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On the distribution of $\arg L(s,\chi)$ in the half-plane $\sigma > \frac{1}{2}$

by

P. D. T. A. Elliott (Nottingham)

1. Introduction. For each complex number $s = \sigma + it$, $\sigma = \operatorname{Re} s$, and non-principal Dirichlet character $\chi \pmod D$, we consider the L-series which is defined to be the analytic continuation of

$$L(s, \chi_D) = \sum_{n=1}^{\infty} \chi(n) n^{-s} \quad (\sigma > 1).$$

For each real number $Q \geqslant 2$ let M_Q denote the sum $\sum (p-1)$ taken over all the odd prime numbers not exceeding Q. We thus count the total number of non-principal characters to prime moduli not exceeding Q. It follows from the prime number theorem that

$$M_Q = \frac{Q^2}{2\log Q} + O\left(\frac{Q^2}{(\log Q)^2}\right).$$

When $L(s,\chi) \neq 0$, $\frac{1}{2} < \sigma \leqslant 1$, let $\arg L(s,\chi)$ denote a value of the argument of $L(s,\chi)$ defined by continuous displacement from the point $s = 2$ along an arc on which $L(s,\chi)$ does not vanish. Thus $\arg L(s,\chi)$ is only defined to within the addition of an integer multiple of 2π. We set

$$\nu_Q\left(\frac{1}{2\pi}\arg L(s,\chi) \leqslant z \pmod 1\right) = M_Q^{-1} \sum_{p \leqslant Q} \sum_{\chi \neq \chi_0} 1$$

where the double sum counts those pairs (p, χ), with χ a non-principal character $\pmod p$, and p an odd prime not exceeding Q, for which $\arg L(s,\chi)$ is defined and has the value $2\pi(n+\alpha)$, n an integer, $0 \leqslant \alpha \leqslant z$.

In order to make the assertion in the following theorem meaningful we recall a few notions concerning distribution functions (mod 1).

A function $G(z)$ is said to be a *distribution function* (mod 1) if and only if it satisfies the following three conditions

(i) It increases in the wide sense.

(ii) It is right continuous, that is $G(z+) = G(z)$ for all z.

(iii) $G(z) = 1$ if $z \geqslant 1$, and $= 0$ if $z < 0$.

A sequence $F_n(z)$ $(n = 1, 2, \ldots)$ of distribution functions of this type will be said to have a *weak limiting distribution* (mod 1) if there exists a further distribution $F(z)$ (mod 1) so that at all points a, β, $0 \leqslant a \leqslant \beta < 1$ which are points of continuity of $F(z)$,

$$F_n(\beta) - F_n(a) \to \big(F(\beta) - F(a)\big) \quad (n \to \infty).$$

Thus in the range $0 \leqslant z < 1$, $F(z)$ is determined only up to an additive constant. If, however,

$$F_n(z) \to F(z) \quad (n \to \infty),$$

holds at every continuity point z of $F(z)$, we say that the $F_n(z)$ *converge strongly* or just *converge*, to the limiting distribution $F(z)$ (mod 1). In this case $F(z)$ is uniquely determined, and with a natural extension if $z < 0$, or $z > 1$, this definition coincides with the usual definition of convergence of distribution functions which are defined on the whole real line.

We can now state our main result.

THEOREM. *At each point s in the half-plane $\sigma > \frac{1}{2}$ the frequencies*

$$\nu_Q\left(\frac{1}{2\pi} \arg L(s, \chi) \leqslant z \,(\mathrm{mod}\, 1)\right) \quad (Q = 2, 3, \ldots)$$

converge to a continuous limiting distribution (mod 1). *Its characteristic function* (= *Fourier transform*) *assumes the form*

$$k \mapsto \prod_p \left(1 + \sum_{m=1}^{\infty} \binom{-k/2}{m} \binom{k/2}{m} p^{-2m\sigma}\right)$$

where $\binom{\pm k/2}{m}$ *denote binomial coefficients. If $\frac{1}{2} < \sigma < 1$ the limiting distribution can be analytically continued from the line segment $0 \leqslant z < 1$ to be an integral function on the whole complex Z-plane.*

Remarks. We note that the form of the limit law depends only upon $\sigma = \mathrm{Re}\, s$, and so is the same on every line $\sigma =$ constant.

In fact we shall prove somewhat more than this. Let δ be a (small) positive constant. Then we shall prove that for each non zero integer k

$$\int_0^1 e^{2\pi i k z} d\nu_Q\left(\frac{1}{2\pi} \arg L(s, \chi) \leqslant z \,(\mathrm{mod}\, 1)\right)$$
$$= \sum_{n=1}^{\infty} c_k(n) c_{-k}(n) n^{-2\sigma} + O\big((\log\log\log Q)^{-1/2}\big)$$

holds uniformly at all points in the rectangle

$$\tfrac{1}{2} + \delta \leqslant \sigma \leqslant 2, \quad |t| \leqslant (\log\log\log Q)^{1/2}.$$

It follows easily from this that there exist limit laws $F(\sigma, z)$ so that

$$\nu_Q\left(\frac{1}{2\pi} \arg L(s, \chi) \leqslant z \,(\mathrm{mod}\, 1)\right) \to F(\sigma, z) \quad (Q \to \infty),$$

holds uniformly for all z, $0 \leqslant z < 1$, and for all points s lying in the above region. At the cost of some complication these estimates can be made uniform for $k \leqslant k_0(Q)$, where $k_0(Q)$ is a function of Q that is unbounded with Q, so that one can measure the rate of convergences of the frequencies in the theorem. Moreover, the height of the rectangle $|t| \leqslant (\log\log\log Q)^{1/2}$ in the region of s uniformity could be considerably increased. A number of the lemmas proved during the proof of the theorem are stated in a form suitable for such applications.

We shall confine our attention to the most interesting cases, and consider points in the semi-infinite strip $\frac{1}{2} < \sigma \leqslant 2$. The extension of the arguments to the cases $\sigma > 2$ is straightforward.

2. Outline of the proof. We shall consider the characteristic functions

$$\int_0^1 e^{2\pi i k z} d\nu_Q\left(\frac{1}{2\pi} \arg L(s, \chi) \leqslant z \,(\mathrm{mod}\, 1)\right) = M_Q^{-1} \sum_{p \leqslant Q} \sum_{\chi \neq \chi_0} e^{it \arg L(s, \chi)},$$

for $k = 0, \pm 1, \pm 2, \ldots$ Here, and without further ado in the similar following double sums which involve $\arg L(s, \chi)$ explicitly, we understand that we count only those pairs (p, χ) for which $\arg L(s, \chi)$ is well defined. These will turn out to be essentially all pairs. We prove that the above double sum converges for each integer k, and so by a generalization of a criterion of H. Weyl there will be a limit law. Its characteristic function turns out to be easily computable, and so we can justify our assertions concerning the nature of the limit law.

For a fixed particular pair (p, χ), and complex number s, with $L(s, \chi)$ non-zero, let $\theta = \arg L(s, \chi)$. One way to approach θ is by means of the relation

$$e^{2i\theta} = L(s, \chi) \big(\overline{L(s, \chi)}\big)^{-1} = L(s, \chi) \big(L(\bar{s}, \bar{\chi})\big)^{-1}.$$

In order to put this into practice we need a representation for $L(s, \chi)^{-1}$. We effect such a representation by first proving that in frequency (= in probability)

$$L(s, \chi) = \prod_{q \leqslant l} \big(1 - \chi(q) q^{-s}\big)^{-1} + O\big(l^{-\frac{1}{2}(\sigma - \frac{1}{2})}\big),$$

where $l = \log\log Q$. We next show that in probability the product which occurs in this estimate exceeds $\exp\big(-(\log l)^{3/4}\big)$ in absolute value, so that for sufficiently large Q, $L(s, \chi)$ does not vanish, and in fact

$$L(s, \chi) = \big(1 + O(l^{-(\sigma - \frac{1}{2})/4})\big) \prod_{q \leqslant l} \big(1 - \chi(q) q^{-s}\big)^{-1}.$$

Clearly, at the same time

$$\overline{L(s,\chi)}^{-1} = \big(1+O(l^{-(\sigma-\frac{1}{2})/4})\big)\prod_{q\le l}\big(1-\overline{\chi}(q)q^{-\bar{s}}\big).$$

These assertions amount to saying that the L-series $L(s,\chi)$ have an Euler product in probability.

Thus, in probability

$$e^{i\arg L(s,\chi)} = \big(1+o(1)\big)\prod_{q\le l}\big(1-\chi(q)q^{-s}\big)^{-1/2}\prod_{q\le l}\big(1-\overline{\chi}(q)q^{-\bar{s}}\big)^{1/2},$$

provided that we can satisfactorily define the operation of taking the square roots.

We wish, in particular, to estimate the mean of the left hand side of this relation taken over pairs (p,χ) with χ non-principal $(\operatorname{mod} p)$, and $p\le Q$, along with the usual proviso. In the above form it is not too clear that one can estimate sums of the form

$$\sum_{p}\sum_{\chi\ne\chi_0}\prod_{q\le l}\big(1-\chi(q)q^{-s}\big)^{1/2}$$

in a simple manner. For each integer $k\ne 0$ we define a multiplicative function $c_k(n)$ by

$$1+\sum_{m=1}^{\infty}c_k(q^m)\chi(q^m)q^{-ms} = \big(1-\chi(q)q^{-s}\big)^{-k/2}.$$

For odd integers k we shall once again need to exercise care in the choice of a square root. It is easy to see that such a function $c_k(n)$ grows slowly with n, and one can prove that in probability

$$\prod_{q\le l}\big(1-\chi(q)q^{-s}\big)^{-k/2} = \sum_{n\le l}c_k(n)\chi(n)n^{-s}+o(1).$$

Our problem then reduces to the estimation of sums of the form

$$M_Q^{-1}\sum_{p\le Q}\sum_{\chi\ne\chi_0}\sum_{m\le l}c_k(m)\chi(m)m^{-s}\sum_{n\le l}c_{-k}(n)\overline{\chi}(n)n^{-\bar{s}}.$$

This is an easy matter, and we can quickly complete the proof of the theorem.

The main difficulty in this argument is the satisfactory definition of

$$\big(L(s,\chi)L(\bar{s},\overline{\chi})^{-1}\big)^{1/2} = e^{i\arg L(s,\chi)},$$

when $\frac{1}{2}<\sigma\le 1$. In this context it is desirable that $L(s,\chi)^{1/2}$ can be analytically continued into a convenient part of the semi-infinite strip $\frac{1}{2}<\sigma\le 1$. To this end we shall prove that most L-series to moduli $p\le Q$ do not vanish in a rectangle $\frac{1}{2}+\delta\le\sigma\le 1$, $|t|\le(\log l)^{1/2}$. This result in fact follows from a theorem of Bombieri [1]. The proof of this theorem is, however, quite complicated, and a result weaker than his, but sufficient for our present purposes, can be readily obtained during our proof if we ensure that certain of our estimates are suitably uniform with respect to s.

3. We begin with the following simple lemma which we shall apply many times.

LEMMA 1. *Let N be a positive integer, and let $a_1,\ldots,a_N$ be N complex numbers, then*

$$\sum_{p\le Q}\sum_{\chi\ne\chi_0}\Big|\sum_{n\le N}a_n\chi(n)\Big|^2 = \Big(M_Q+O\Big(\frac{QN}{\log Q}+\frac{N^2}{\log N}\Big)\Big)\sum_{n\le N}|a_n|^2.$$

Proof. This estimate can be justified by expanding the sum over n, and inverting the order of summation. A detailed proof can be found in [3], Theorem 1.

We fix a (small) real number δ once and for all to satisfy $0<2\delta<1$, and denote by $\mathfrak{A}=\mathfrak{A}(Q,\delta)$ the rectangle in the complex s-plane given by

$$\tfrac{1}{2}+\delta\le\sigma\le 2,\qquad |t|\le(\log l)^{1/2}.$$

LEMMA 2. (Truncation lemma).

$$\nu_Q\Big(\max_{s\in\mathfrak{A}}\Big|L(s,\chi)-\sum_{n\le l}\chi(n)n^{-s}\Big|>l^{-\delta/2}\Big)\ll l^{-\delta}(\log l).$$

Remark. This lemma is a model for several others which follow it. The truncation in this lemma is very severe, so as to simplify the following arguments. In order to obtain a better measure of the rate of convergence of the frequencies in the theorem one would use a parameter somewhat larger than $l=\log\log Q$.

Proof. Consider first the sum

$$\sum_{l<n\le Q}\chi(n)n^{-s}.$$

Integration by parts shows that it can be expressed as

$$s\int_l^Q y^{-s-1}\sum_{l<n\le y}\chi(n)\,dy.$$

Then by the inequality of Cauchy–Schwarz

$$\max_{s\in\mathfrak{A}}\Big|\sum_{l<n\le Q}\chi(n)n^{-s}\Big|^2\le\max_{s\in\mathfrak{A}}|s|\int_l^Q y^{-\sigma-1}\,dy\int_l^Q y^{-\sigma-1}\Big|\sum_{l<n\le y}\chi(n)\Big|^2dy$$
$$\le c_1(\log l)^{1/2}l^{-\delta}\int_l^Q y^{-\delta-1}\Big|\sum_{l<n\le y}\chi(n)\Big|^2dy.$$

Hence

$$M_Q^{-1}\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\max_{s\in\mathfrak{A}}\Big|\sum_{l<n\leqslant Q}\chi(n)n^{-s}\Big|^2$$
$$\leqslant c_1(\log l)l^{-\delta}\int_l^Q y^{-\delta-1}M_Q^{-1}\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\Big|\sum_{l<n\leqslant y}\chi(n)\Big|^2dy$$

which by Lemma 1 is at most

$$c_2 l^{-\delta}(\log l)\int_l^Q y^{-\delta-1}\big(1+Q^{-1}y+M_Q^{-1}y^2(\log y)^{-1}\big)dy\ll l^{-2\delta}(\log l).$$

Moreover, for each s in $\mathfrak{A}$, by the Polya–Vinogradov inequality, if χ is a non-principal character (mod p), $p\leqslant Q$:

$$\sum_{n>Q}\chi(n)n^{-s} = s\int_Q^\infty y^{-\sigma-1}O(p^{1/2}\log p)\,dy$$
$$= O\big(Q^{1/2}\log Q(\log l)^2Q^{-\delta-1/2}\big) = O\big(l^{-2\delta}(\log l)\big).$$

Altogether, therefore

$$M_Q^{-1}\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\max_{s\in\mathfrak{A}}\Big|\sum_{n>l}\chi(n)n^{-s}\Big|^2\ll l^{-2\delta}(\log l),$$

and by a standard argument of Tchebycheff Lemma 2 is proved.

Lemma 3. *Let N be a positive integer not exceeding Q, and let $b_1, b_2, \ldots, b_N$ be a set of N complex numbers. Let further there be constants $A, E>0$ so that each b_n satisfies*

$$|b_n|\leqslant A\tau(n)^E\quad (n=1,\ldots,N)$$

where $\tau(n)$ denotes the number of divisors of the integer n. Then for each real number w, $1\leqslant w\leqslant N$,

$$\nu_Q\Big(\max_{s\in\mathfrak{A}}\Big|\sum_{w<n\leqslant N}b_n\chi(n)n^{-s}\Big|>z\Big)\ll z^{-2}w^{-2\delta}(\log\max(w,l))^F,$$

with $F=4^E$.

Proof. This result is obtained in a manner precisely similar to that used in the first part of the proof of Lemma 2. We note only that in the application of Lemma 1 we obtain the upper bound

$$\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\Big|\sum_{w<n\leqslant y}\chi(n)b_n\Big|^2\leqslant\Big(M_Q+O\Big(\frac{Qy}{\log Q}+\frac{y^2}{\log y}\Big)\Big)\sum_{w<n\leqslant y}|b_n|^2.$$

To estimate the sum involving the $|b_n|^2$ we note that by a well known estimate

$$\sum_{n\leqslant y}|b_n|^2\leqslant A^2\sum_{n\leqslant y}\tau(n)^{2E}\leqslant c_3y(\log y)^{F-1},$$

where $F=2^{2E}$. Everything now goes through as for Lemma 1 with the additional factor of a power of $\log y$.

We now apply Lemma 3 to show that in probability

$$\sum_{n\leqslant l}\chi(n)n^{-s}\quad\text{and}\quad\prod_{q\leqslant l}\big(1-\chi(q)q^{-s}\big)^{-1}$$

are approximately equal for every value of s in $\mathfrak{A}$.

Lemma 4. *We have*

$$\nu_Q\Big(\max_{s\in\mathfrak{A}}\Big|\sum_{n\leqslant l}\chi(n)n^{-s}-\prod_{q\leqslant l}\big(1-\chi(q)q^{-s}\big)^{-1}\Big|>l^{-\delta/2}\Big)\ll l^{-\delta}(\log l).$$

Proof. We begin by defining numbers b_n, for $l<n\leqslant Q$, so that

$$\sum_{n\leqslant l}\chi(n)n^{-s}-\prod_{q\leqslant l}\big(1-\chi(q)q^{-s}\big)^{-1}=\sum_{n>l}b_n\chi(n)n^{-s},\qquad\sigma>0.$$

It is clear that b_n is zero unless n is made up of primes q not exceeding l, and that for every integer $n>l$, $|b_n|\leqslant 1$.

By Lemma 3 with $E=0$, $w=l$, $z=l^{-\delta/2}$, we see that

$$\nu_Q\Big(\max_{s\in\mathfrak{A}}\Big|\sum_{l<n\leqslant Q}b_n\chi(n)n^{-s}\Big|>l^{-\delta/2}\Big)\ll l^{-\delta}(\log l).$$

We shall now show that uniformly for all s in $\mathfrak{A}$, the sum

$$\Big|\sum_{n>Q}b_n\chi(n)n^{-s}\Big|\leqslant\sum_{n>Q}|b_n|n^{-1/2-\delta}$$

is small.

Consider first those integers n for which $\omega(n)$, the number of distinct prime divisors of n, has the value k. Clearly their contribution to the above sum is at most

$$\frac{1}{k!}\Big(\sum_{r=1}^\infty q^{-(1/2+\delta)r}\Big)^k\leqslant\frac{1}{k!}(c_3l^{1/2-\delta})^k.$$

For a suitable positive constant c_4, by an application of Stirling's formula

$$\sum_{\omega(n)>c_4l^{1/2-\delta}}|b_n|n^{-1/2-\delta}\leqslant c_5\sum_{k>c_4l^{1/2-\delta}}2^{-k}\ll\exp(-c_4l^{1/2-\delta}).$$

Any remaining integer n at which b_n is non-zero has the properties

(i) $\omega(n) \leqslant c_4 l^{1/2-\delta}$,

(ii) $q \mid n \Rightarrow q \leqslant l$,

(iii) $n > Q$.

We denote summation over integers n which satisfy conditions (i) and (ii) by $''$. Let a_0 be the maximum power to which any prime divisor q of n occurs. Then

$$Q \leqslant \Big(\prod_{q\mid n} q\Big)^{a_0} \leqslant \exp\Big(a_0 \sum_{q\mid n} \log q\Big) \leqslant \exp(a_0 c_6 l)$$

so that

$$a_0 \geqslant c_6^{-1} l^{-1} \log Q = a_1,$$

say. We can therefore deduce that

$$\sum_{n>Q}{}'' |b_n| n^{-1/2-\delta} \leqslant \sum_{q\leqslant l} \sum_{\substack{q^{a_1}\mid n\\ n>Q}} |b_n| n^{-1/2-\delta}$$
$$\leqslant \sum_{q\leqslant l} q^{-a_1(1/2+\delta)} \sum_{n>l}{}'' |b_n| n^{-1/2-\delta} \leqslant \sum_{q\leqslant l} q^{-a_1/2} \sum_{n>l} |b_n| n^{-1/2-\delta},$$

where

$$\sum_{q\leqslant l} q^{-a_1/2} \leqslant 2^{-a_1/2} + \int_2^\infty y^{-a_1/2}\, dy \leqslant c_7 2^{-a_1/2}.$$

Altogether, therefore,

$$\sum_{n>Q}{}'' |b_n| n^{-1/2-\delta} \ll 2^{-a_1/2} \prod_{q\leqslant l} (1+q^{-1/2-\delta}) \ll \exp\Big(-a_1 \frac{\log 2}{2} + c_8 l^{1/2-\delta}\Big)$$
$$\ll \exp(-\sqrt{\log Q}),$$

and so

$$\max_{s\in\mathfrak{A}} \Big|\sum_{n>Q} b_n \chi(n) n^{-s}\Big| \ll \exp(-c_4 l^{1/2-\delta}).$$

The assertion of Lemma 4 now follows easily.

We have now shown that for all but a frequency of at most $O(l^{-\delta}(\log l)^5)$ we have

$$L(s, \chi) = \prod_{q\leqslant l} (1-\chi(q) q^{-s})^{-1} + O(l^{1-\delta/2})$$

uniformly for $\frac{1}{2}+\delta \leqslant \sigma \leqslant 2$, $|t| \leqslant (\log l)^{1/2}$.

Our next step is to prove that in probability $L(s, \chi)$ is also non-zero in this same region.

Lemma 5.

$$\nu_Q\Big(\max_{s\in\mathfrak{A}} \Big|\sum_{q\leqslant l} \chi(q) q^{-s}\Big| > (\log l)^{3/4}\Big) \ll (\log l)^{-1/2}.$$

Proof. We apply Lemma 3 with $b_n = 0$ unless n is a prime $q \leqslant l$, when we set $b_n = 1$. Thus we can take $E = 0$, $w = 1$, $z = (\log l)^{3/4}$, and obtain the present result.

It follows from the previous lemmas that in probability

$$|L(s, \chi)| \geqslant \Big|\prod_{q\leqslant l} (1-\chi(q) q^{-s})^{-1}\Big| - c_9 l^{-\delta/2} > c_{10} \exp(-(\log l)^{3/4}) > 0$$

holds uniformly for s in $\mathfrak{A}$. We can thus summarize our results so far as: in probability

$$L(s, \chi) = (1+O(l^{-\delta/4})) \prod_{q\leqslant l} (1-\chi(q) q^{-s})^{-1},$$
$$L(s, \chi)^{-1} = (1+O(l^{-\delta/4})) \prod_{q\leqslant l} (1-\chi(q) q^{-s})$$

both hold uniformly in the rectangle $\frac{1}{2}+\delta \leqslant \sigma \leqslant 2$, $|t| \leqslant \sqrt{\log l}$. Moreover, the frequency of those which are exceptional in the sense that this fails to be true is at most $O((\log l)^{-1/2})$.

We now consider the characteristic functions of the frequencies

$$\nu_Q\Big(\frac{1}{2\pi} \arg L(s, \chi) \leqslant z \ (\mathrm{mod}\ 1)\Big) \quad (Q = 2, 3, \ldots).$$

Let k be a rational integer. We shall assume it to be non-zero in what follows, since the case $k = 0$ will be trivial. Consider the characteristic function

$$M_Q^{-1} \sum_{p\leqslant Q} \sum_{\chi\neq\chi_0} e^{ik \arg L(s,\chi)}.$$

For convenience of notation we temporarily set $\theta = \theta(p, \chi) = \arg L(s, \chi)$. Then, as stated in the introduction, we shall use the representation

$$\text{(ii)} \quad e^{ik\theta} = \frac{L(s, \chi)^{k/2}}{L(\bar{s}, \bar{\chi})^{k/2}} = (1+O(l^{-\delta/4})) \prod_{q\leqslant l} \Big(1-\frac{\chi(q)}{q^s}\Big)^{-k/2} \prod_{q\leqslant l} \Big(1-\frac{\bar{\chi}(q)}{q^{\bar{s}}}\Big)^{k/2}.$$

If k is even this representation is clear. If k is odd, however, we have to specify the value of the square roots.

Consider first those pairs (p, χ) which are not ruled out by any of the Lemmas 2, 4 and 5. For these $L(s, \chi)$ does not vanish in the rectangle $\mathfrak{A}$, and we can define

$$L(s, \chi)^{k/2} = \exp\Big(\frac{k}{2} \log L(s, \chi)\Big)$$

where the value of the logarithm is obtained continuously from the principal value on the real axis. Whatever this value is

$$L(s, \chi)^{k/2} L(\bar{s}, \bar{\chi})^{-k/2}$$

can be interpreted as $\exp\{k(\log L(s,\chi)\}$ where the principal value of the argument is chosen, so that the first equality in (ii) has a meaning.

Set

$$L(s,\chi) = g(s)\prod_{q\leqslant l}\big(1-\chi(q)q^{-s}\big)^{-1}$$

so that

$$g(s) = 1+O(l^{-\delta/4}).$$

Then if we take the principal value (of the understood logarithm) in each factor

$$\big(1-\chi(q)q^{-s}\big)^{-k/2}$$

we shall have

$$L(s,\chi)^{k/2}\prod_{q\leqslant l}\big(1-\chi(q)q^{-s}\big)^{k/2} = g(s)^{k/2}$$

for some value of the right hand side. But, once again, whatever this value

$$g(s)^{k/2}\overline{g(s)}^{-k/2} = \big(g(s)\overline{g(s)}^{-1}\big)^{k/2}$$

where on the right hand side we take the principal value. Here $g(s)g(s)^{-1} = 1+O(l^{-\delta/4})$ so that

$$g(s)^{k/2}\overline{g(s)}^{-k/2} = 1+O(l^{-\delta/4}).$$

Thus, we can understand (ii) to hold with every (implied) logarithm having its principal value.

Let us call the set of pairs (p,χ) for which we have (ii), Π_1. We now prove that for most pairs (p,χ) in Π_1 we can replace the products over the primes q by a finite sum.

For each prime p, and character χ, we have

$$\text{(iii)}\quad \big(1-\chi(q)q^{-s}\big)^{-k/2} = 1+\sum_{m=1}^{\infty}\binom{-k/2}{m}\chi(q^m)q^{-ms} = \sum_{m=1}^{\infty}c_k(q^m)\chi(q^m)q^{-ms},$$

where

$$c_k(q^m) = \binom{-k/2}{m}\qquad (m = 0,1,\ldots)$$

is the function defined in the introduction. For we notice that this is certainly true (with the principal value on the left hand side of (iii)), if $\operatorname{Re}s$ is sufficiently large, and if $\sigma\geqslant\frac12$, $|\chi(q)q^{-s}|\leqslant 1/\sqrt2$, so that it holds in the half-plane $\sigma>\frac12$ by analytic continuation.

Clearly

$$|c_k(q^m)|\leqslant\frac{|k|}{2}\left(\frac{|k|}{2}+1\right)\ldots\left(\frac{|k|}{2}+m-1\right)\frac{1}{m!}\leqslant\frac{|k|}{2}\prod_{\nu=1}^{m-1}\left(1+\frac{|k|}{2\nu}\right)$$

$$\leqslant\frac{|k|}{2}\exp\left(\sum_{\nu=1}^{m-1}\frac{|k|}{2}\nu^{-1}\right)\leqslant\frac{|k|}{2}\exp\left(\frac{|k|}{2}\left(\int_1^{\infty}\frac{dy}{y}+1\right)\right)\leqslant cm^{|k|/2}$$

$$\leqslant(m+1)^E\qquad(m = 1,2,\ldots),$$

holds for a suitable constant $E\geqslant 0$ depending upon k only. It follows that if $n = \prod q_i^{m_i}$ is the canonical factorization of an integer n,

$$|c_k(n)|\leqslant\prod_i(m_i+1)^E = \tau(n)^E.$$

We can now apply Lemma 3 with $b_n = c_k(n)$, $w = l$, $l<n\leqslant Q$, $z = l^{-\delta/2}$ to deduce that if $\sim$ denotes summation over integers n made up only of prime factors $q\leqslant l$

$$\nu_Q\Big(\Big|\prod_{q\leqslant l}\big(1-\chi(q)q^{-s}\big)^{-k/2}-\sideset{}{^\sim}\sum_{l<n\leqslant Q}c_k(n)\chi(n)n^{-s}\Big|>l^{-\delta/2}\Big)\ll l^{-\delta}(\log l)^F.$$

By an argument exactly similar to that used in the proof of Lemma 4 we see that

$$\nu_Q\Big(\Big|\sideset{}{^\sim}\sum_{n>Q}c_k(n)\chi(n)n^{-s}\Big|>l^{-\delta/2}\Big)\ll l^{-\delta}(\log l)^F.$$

In a like manner we have that

$$\nu_Q\Big(\Big|\prod_{q\leqslant l}\big(1-\chi(q)q^{-\bar s}\big)^{-k/2}-\sideset{}{^\sim}\sum_{n>l}c_{-k}(n)\bar\chi(n)n^{-\bar s}\Big|>l^{-\delta/2}\Big)\ll l^{-\delta}(\log l)^F.$$

We now denote by Π_2 the set of all pairs (p,χ) in Π_1 which are not exceptional in the above senses. Then taking into account the cardinality of all sets of pairs so far deemed exceptional we have

$$\begin{aligned}
&M_Q^{-1}\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}e^{ik\arg L(s,\chi)}\\
&\quad= M_Q^{-1}\sum_{\substack{p\leqslant Q\\(p,\chi)\in\Pi_1}}\sum_{\chi\neq\chi_0}\Big(\sum_{m\leqslant l}c_k(m)\chi(m)m^{-s}+O(l^{-\delta/2})\Big)\times\\
&\qquad\qquad\times\Big(\sum_{n\leqslant l}c_{-k}(n)\bar\chi(n)n^{-\bar s}+O(l^{-\delta/2})\Big)+O\big((\log l)^{-1/2}\big)\\
&\quad= M_Q^{-1}\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\sum_{m\leqslant l}c_k(m)\chi(m)m^{-s}\sum_{n\leqslant l}c_{-k}(n)\bar\chi(n)n^{-\bar s}+\\
&\quad+O\Big(l^{-\delta/2}M_Q^{-1}\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\Big(\Big|\sum_{m\leqslant l}c_k(m)\chi(m)m^{-s}\Big|+\Big|\sum_{n\leqslant l}c_{-k}(n)\bar\chi(n)n^{-\bar s}\Big|\Big)\Big)+\\
&\quad+O\Big(M_Q^{-1}\sum_{\substack{p\leqslant Q,\chi\neq\chi_0\\(p,\chi)\notin\Pi_1}}\sum\Big|\sum_{m\leqslant l}c_k(m)\chi(m)m^{-s}\Big|\cdot\Big|\sum_{n\leqslant l}c_{-k}(n)\bar\chi(n)n^{-\bar s}\Big|\Big)+O\big((\log l)^{-1/2}\big)\\
&\quad= \Sigma_1+\Sigma_2+\Sigma_3+O\big((\log l)^{-1/2}\big),
\end{aligned}$$

say.

We can give an upper bound for the sums occuring in Σ_2 by applying first the inequality of Cauchy–Schwarz, and then Lemma 1. In this we obtain, for example

$$\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\Big|\sum_{m\leqslant l}c_k(m)\chi(m)m^{-s}\Big|^2 \ll M_Q\sum_{n=1}^{\infty}|c_k(n)|^2 n^{-2\sigma} \ll M_Q$$

and consequently $\Sigma_2 = O(l^{-\delta/2})$.

An upper bound for Σ_3 can also be obtained by applying the Cauchy–Schwarz inequality twice, and then Lemma 1. Thus we have

$$\Sigma_3^4 \ll (\log l)^{-1/2}M_Q^{-1}\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\Big|\sum_{m\leqslant l}c_k(m)\chi(m)m^{-s}\Big|^4 + \\ +(\log l)^{-1/2}M_Q^{-1}\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\Big|\sum_{n\leqslant l}c_{-k}(n)\bar{\chi}(n)n^{-\bar{s}}\Big|^4.$$

Setting

$$\Big(\sum_{m\leqslant l}c_k(m)\chi(m)m^{-s}\Big)^2 = \sum_{r\leqslant l^2}d_r\chi(r)r^{-s} \qquad (\sigma>0),$$

and so defining the numbers d_r, where

$$|d_r| = \Big|\sum_{d|n}c_k(d)\,d_k\Big(\frac{n}{d}\Big)\Big| \leqslant \sum_{d|n}\tau(n)^E\tau\Big(\frac{n}{d}\Big)^E \leqslant \tau(n)^{2E+1}$$

and noting that

$$\sum_{n=1}^{\infty}\tau(n)^{2E+1}n^{-2\sigma} < \infty,$$

we can apply Lemma 1 to deduce that

$$\Sigma_3 = O\Big(\Big((\log l)^{-1/2}\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\Big|\sum_{r\leqslant l^2}d_r\chi(r)r^{-s}\Big|^2\Big)^{1/4}\Big) \\ = O\Big((\log l)^{-1/8}\Big(\sum_{r\leqslant 2l}|d_r|^2 r^{-2\sigma}\Big)^{1/4}\Big) = O\big((\log l)^{-1/8}\big).$$

Finally, we can invert the order of summation in Σ_1, to obtain

$$\Sigma_1 = \sum_{m\leqslant l}c_k(m)m^{-s}\sum_{n\leqslant l}c_{-k}(n)n^{-\bar{s}}M_Q^{-1}\sum_{p\leqslant Q}\sum_{\chi\neq\chi_0}\chi(m)\bar{\chi}(n).$$

If $m=n$ the inner double-sum has the value $M_Q-\sum(p-1)$, the sum being taken over the prime divisors p of n. The such contribution of such pairs to Σ_1 is thus

$$\sum_{n=1}^{l}c_k(n)c_{-k}(n)n^{-2\sigma}\big(1+O(M_Q^{-1}l^2)\big) = \sum_{n=1}^{\infty}c_k(n)c_{-k}(n)n^{-2\sigma}+O(l^{-\delta}).$$

For the remaining pairs this inner double sum is

$$\sum_{p\leqslant Q,\,p|mn}(-1) = O\big(Q(\log Q)^{-1}\big),$$

so that their contribution to Σ_1 does not exceed

$$O\Big(Q^{-1}\Big(\sum_{m\leqslant l}|c_k(m)m^{-s}|\Big)^2\Big) = O\Big(Q^{-1}l^{1/2}\Big(\sum_{m\leqslant l}|c_k(m)|\,m^{-2\sigma}\Big)^{1/2}\Big) = O(Q^{-1/2}).$$

Putting these results together we see that

$$\text{(iv)}\qquad \int_0^1 e^{2\pi ikz}\,d\nu_Q\Big(\frac{1}{2\pi}\arg L(s,\chi)\leqslant z\ (\mathrm{mod}\ 1)\Big) \\ = \sum_{n=1}^{\infty}c_k(n)c_{-k}(n)n^{-2\sigma}+O\big((\log l)^{-1/8}\big),$$

holds uniformly at all points in the rectangle

$$\tfrac12+\delta\leqslant\sigma\leqslant 2,\qquad |t|\leqslant\sqrt{\log l}.$$

We can now appeal to the fact that a sequence of distribution functions $F_n(z)$, $n=1,2,\ldots$ possesses a limiting distribution (mod 1) if and only if for each integer k

$$\text{(v)}\qquad \lim_{n\to\infty}\int_0^1 e^{2\pi ikz}\,dF_n(z)$$

exists. Assuming it to exist, let us call this limiting distribution $F(z)$. Then $F(z)$ is continuous if and only if (see for example Edwards [2], pp. 129–130) the Fourier coefficients $\varphi(2\pi k)$ of $F(z)$ satisfy

$$N^{-1}\sum_{|k|\leqslant N}|\varphi(2\pi k)| \to 0 \qquad (N\to\infty).$$

Since we have a quantitative estimate of the rate of convergence of the relevant frequencies we can appeal to the following quantitative form of (3), which is analogous to the classical theorem of Esseen concerning distribution functions defined on the whole real line.

LEMMA 6. *Let $F(z)$, $G(z)$ be two distributions* (mod 1), *and let $F(z)$ be continuous at the points $z=0,1$. Suppose further that $F(z)$ is differentiable, and that there exists a positive constant A so that $|F'(z)|\leqslant A$, $0\leqslant z\leqslant 1$, is satisfied. Let their respective Fourier transforms be $\varphi(2\pi\nu)$ and $\psi(2\pi\nu)$. Then there is a positive constant B, depending upon A, so that*

$$|F(z)-G(z)| \leqslant \sum_{0<|k|\leqslant m}k^{-1}|\varphi(2\pi k)-\psi(2\pi k)|+Bm^{-1}$$

holds uniformly for all real numbers z, and $m\geqslant 1$.

Proof. A proof of this, and in fact of an equivalent result, but under weaker restrictions, can be found in [4], Theorem 2.

In our case we define $F(\sigma, z)$ to be the continuous distribution whose characteristic function is given by

$$k \mapsto \varphi(2\pi k) = \sum_{n=1}^{\infty} c_k(n) c_{-k}(n) n^{-2\sigma} \quad (k = 0, \pm 1, \pm 2, \ldots).$$

Then letting

$$G(z) = \nu_Q\left(\frac{1}{2\pi}\arg L(s, \chi) \leqslant z \ (\mathrm{mod}\ 1)\right) \quad (Q = 2, 3, \ldots)$$

and appealing to the estimates (iv) we deduce that

$$\nu_Q\left(\frac{1}{2\pi}\arg L(s, \chi) \leqslant z \ (\mathrm{mod}\ 1)\right) = \big(1+o(1)\big) F(\sigma, z)$$

holds uniformly for s in the region $\frac{1}{2}+\delta \leqslant \sigma \leqslant 2$, $|t| \leqslant \sqrt{\log l}$. With a little extra attention the rate of convergence here can be estimated in terms of Q, δ. This justifies the main assertion of the theorem.

To complete the proof of the theorem we remark that since the $c_{\pm k}(n)$ are multiplicative functions of n,

$$\varphi(2\pi k) = \prod_p \Big(1+\sum_{m=1}^{\infty} c_k(p^m) c_{-k}(p^m) p^{-2m\sigma}\Big).$$

Here each factor of the product is a characteristic function, and so satisfies

$$\Big|1+\sum_{m=1}^{\infty} c_k(p^m) c_{-k}(p^m) p^{-2m\sigma}\Big| \leqslant 1.$$

This can be readily seen since a typical factor has the value

$$\begin{aligned}
&\lim_{Q\to\infty} M_Q^{-1} \sum_{p \leqslant Q} \sum_{\chi \neq \chi_0} \sum_{m=1}^{\infty} c_k(q^m)\chi(q^m) q^{-m\sigma} \sum_{n=1}^{\infty} c_{-k}(q^n)\bar{\chi}(q^n) q^{-n\sigma} \\
&\qquad = \lim_{Q\to\infty} M_Q^{-1} \sum_{p \leqslant Q} \sum_{\chi \neq \chi_0} \big(1-\chi(q) q^{-s}\big)^{-k/2} \big(1-\bar{\chi}(q) q^{-s}\big)^{k/2} \\
&\qquad = \lim_{Q\to\infty} M_Q^{-1} \sum_{p \leqslant Q} \sum_{\chi \neq \chi_0} e^{ik\arg(1-\chi(q)q^{-s})}.
\end{aligned}$$

Next, we note that for each prime power p^m,

$$\begin{aligned}
|c_k(p^m) c_{-k}(p^m)| &= \left|\binom{-k/2}{m}\binom{k/2}{m}\right| \\
&\leqslant \prod_{j=0}^{m-1} \left(\frac{|k|}{2}+j\right)^2 \frac{1}{m!^2} \leqslant \prod_{j=0}^{m-1}\left(\frac{|k|}{2j}+1\right)^2 \\
&\leqslant (|k|+1)^{2m} \quad (m = 0, 1, \ldots),
\end{aligned}$$

so that if $p^\sigma \geqslant 16(|k|+1)$,

$$\sum_{m=2}^{\infty} |c_k(p^m) c_{-k}(p^m)| p^{-2m\sigma} \leqslant \sum_{m=2}^{\infty} \frac{(|k|+1)^{2m}}{p^{2m\sigma}} \leqslant 2\left(\frac{|k|+1}{p^\sigma}\right)^4 \leqslant \frac{k^2}{8p^{2\sigma}}.$$

Hence

$$0 \leqslant 1+\sum_{m=1}^{\infty} c_k(p^m) c_{-k}(p^m) p^{-2m\sigma} \leqslant 1-\frac{k^2}{4p^{2\sigma}}+\frac{k^2}{8p^{2\sigma}} = 1-\frac{k^2}{8p^{2\sigma}}$$

and

$$\begin{aligned}
|\varphi(2\pi k)| &\leqslant \prod_{p^\sigma \geqslant 16(|k|+1)} \left(1-\frac{k^2}{8p^{2\sigma}}\right) \leqslant \exp\left(-\frac{k^2}{8}\sum_{p^\sigma \geqslant 16(|k|+1)} \frac{1}{p^{2\sigma}}\right) \\
&\leqslant \exp\left(-c|k|^{1/\sigma}\right) \quad (k = 0, \pm 1, \pm 2, \ldots).
\end{aligned}$$

Finally we note that $F(z)$ has a Fourier representation

$$F(z) = -\frac{1}{2\pi i} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\varphi(2\pi k)}{k} e^{-2\pi i k z} + \text{const},$$

valid in the interval $0 \leqslant z \leqslant 1$. In view of the above estimate for the Fourier coefficients this series will clearly give an analytic continuation of $F(z)$ to the complex z-plane, which is an integral function. This continuation is, of course, at odds with the (completely arbitrary) extension in (iii), in the introduction, of the definition of $F(z)$ to the whole real line.

This completes the proof of the theorem.

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UNIVERSITY OF COLORADO
Boulder, Colorado

UNIVERSITY OF NOTTINGHAM
Nottingham, England

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