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On a theorem of Ramachandra

by

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§ 1. Let $S(H)$ denote the set of all triplets of rational numbers $\alpha_1, \alpha_2, \beta_1$ satisfying (i) $\alpha_1 > 0$, $\alpha_2 > 0$ are multiplicatively independent, (ii) the heights of $\alpha_1, \alpha_2, \beta_1$ respectively do not exceed H, H and $(\log H)^{100}$. The object of this paper is to prove

THEOREM 1. *Let $H \geqslant e^e$. Then the minimum of $|\beta_1 \log \alpha_1 - \log \alpha_2|$ as $\alpha_1, \alpha_2, \beta_1$ run through $S(H)$ exceed*

$$\exp\left(-C(\log H)^3(\log\log H)^2\right) \tag{1}$$

where C is an absolute constant.

This is an improvement of a similar theorem proved recently by K. Ramachandra [5] where one has $C(\varepsilon)\exp\left(-(\log H)^{4+\varepsilon}\right)$ where $\varepsilon > 0$ and $C(\varepsilon)$ depends only on ε.

§ 2. Since it is convenient to use the notion of size in the proof, we give the definition of size of an algebraic number. The *size* of an algebraic number α is $\overline{|\alpha|} + d(\alpha)$ where $\overline{|\alpha|}$ denotes the maximum of the absolute values of the conjugates of α and $d(\alpha)$ is the least natural number for which $\alpha d(\alpha)$ is an algebraic integer.

The height $H(\alpha)$ and the size $S(\alpha)$ of an algebraic number α of degree not exceeding h are related by the inequalities:

$$S(\alpha) \leqslant 2^h\left(H(\alpha)\right)^h, \quad H(\alpha) \leqslant 2^h\left(S(\alpha)\right)^{2h}$$

(see [6], page 76); it is immaterial whether we state the theorem in terms of size or height.

Our notation is the same as that of Ramachandra's paper [5] except when specified explicitly. In § 5 however we follow a different notation and in particular k should not be confused with the k which occurs in [5]. S_1 will stand for a number $\geqslant e^e$.

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Without loss of generality, we assume that S_1 is large enough. Denote by B a large positive constant.

We set

$$h = [B^{7/6}\log\log S_1], \quad k = [\tfrac{1}{4}B^{3/2}(\log S_1)^2(\log\log S_1)],$$

$$L = [2B^{4/3}(\log S_1)(\log\log S_1)].$$

Further define

$$h_1 = h, \quad h_2 = 1041h_1.$$
$$k_1 = k, \quad k_2 = [k_1/2].$$

Let $(\alpha_1, \alpha_2, \beta_1)$ be an arbitrary triple of rational numbers satisfying (i) $\alpha_1 > 0$, $\alpha_2 > 0$ are multiplicatively independent, (ii) the size of $\alpha_1, \alpha_2, \beta_1$ respectively denote exceed S_1, S_1 and $(\log S_1)^{100}$. Define

$$\beta = |\beta_1\log\alpha_1 - \log\alpha_2|.$$

§ 3. We shall require the following lemmas.

LEMMA 1. *Let m, s and t be positive integers and set $r = st$. Le, $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ and $\beta_0, \beta_1, \ldots, \beta_{s-1}$ be m and s distinct complex numberst respectively, and let*

$$a = \max_{0\le\nu<m}(|\alpha_\nu|, 1), \qquad b = \max_{0\le\sigma<s}(|\beta_\sigma|, 1),$$

$$a_0 = \min_{\substack{0\le\nu<m\\0\le\mu<m\\\mu\ne\nu}}(|\alpha_\mu-\alpha_\nu|, 1), \qquad b_0 = \min_{\substack{0\le\sigma<s\\0\le\varrho<s\\\varrho\ne\sigma}}(|\beta_\sigma-\beta_\varrho|, 1).$$

Put for arbitrary complex numbers A_ν

$$E(z) = \sum_{\nu=0}^{m-1}A_\nu e^{\alpha_\nu z}$$

and

$$A = \max_{0\le\nu<m}|A_\nu|, \quad E = \max_{\substack{0\le\varrho<t\\0\le\sigma<s}}|E^{(\varrho)}(\beta_\sigma)|.$$

Assume that

$$r \geqslant 2m + 13ab.$$

Then

$$A \leqslant s\sqrt{m!}\,e^{7ab}\left(\frac{1}{2a_0b}\right)^{m-1}\left(\frac{72b}{b_0\sqrt{s}}\right)^r E.$$

This is due to Tijdeman [9].

LEMMA 2. $L^2 \geqslant 2h(k+1)$.

This is inequality 3 in [4] with $n = f = d = 1$.

Proof. The lemma follows directly from the definition of k and L.

LEMMA 3. *Let $\beta \leqslant \exp(-\frac{1}{2}h_1k_1\log h_2)$. Then the following inequality holds*

$$2^{\frac{1}{2}h_1k_1} \geqslant C_1^{13Lh_2+4k\log(SL)(\log C_1)^{-1}+8k(\log\log S_1)(\log C_1)^{-1}}\left(1+\beta\exp(\tfrac{1}{2}h_1k_1\log h_2)\right)$$

where $C_1 = S_1^2$ and $S = (\log S_1)^{100}$.

This is an inequality immediately before inequality numbered (7) in [5].

Proof. We must satisfy

$$2^{\frac{1}{2}h_1k_1} > 2S_1^{26Lh_2}(SL)^{4k}(\log S_1)^{8k},$$

i.e.,

$$\tfrac{1}{2}h_1k_1\log 2 > \log 2 + 26Lh_2\log S_1 + 4k\log S + 4k\log L + 8k\log\log S_1.$$

As $\frac{1}{2}\log 2 > \frac{1}{4}$ and L may be assumed large enough, it is enough to have

$$h_1k_1 > 104Lh_2\log S_1 + 16k\log S + 17k\log L + 32k\log\log S_1.$$

As $S = (\log S_1)^{100}$, it is sufficient to get

(2) $$h_1k_1 > 104Lh_2\log S_1 + 1632k\log\log S_1 + 17k\log L.$$

Now

$$17k\log L \leqslant \tfrac{17}{4}(\log\log S_1)(\log S_1)^2\cdot 2B^{3/2}(\log\log S_1)$$
$$= \tfrac{17}{2}B^{3/2}(\log S_1)^2(\log\log S_1)^2,$$
$$1632k\log\log S_1 \leqslant 408B^{3/2}(\log S_1)^2(\log\log S_1)^2,$$
$$104Lh_2\log S_1 \leqslant 2\cdot 104\cdot 1041\cdot B^{5/2}(\log S_1)^2(\log\log S_1)^2$$

and

$$h_1k_1 > \tfrac{1}{16}B^{8/3}(\log S_1)^2(\log\log S_1)^2.$$

As $8/3 > 5/2$, choose B large enough so that (2) is satisfied and Lemma 3 is proved.

§ 4. Proof of Theorem 1. Assume that

(3) $$\beta < \exp(-\tfrac{1}{2}h_1k_1\log h_2).$$

Follow Ramachandra's paper [5] to conclude the following from Lemma 2 and Lemma 3

$$\sum_{\lambda_1=0}^{L}\sum_{\lambda_2=0}^{L}p(\lambda_1, \lambda_2)\alpha_1^{\lambda_1 l}\alpha_2^{\lambda_2 l}\gamma_1^m = 0, \quad 1 \leqslant l \leqslant h_2, \; 0 \leqslant m \leqslant k_2.$$

Hence

$$|\Phi^{(m)}(l)| \leqslant \beta(28S_1)^{8Lh_2}(2SL)^{3k}(3\log S_1)^{2k}, \quad 1 \leqslant l \leqslant h_2,\ 0 \leqslant m \leqslant k_2.$$

(See [5], inequality (3).)

Observe the following:

1) If λ_1', λ_2' are rational integers, not both zero, and in absolute value $\leqslant L$, then

$$|\lambda_1'\log\alpha_1 + \lambda_2'\log\alpha_2| > \tfrac{1}{2}S_1^{-4L}$$

and

$$\begin{aligned}|(\lambda_1'+\lambda_2'\beta_1)\log\alpha_1| &= |(\lambda_1'+\lambda_2'\beta_1)\log\alpha_1 - (\lambda_1'\log\alpha_1+\lambda_2'\log\alpha_2) + \\ &\qquad + (\lambda_1'\log\alpha_1+\lambda_2'\log\alpha_2)| \\ &\geqslant |\lambda_1'\log\alpha_1+\lambda_2'\log\alpha_2| - L\beta \geqslant \tfrac{1}{2}S_1^{-4L} - L\beta.\end{aligned}$$

Assume that

$$\beta < \frac{1}{4L}S_1^{-4L}. \tag{4}$$

We have

$$(\lambda_1'+\lambda_2'\beta_1)\log\alpha_1 \neq 0.$$

In particular, $(\lambda_1+\lambda_2\beta_1)\log\alpha_1$, $0 \leqslant \lambda_1, \lambda_2 \leqslant L$ are distinct for distinct tuples (λ_1, λ_2).

Further notice that

$$|(\lambda_1'+\lambda_2'\beta_1)\log\alpha_1| > \frac{1}{S_1^u} \tag{5}$$

where u is an absolute constant.

2) For $0 \leqslant \lambda_1, \lambda_2 \leqslant L$

$$\begin{aligned}|(\lambda_1+\lambda_2\beta_1)\log\alpha_1| &= |(\lambda_1+\lambda_2\beta_1)\log\alpha_1 - (\lambda_1\log\alpha_1+\lambda_2\log\alpha_2) + \\ &\qquad + (\lambda_1\log\alpha_1+\lambda_2\log\alpha_2)| \\ &\leqslant L\beta + |\lambda_1\log\alpha_1+\lambda_2\log\alpha_2| \leqslant 3L\log S_1.\end{aligned} \tag{6}$$

In Lemma 1, set

$$E(z) = \Phi(z) = \sum_{\lambda_1=0}^{L}\sum_{\lambda_2=0}^{L} p(\lambda_1, \lambda_2)\exp\big((\lambda_1+\lambda_2\beta_1)\log\alpha_1\cdot z\big).$$

Arrange

$$(\lambda_1+\lambda_2\beta_1)\log\alpha_1,\ 0 \leqslant \lambda_1, \lambda_2 \leqslant L \quad \text{as} \quad \alpha_0, \alpha_1, \ldots, \alpha_{(L+1)^2-1},\ m = (L+1)^2.$$

Arrange

$$1, 2, \ldots, h_2 \quad \text{as} \quad \beta_0, \beta_1, \ldots, \beta_{s-1},\ s = h_2.$$

Set

$$t = k_2+1, \quad r = st = h_2(k_2+1).$$

In the notation of Lemma 1, we have

$$a \leqslant 3L\log S_1 \text{ (see (6))}, \quad b = h_2.$$

$$a_0 \geqslant \frac{1}{S_1^u} \text{ (see (5))}, \quad b_0 = 1.$$

$$A = \max_{0\leqslant\lambda_1,\lambda_2\leqslant L}|p(\lambda_1, \lambda_2)| \geqslant 1.$$

$$E \leqslant \beta(28S_1)^{8Lh_2}(2SL)^{3k}(3\log S_1)^{2k}.$$

Further we show that the inequality

$$r \geqslant 2m+13ab$$

is satisfied.

It is sufficient to show that

$$h_2(k_2+1) \geqslant 4L^2+39Lh_2\log S_1,$$

i.e.,

$$h_2k \geqslant 12L^2+117Lh_2\log S_1.$$

Now

$$h_2k > \tfrac{1041}{16}B^{8/3}(\log S_1)^2(\log\log S_1)^2,$$

$$12L^2 \leqslant 48B^{8/3}(\log S_1)^2(\log\log S_1)^2$$

and

$$117Lh_2\log S_1 \leqslant 2\cdot 117\cdot 1041\cdot B^{5/2}(\log S_1)^2(\log\log S_1)^2.$$

So it is sufficient to prove that

$$\begin{aligned}&\tfrac{1041}{16}B^{8/3}(\log S_1)^2(\log\log S_1)^2 \\ &\quad > 48B^{8/3}(\log S_1)^2(\log\log S_1)^2 + 2\cdot 117\cdot 1041\cdot B^{5/2}(\log S_1)^2(\log\log S_1)^2.\end{aligned} \tag{7}$$

Since $1041/16 > 49$ and $8/3 > 5/2$, (7) is satisfied if B is large enough. Hence by Lemma 1, we get

$$1 \leqslant A \leqslant h_2(4L)^{4L^2}\big(\exp(21Lh_2\log S_1)\big)(72h_2)^{h_2k}S_1^{4uL^2}\beta(28S_1)^{8Lh_2}(2SL)^{3k}(3\log S_1)^{2k}.$$

Notice that

$$h_2(4L)^{4L^2}\big(\exp(21Lh_2\log S_1)\big)(72h_2)^{h_2k}(28S_1)^{8Lh_2}(2SL)^{3k}(3\log S_1)^{2k} < \exp\big((\log S_1)^3\big)$$

and

$$S_1^{4uL^2} = \exp(4uL^2\log S_1) \leqslant \exp\big(16uB^{8/3}(\log S_1)^3(\log\log S_1)^2\big).$$

Assume that

$$\beta < \exp\big(-18uB^{8/3}(\log S_1)^3(\log\log S_1)^2\big).$$

Notice that (3) and (4) are satisfied. Further $1 \leqslant A < 1$, which is not possible. Hence

$$\beta > \exp\left(-C(\log S_1)^3(\log\log S_1)^2\right)$$

where C is a large positive constant. This completes the proof of Theorem 1.

Remarks. (i) It is easy to see that if either a_1 or a_2 has denominator $\leqslant (\log S_1)^{A_1}$, A_1 any positive constant, then

$$\beta > \exp\left(-C_1(\log S_1)^2(\log\log S_1)^3\right)$$

where C_1 is an absolute constant.

(ii) The constant 100 in Theorem 1 can be replaced by any positive constant.

§ 5. Relation to the problem of Erdös. Let $k \geqslant 2$ be a natural number and let $n_1, n_2, \ldots$ the sequence of all natural numbers in increasing order which have at least one prime factor exceeding k. Then it is easy to see that $n_{i+1}-n_i$ does not exceed $2k$. The problem of Erdös consists in improving this upper bound. Erdös [3] gave the bound $\left(3+o(1)\right)\dfrac{k}{\log k}$ and Ramachandra [7] improved it to $\left(1+o(1)\right)\dfrac{k}{\log k}$. In this direction, Ramachandra [5] proved some partial results in the direction of the estimate $\dfrac{k}{2\log k}\left(1+o(1)\right)$. Combining these results of K. Ramachandra with an ingenious lemma of his own, R. Tijdeman [10] proved the following:

THEOREM 2.

$$\max(n_{i+1}-n_i) \leqslant \frac{k}{2\log k}\left(1+o(1)\right)$$

where the maximum is taken over all n_i.

It may be stated that Ramachandra's argument shows that our Theorem 1 implies

THEOREM 3.

$$\max(n_{i+1}-n_i) \leqslant \frac{k}{2\log k}\left(1+o(1)\right)$$

where the maximum is taken over all $n_i \geqslant \exp\left(C_1(\log k)^3(\log\log k)^2\right)$ *where* C_1 *is some positive constant.*

We mention a result, due essentially to Ramachandra, in this direction.

THEOREM 4.

$$\max(n_{i+1}-n_i) \leqslant \frac{k}{3\log k}\left(1+o(1)\right)$$

where the maximum is taken over all $n_i \geqslant \exp\left((\log k)^{9+\varepsilon}\right)$ *where* $\varepsilon > 0$ *is an arbitrary fixed positive constant.*

Ramachandra's arguments in [5] shows that Theorem 4 follows from the following

THEOREM 5. *Let* $a_1, a_2, a_3, \beta_1, \beta_2$ *be rational numbers satisfying* (i) $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ *are multiplicatively independent,* (ii) *the size of* $a_1, a_2, a_3, \beta_1, \beta_2$ *respectively do not exceed* $S_1, S_1, S_1, (\log S_1)^{100}$ *and* $(\log S_1)^{100}$. *Then*

(8) $$|\beta_1 \log a_1 + \beta_2 \log a_2 - \log a_3| > D(\varepsilon)\exp\left(-(\log S_1)^{9+\varepsilon}\right)$$

where $\varepsilon > 0$ *and* $D(\varepsilon)$ *depends only on* ε.

The proof of Theorem 5 runs exactly like that of Theorem 5 of [5], except for obvious changes.

Added in proof. It has recently been proved by the author that, given $\varepsilon > 0$, one can have on the right hand sides of (1) and (8) the expressions $E(\varepsilon)\exp\left(-(\log S_1)^{2+\varepsilon}\right)$ and $E(\varepsilon)\exp\left(-(\log S_1)^{8+\varepsilon}\right)$ respectively, where $E(\varepsilon)$ is an effectively computable positive constant depending only on ε. In Theorem 4, the maximum can be taken over all n_i such that $n_i \geqslant \exp\left((\log k)^{8+\varepsilon}\right)$.

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