On real characters.

By

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Let $\chi(n)$ denote a real non-principal character. We define

$$S_1(x) = \sum_{n \leq x} \chi(n), \quad S_m(x) = \sum_{n \leq x} S_{m-1}(n) \qquad \text{for } m \geqq 2.$$

S. Chowla [1]) considered the hypothesis that

$$(1) \qquad S_m(x) \geqq 0 \qquad \text{for } x \geqq 1,$$

if $m \geqq m_0(\chi)$.

It will be shewn in this paper that this is not the case for all real characters χ.

Put

$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{n=1}^{\infty} \lambda(n) n^{-s} \qquad \text{for } s > 1$$

and

$$f(y) = \sum_{n=1}^{\infty} \lambda(n) e^{-ny} \qquad \text{for } y > 0.$$

Then

$$\Gamma(s) \frac{\zeta(2s)}{\zeta(s)} = \int_0^{\infty} y^{s-1} f(y)\, dy$$

[1]) Acta Arithmetica, Vol. 1, p. 113.

for $s > 1$, and the integral tends to 0 as $s \to 1$. Hence, $f(y)$ not being identically 0,

$$f(y) < 0$$

for some suitably chosen $y > 0$.

We choose an integer $a > 0$ such that

$$(2) \qquad f(y) + 2 \sum_{n=a+1}^{\infty} e^{-ny} = f(y) + 2 \frac{e^{-ay}}{e^y - 1} < 0$$

and an integer d (positive or negative) such that

$$\left(\frac{d}{p}\right) = -1$$

for all primes $p \leqq a$. Here $\left(\frac{d}{p}\right)$ denotes the Legendre-Kronecker symbol. Then

$$\chi(n) = \left(\frac{d}{n}\right) = \lambda(n) \qquad \text{for } n \leqq a,$$

and by (2)

$$(3) \qquad \sum_{n=1}^{\infty} \chi(n) e^{-ny} \leqq f(y) + 2 \sum_{n=a+1}^{\infty} e^{-ny} < 0.$$

As

$$\sum_{n=1}^{\infty} \chi(n) e^{-ny} = (1 - e^{-y})^m \sum_{n=1}^{\infty} S_m(n) e^{-ny},$$

(3) shews that (1) is not true for any value of m.

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