, m+1, \ldots, m+f(m, n)$ can be partitioned into n sum-free sets. What can be said about $f(m, n)$?

It is clear that

$$f(1, n) = f(n)-1$$

and since the integers $m, 2m, \ldots, m\big(f(n)+1\big)$ cannot be partitioned into n sum-free sets, that

$$f(m, n) \leq mf(n)-1. \tag{4.1}$$

Using (1.5) we have

$$f(m, n) \leq m[n!\,e]-m-1.$$

Turán considered the function $f(m, 2)$ and proved that $f(m, 2) = 4m-1$. H. L. Abbott [1] observed that in fact we have equality in equation (4.1) for $m = 1, 2, 3$ and that

$$f(m, n+1) \geq 3f(m, n)+m+2$$

and consequently

$$f(m, n) \geq \frac{m3^n-m-2}{2}. \tag{4.2}$$

S. Znam has also studied the function $f(m, n)$, [14], but does not obtain any improvements on the results of Abbott. In [1] Abbott asks whether there exists a constant $c > 3$ such that

$$f(m, n) > mc^n$$

for all m and all n sufficiently large? We can now answer this question in the affirmative.

First we prove the following:

THEOREM 4.1. *For any positive integer n, define $g(n)$ to be the largest positive integer such that the integers $1, 2, \ldots, g(n)$ may be partitioned into n classes, none of which contain a solution of either of the equations*

$$\begin{aligned} x_1+x_2 &= x_3, \\ x_1+x_2+1 &= x_3. \end{aligned} \tag{4.3}$$

We will call such classes strongly sum-free. Then for any positive integer m

$$g(n+m) \geq 2f(m)g(n)+f(m)+g(n)$$

where $f(m)$ is the Schur function for the equation $x_1+x_2 = x_3$.

Proof. Given a partitioning of $1, 2, \ldots, f(m)$ into m sum-free classes $A_1, A_2, \ldots, A_m$, partition the integers $1, 2, \ldots, 2f(m)+1$ into $m+1$ classes $B_1, B_2, \ldots, B_{m+1}$ as follows:

$$B_i = \{2a \mid a \in A_i\}, \quad i = 1, 2, \ldots, m,$$
$$B_{m+1} = \{1, 3, 5, \ldots, 2f(m)+1\}.$$

The classes B_i, for $i = 1, 2, \ldots, m$ are strongly sum-free and B_{m+1} is sum-free.

Partition the integers $1, 2, \ldots, g(n)$ into n strongly sum-free classes $C_1, C_2, \ldots, C_n$. Construct $m+n$ classes $D_j, j = 1, 2, \ldots, m+n$, as follows: For $j = 1, 2, \ldots, m$

$$D_j = \{(2a-1)g(n)+a+b \mid 2a \in B_j,\ b = 0, 1, \ldots, g(n)\}$$

and for $j = 1, 2, \ldots, n$

$$D_{m+j} = \{2ag(n)+a+b \mid 2a+1 \in B_{m+1},\ b \in C_j\}.$$

Then the classes $D_1, D_2, \ldots, D_{m+n}$ contain the integers

$$1, 2, \ldots, 2f(m)g(n)+f(m)+g(n)$$

and it remains to be shown that they are strongly sum-free.

Suppose that for some $j, 1 \leqslant j \leqslant m$, D_j is not strongly sum-free. Then either

$$(4.4)\quad (2a_1-1)g(n)+a_1+b_1+(2a_2-1)g(n)+a_2+b_2 = (2a_3-1)g(n)+a_3+b_3$$

or

$$(4.5)\quad (2a_1-1)g(n)+a_1+b_1+(2a_2-1)g(n)+a_2+b_2+1 = (2a_3-1)g(n)+a_3+b_3$$

where in each case $a_1, a_2, a_3 \in A_j$ and $0 \leqslant b_1, b_2, b_3 \leqslant f(n)$. Now (4.4) implies

$$(4.6)\quad \big(2g(n)+1\big)(a_1+a_2-a_3) = g(n)+b_3-b_1-b_2.$$

Since A_j is sum-free, $a_1+a_2-a_3 \neq 0$. Therefore the left side of (4.6) is at least $2g(n)+1$ in absolute value, while the right side is at most $2g(n)$. Hence (4.4) cannot hold. A similar argument shows that (4.5) cannot hold and thus D_j is strongly sum-free for $j = 1, 2, \ldots, m$.

Now suppose some class D_{m+j}, $1 \leqslant j \leqslant n$, is not strongly sum-free. Then either

$$(4.7)\quad 2a_1g(n)+a_1+b_1+2a_2g(n)+a_2+b_2 = 2a_3g(n)+a_3+b_3$$

or

$$(4.8)\quad 2a_1g(n)+a_1+b_1+2a_2g(n)+a_2+b_2+1 = 2a_3g(n)+a_3+b_3$$

where in each case $2a_1+1, 2a_2+1, 2a_3+1 \in B_{m+1}$ and $b_1, b_2, b_3 \in C_j$. Now (4.7) implies

$$(4.9)\quad \big(2g(n)+1\big)(a_1+a_2-a_3) = b_3-b_1-b_2.$$

But, since $b_1, b_2, b_3 \in C_j$, (4.9) implies $a_1+a_2-a_3 = 0$ and thus $b_1+b_2 = b_3$ which contradicts the definition of $g(n)$. A similar argument shows that (4.8) cannot hold. Hence D_{m+j} is strongly sum-free for $j = 1, 2, \ldots, n$ and the theorem is proved.

THEOREM 4.2. *For any positive integers m and n*

$$f(m, n) \geqslant mg(n)-1.$$

Proof. Partition the integers $1, 2, \ldots, g(n)$ into n strongly sum-free classes $C_1, C_2, \ldots, C_n$. Now partition the integers $m, m+1, \ldots, mg(n)+ +m-1$ into n classes $C_1', C_2', \ldots, C_n'$ by placing $am+b$ in C_i' whenever $a \in C_i$, where $a = 1, 2, \ldots, g(n)$ and $b = 0, 1, \ldots, m-1$.

Suppose for some $j, 1 \leqslant j \leqslant n$, C_j' is not sum-free. Then we must have

$$(4.10)\quad a_1m+b_1+a_2m+b_2 = a_3m+b_3$$

where $a_1, a_2, a_3 \in C_j$ and $0 \leqslant b_1, b_2, b_3 \leqslant m-1$. Equation (4.10) implies

$$(4.11)\quad m(a_1+a_2-a_3) = b_3-b_1-b_2.$$

But, since C_j is strongly sum-free, the left hand side of (4.11) is either at least m or at most $-2m$. It now follows since $0 \leqslant b_1, b_2, b_3 \leqslant m-1$ that C_j' is sum-free and the theorem is proved.

We may now obtain a much stronger result than that given by (4.2).

COROLLARY 4.1. *For any positive integer m and n*

$$f(m, n) \geqslant m\big(3f(n-1)+1\big)-1.$$

Proof. Let $n = 1$ and $m = n-1$ in Theorem 4.1 and we have

$$g(n) \geqslant 3f(n-1)+1$$

and the result now follows from Theorem 4.2.

COROLLARY 4.2. *For any positive integers m and n*

$$f(m, n) > cm89^{n/4}$$

for some absolute constant c.

Proof. This is an immediate consequence of Corollary 4.1 and Corollary 2.2.

§ 5. Some related questions. An analogous problem to that of sum-free sets is that of product free sets, i.e., what is the largest positive integer $l(n)$ such that the integers $2, 3, \ldots, l(n)$ can be partitioned into n classes, no class containing a solution to the equation $x_1x_2 = x_3$? It is easy to see that

$$(5.1)\quad 2^{\frac{3^n+1}{2}-1} \leqslant l(n) \leqslant 2^{f(n)+1}-1$$

where $f(n)$ is the Schur function for sum-free sets.

If we partition the integers $2^1, 2^2, \ldots, 2^{g(n)}$ into n classes $C_1, C_2, \ldots, C_n$, where $g(n)$ is the function defined in Theorem 4.1, and place 2^k+j in class C_i whenever $2^k \in C_i$ and $j = 0, 1, \ldots, 2^k-1$, then it is easy to see that

THEOREM 5.1. *For any positive integer* n

$$2^{g(n)+1}-1 \leqslant l(n) \leqslant 2^{f(n)+1}-1.$$

COROLLARY 5.1. *For any positive integer* n

$$l(n) \geqslant 2^{3f(n-1)+1}-1.$$

These results clearly are substantial improvements of (5.1).

We now consider an analogous problem in set theory: Given a positive integer n, what is the minimum number, $k(n)$, such that the 2^n subsets of a set S of n elements can be partitioned into $k(n)$ union-free classes? That is, no class contains a solution to $A \cup B = C$, A, B and C distinct.

Consider the following partitioning of the integers $1, 2, \ldots, n$:

$$C_1 = \{1, 3, 7, \ldots\},$$

$$C_i = \{2(i-1), 4(i-1)+1, 8(i-1)+3, \ldots\}, \quad i = 2, 3, \ldots, [n/2]+1.$$

If we now place all the subsets of S of order k in a class C_i' whenever $k \in C_i$ it is easy to see that

$$k(n) \leqslant \left[\frac{n}{2}\right]+1. \tag{5.2}$$

On the other hand, D. Kleitman [7] has shown, for some constant c, that no union-free class can contain more than $c\dfrac{2^n}{\sqrt{n}}$ subsets of S. Therefore it follows that

$$k(n) > c\sqrt{n} \tag{5.3}$$

for some constant c.

At the present time we have not been able to improve either of these results even though one might expect $k(n)$ to be closer to (5.2) than to (5.3).

One can also raise similar questions about sum-free sets in Abelian groups. Let G be an Abelian group of order n and denote by $f(G)$ the least number of sum-free sets into which $G-\{e\}$ can be partitioned and denote by $f(n)$ the maximum of $f(G)$ where the maximum is taken over all Abelian groups of order n. Then the original Schur argument can be modified to give

$$f(n) > \frac{c\log n}{\log\log n}$$

for some constant c and all sufficiently large n.

We can prove that

$$f(n) < c_1 \log n$$

for some constant c_1 and all sufficiently large n. We have not been able to sharpen the bounds given above. In fact we cannot even evaluate $f(G)$ for Abelian groups of "small" order.

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