ninth powers. Hence by Lemma 30, all in $K$ are represented by [6, 54]. Thus all in $(E, E+2\cdot 2^n)$ are represented by [7, 54]. The values of $G_3, \ldots, G_{10}$ are 38, 13, 7, 5, 4, 3, 2, 2. Thus the $R_i$ are 0, 1, 1, 0, 1, 0, 0 and the $Q_i$ are 19, 6, 3, 2, 2, 1, 1, 1. Hence by Lemma 6 with $m=10$, all in $(E, L_0 = E + 10^9)$ are represented by [14, 89]. Here $F_0 = [36, 256]$. We may take $k = 86$ (since $k_0 = 56$ and $3k-2=256$), $z = .0000105$. We get $-\log b_4 < 3.363842\, n^3$, $\log C_1 < .4890613\, n^2$, $\log C_1/c < 3.4347593\, n^3$. $-J = .0241104$, $\operatorname{Log} P \geqq 15.82892\, n^4$, $\operatorname{Log} N \geqq 15.8290\, n^5$. The final factor in (10) is $.1107101$. Hence $t + T \geqq 136$. Take $T = 22$, $t = 114$. Then $[T+14, t+89] = [36, 203] < F_0$.

In Lemma 8, $f = g = 1$, $d = 2$, and all in $(2^n q, 2^n q + 2^n)$ are represented by [3, 141]. By Lemma 6 with $m=7$, $E = 2^n q$, all $(2^n q, 2^n q + 7^9)$ are represented by [8, 173] and hence by $F_0$.

This proves Theorem 2. The proof of Theorem 4 is omitted.

**16.** For $a \leqq n$, $n \geqq 11$, we get $\operatorname{Log} N \geqq .5335973\, n^6$.

**17. Case** $a=4$. We take $Y=y=0$ since (25) and (28) are then positive. Take $m=n+3$. We omit the final term in Lemma 27. In (29), we decrease each $R_i$ to 0 and get $E' = 4Z + \Sigma G_i + 37$.

When $n=11$, we get $Z=151$, $X<93$, $E'<734$. The limits (32) are $.35840$ and $.59546$. But $d(11)$ lies between them.

When $n=12$, $Z=171$, $X=110$, $E'=831$. The limits (32) are $.20288$ and $.76349$. But $d(12)$ and $d(13)$ lie between them.

When $n=14$, $Z=213$, $\sum_4 G_i = 130$, $\sum_3 R_i = 25$, and (29) gives $E=994$, The limits (32) are $.060669$ and $.921025$. For $15 \leqq n \leqq 28$, $d(n)$ lies between them. The same holds for $n \geqq 29$ and the new limits.

Let $n=14$. For $E = 5^n + 5\cdot 6^n + 7^n$ all integers in $K = (E, E+2^n)$ are sums of 1079 + 7 14-th powers.[10] Hence by Lemma 30, all in $K$ are represented by [19, 271]. Hence all in $(E, E + a\cdot 2^n)$ are represented by [22, 271]. Then by Lemma 6, all in $(E, E + 17^{14})$ are represented by [64, 370]. Since $v = 1/n$ to the sixth decimal place, we may use $Z=213$ as above. Take $T=0$. Then $[T+64, 370+t] = [64, 583] \leqq [64, 4151]$, viz., (23) for $Y=y=2$. By using an interval longer than $2^n$ and the paper cited, we reach (72) and hence retain $Y=0$.

(Received 5 August, 1936.)

[10]) Dickson, Monatshefte Math. Phys., vol. 43 (1936), p. 393, tablette $A=0$.

# On Waring's problem for fourth and higher powers.

By

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1. Introduction.

Let $k$ be a positive integer. Then $G(k)$ denotes the least number $s$ such that every sufficiently large integer is a sum of $s$ $k$-th powers (of positive integers). This notation was introduced by Hardy and Littlewood[1]) and is now generally accepted. In my paper "Proof that every large integer is a sum of seventeen biquadrates"[2]), hereafter quoted as I, I proved (simultaneously with Davenport and Heilbronn[3]) that $G(4) \leqq 17$, and conjectured that the same method could be applied to the case $k>4$ with the following result:

Let

$$m_0 = \left[\frac{(k-2)\log 2 + \log(k-2) - \log k}{\log k - \log(k-1)}\right] \tag{1}$$

and

$$s = 2m_0 + 7 + [2^{k-1}(k-2)(1-k^{-1})^{m_0+1}], \tag{2}$$

where $[x]$ denotes the integral part of $x$. Then

$$G(k) \leqq s. \tag{3}$$

The object of the present paper is to prove this conjecture.

When $k=4$, it follows from (1) and (2) that $s=17$, so that (3) is

[1]) Some problems of partitio numerorum I, Göttinger Nachrichten (1920), 33—54.

[2]) Proc. London Math. Soc. (2), 41 (1936), 126—142.

[3]) Ibid. 143—150.

true in this case, but it is no easier to prove (3) for $k>4$ than on the wider assumption

(4) $$k \geqq 4,$$

which will be made throughout this paper.

**2.** Notation.

In what follows, $h$, $m$, and $r$ are integers; $l$, $n$, $q$, and $v$ are positive integers; $t$, $x$, $y$, $\Theta$, and $\vartheta$ are real numbers; $u$ is a number greater than 1.

$\delta$ is a sufficiently small positive number, depending only on $k$. The precise meaning of "sufficiently small" will become clear from the context.

$\varepsilon$ is, as usual, a positive number which may be taken as small as we please.

$B_1, B_2, \ldots, C_1, C_2,$ are suitable (sufficiently large) positive numbers, the $B$'s depending at most on $k$, the $C$'s at most on $k$ and $\varepsilon$.

**3.** In order to prove (3), it is sufficient to show that, if $n$ is large enough, then the equation

(5) $$\sum_{v=1}^{s} r_v^k = n$$

has at least one solution in positive integers $r_1, r_2, \ldots r_s$. To this end, I shall obtain an asymptotic formula for the number of those solutions of (5) which satisfy the inequalities

(6) $$n^{1/k-\delta} \leqq r_v < n^{1/k} \qquad (v=1, 2),$$

(7) $$\frac{1}{2} u_m \leqq r_v < u_m \qquad (v = 2m+1,\ 2m+2;\ m=1, 2, \ldots, m_0),$$

and

(8) $$r_v < u_{m_0+1} \qquad (v = 2m_0+3,\ 2m_0+4,\ 2m_0+5,\ 2m_0+6),$$

where

(9) $$u_m = n^{k^{-1}(1-k^{-1})^m},$$

These inequalities do not restrict $r_v$ for $2m_0+7 \leqq v \leqq s$, but (5) clearly implies

(10) $$r_v < n^{1/k}.$$

**4.** I put (cf. I, 1)

(11) $$\xi_q = e^{2\pi i/q},$$



(12) $$f(\Theta, u) = \sum_{v<u} e^{2\pi i \Theta v^k},$$

(13) $$g(\vartheta, u) = \int_0^u e^{2\pi i \vartheta y^k}\, dy,$$

(14) $$S_{h,q} = \sum_{v=1}^{q} \xi_q^{hv^k},$$

(15) $$A_q = q^{-s} \sum_h{}' S_{h,q}^s \xi_q^{-nh},$$

where the accent stands for the two conditions $0 < h \leqq q$ and

(16) $$(h, q) = 1,$$

the latter of which will be assumed throughout this paper, and

(17) $$S = \sum_{q=1}^{\infty} A_q.$$

**5.** We have

(18) $$|S_{h,q}| \leqq B_1 q^{1-1/k}$$

(Hardy and Littlewood[4]), Lemma 3), and it easily follows from (13) that

(19) $$|g(\vartheta, u)| \leqq \min(u, B_2 |\vartheta|^{-1/k}) \leqq B_3 (u^{-k} + |\vartheta|)^{-1/k}.$$

**6.** If

(20) $$\Theta = h/q + \vartheta$$

and

(21) $$q \leqq u^{1-\varepsilon},\ |\vartheta| \leqq q^{-1} u^{1-k-\varepsilon},$$

then

(22) $$|f(\Theta, u) - S_{h,q} q^{-1} g(\vartheta, u)| \leqq C_1 q^{1-2^{1-k}+\varepsilon}.$$

This is a straightforward generalization of **I**, (2, 71).

**7.** If $u$ and $\Theta$ are such that the conditions (16), (20), and

(23) $$q \leqq u^{2^{1-k}k}, \qquad |\vartheta| \leqq q^{-1} u^{-k+2^{1-k}k}$$

are not simultaneously satisfied by any values of $h$, $q$, and $\vartheta$, then

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[4]) Some problems of partitio numerorum IV, Math. Zeitschrift, 12 (1922), 161—188.

$$(24) \qquad |f(\Theta, u)| \leqq C_2 u^{1-2^{1-k}+\varepsilon},$$

Proof. It is known that, corresponding to any $y \geqq 1$, there exist numbers $h$, $q$, and $\vartheta$ such that

$$\Theta = h/q + \vartheta, \quad (h, q) = 1, \quad q \leqq y, \quad \text{and} \quad |\vartheta| \leqq (qy)^{-1}.$$

In particular, we can choose $h$, $q$, and $\vartheta$ so that (20) and (16) hold and

$$(25) \qquad q \leqq u^{k-1+\varepsilon}, \quad |\vartheta| \leqq q^{-1} u^{1-k-\varepsilon}.$$

Then, since (23) does not hold, at least one of the following three conditions is satisfied:

$$\text{(i)} \qquad u^{1-\varepsilon} < q \leqq u^{k-1+\varepsilon},$$

$$\text{(ii)} \qquad u^{2^{1-k}k} < q \leqq u^{1-\varepsilon},$$

$$\text{(iii)} \qquad q \leqq u^{1-\varepsilon}, \quad |\vartheta| > q^{-1} u^{-k+2^{1-k}k}.$$

Suppose, first, that (i) holds. Then, putting

$$(26) \qquad s_v = \sum_{m=1}^{v} \xi_q^{hm^k},$$

and defining $\lambda$ as the greatest integer less than $u$, we have, by (12), (20), and (11),

$$f(\Theta, u) = \sum_{r=1}^{\lambda} \xi_q^{hr^k} e^{2\pi i \vartheta r^k} = s_1 e^{2\pi i \vartheta} + \sum_{r=2}^{\lambda} (s_r - s_{r-1}) e^{2\pi i \vartheta r^k}$$

$$= \sum_{v=1}^{\lambda-1} s_v \left(e^{2\pi i \vartheta v^k} - e^{2\pi i \vartheta (v+1)^k}\right) + s_\lambda e^{2\pi i \vartheta \lambda^k},$$

so that

$$(27) \qquad |f(\Theta, u)| \leqq \sum_{v=1}^{\lambda-1} |s_v| \, 2\pi |\vartheta| \, \{(v+1)^k - v^k\} + |s_\lambda|.$$

Now, by (26), (11), (16), and Weyl's inequality [5],

$$|s_v|^{2^{k-1}} \leqq C_3 q^{\varepsilon} v^{\varepsilon} \left(v^{2^{k-1}-1} + v^{2^{k-1}-k} q + v^{2^{k-1}} q^{-1}\right).$$

Hence, by (i),

[5] See, e. g., Landau, Vorlesungen über Zahlentheorie, 1 (Leipzig, 1927), Theorem 267.



$$|s_v| \leqq C_4 u^{1-2^{1-k}+\varepsilon} \quad (v < u),$$

and, by (25) and (i), $|\vartheta| < u^{-k}$. Hence, by (27) and the definition of $\lambda$,

$$|f(\Theta, u)| \leqq C_4 u^{1-2^{1-k}+\varepsilon} \left(2\pi u^{-k} \sum_{v=1}^{\lambda-1} \{(v+1)^k - v^k\} + 1\right)$$

$$< C_4 u^{1-2^{1-k}+\varepsilon} (2\pi + 1),$$

which implies (24).

Next suppose that (ii) holds. Then, by **6**, (18), and (19),

$$|f(\Theta, u)| \leqq B_1 q^{-1/k} u + C_1 q^{1-2^{1-k}+\varepsilon},$$

which, together with (ii), proves (24).

Finally suppose that (iii) holds. Then, by **6**, (18), and (19),

$$|f(\Theta, u)| \leqq B_1 q^{-1/k} B_2 |\vartheta|^{-1/k} + C_1 q^{1-2^{1-k}+\varepsilon},$$

which, together with (iii), again proves (24).

**8.** On the other hand, if (16), (20), and (23) are satisfied, then

$$(28) \qquad |f(\Theta, u)| \leqq B_4 q^{-1/k} (u^{-k} + |\vartheta|)^{-1/k}.$$

This follows easily from **6**, (18), and (19).

**9.** I put

$$(29) \qquad f_0(\Theta) = f(\Theta, n^{1/k}) - f(\Theta, n^{1/k-\delta}),$$

$$(30) \qquad f_m(\Theta) = f(\Theta, u_m) - f\left(\Theta, \frac{1}{2} u_m\right) \qquad (m \geqq 1),$$

$$(31) \qquad F_1(\Theta) = f^2(\Theta, u_{m_0+1}) \prod_{m=1}^{m_0} f_m(\Theta),$$

$$(32) \qquad F_2(\Theta) = f_0(\Theta) F_1(\Theta),$$

and

$$(33) \qquad F_3(\Theta) = F_2^2(\Theta) f^{s-2m_0-6}(\Theta, n^{1/k}).$$

Then, by (12),

$$(34) \qquad F_3(\Theta) = \sum_{r=1}^{sn} a_r e^{2\pi i \Theta r},$$

where $a_r$ is the number of solutions of the equation

$$\sum_{v=1}^{s} r_v^k = r$$

in positive integers $r_1, r_2, \dots, r_s$ satisfying the inequalities (6), (7), (8), and (10). Hence (cf. 3) it is sufficient to show that, if $n$ is large enough, then $a_n \neq 0$.

**10.** By (12), (29), (30), (31), and (32),

$$F_2(\theta) = \sum_{r=1}^{sn} b_r e^{2\pi i \theta r}, \tag{35}$$

where $b_r$ is the number of solutions of the equation

$$\sum_{m=0}^{m_0+2} r_m^k = r$$

in positive integers $r_0, r_1, \dots, r_{m_0+2}$ satisfying the inequalities

$$n^{1/k-\delta} \leqq r_0 < n^{1/k},$$

$$\frac{1}{2} u_m \leqq r_m < u_m \qquad (m = 1, 2, \dots, m_0),$$

and

$$r_m < u_{m_0+1} \qquad (m = m_0+1, m_0+2).$$

It easily follows (cf. I, (3 · 14)) that

$$0 \leqq b_r \leqq C_5 n^{(k-1)\delta+\varepsilon},$$

which implies

$$0 \leqq b_r \leqq B_5 n^{k\delta}.$$

From this and (35) it follows that

$$\int_0^1 |F_2(\theta)|^2 d\theta = \sum_{r=1}^{sn} b_r^2 \leqq B_5 n^{k\delta} \sum_{r=1}^{sn} b_r = B_5 n^{k\delta} F_2(0). \tag{36}$$

Now, by (12),

$$|f(\theta, u)| < u, \tag{37}$$

and, by (12), (29), and (30),

$$|f_m(\theta)| < u_m \qquad (m = 0, 1, \dots), \tag{38}$$

where $u_0 = n^{1/k}$, so that (9) holds also for $m = 0$. Hence, by (31) and (32),

$$|F_2(\theta)| < u_{m_0+1}^2 \prod_{m=0}^{m_0} u_m. \tag{39}$$



Putting

$$\alpha = 2k^{-1}(1-k^{-1})^{m_0+1} + k^{-1} \sum_{m=0}^{m_0} (1-k^{-1})^m, \tag{40}$$

so that

$$\alpha = 1 - (1-2k^{-1})(1-k^{-1})^{m_0+1}, \tag{41}$$

we have, by (39) and (9),

$$|F_2(0)| < n^\alpha,$$

and hence, by (36),

$$\int_0^1 |F_2(\theta)|^2 d\theta < B_5 n^{\alpha+k\delta}. \tag{42}$$

**11.** Let $E$ denote the set of those numbers $\theta$ between 0 and 1 for which the conditions (20), (16), and

$$q \leqq n^{2^{1-k}}, \quad |\vartheta| \leqq q^{-1} n^{-1+2^{1-k}} \tag{43}$$

are not satisfied by any values of $h$, $q$, and $\vartheta$. Then, by (34),

$$a_n = \int_{t_0/n}^{1+t_0/n} F_3(\theta) e^{-2\pi i \theta n} d\theta \tag{44}$$

$$= \int_E F_3(\theta) e^{-2\pi i \theta n} d\theta + \sum_{q \leqq t_0} \sum_h{}' \int_{h/q-\vartheta_0}^{h/q+\vartheta_0} F_3(\theta) e^{-2\pi i \theta n} d\theta,$$

where

$$t_0 = n^{2^{1-k}}, \quad \vartheta_0 = q^{-1} n^{-1+2^{1-k}} = t_0 (qn)^{-1}$$

(for the meaning of the accent see 4). Also, by 7,

$$|f(\theta, n^{1/k})| \leqq C_2 n^{k^{-1}(1-2^{1-k}+\varepsilon)} \qquad (\theta \text{ in } E),$$

so that, by (33) and (42),

$$\left| \int_E F_3(\theta) e^{-2\pi i \theta n} d\theta \right| \leqq C_6 n^{(s-2m_0-6)k^{-1}(1-2^{1-k}+\varepsilon)+\alpha+k\delta},$$

which implies

$$\left| \int_E F_3(\theta) e^{-2\pi i \theta n} d\theta \right| \leqq B_6 n^{(s-2m_0-6)k^{-1}(1-2^{1-k})+\alpha+2k\delta}. \tag{45}$$

Now let

$$\beta = 2\alpha - 1 + (s - 2m_0 - 6)/k. \tag{46}$$

Then, by (41) and (2),

$$(s-2m_0-6)k^{-1}(1-2^{1-k}) + \alpha < \beta,$$

and so, since $\delta$ is „sufficiently" small, it follows from (45) that

$$\left|\int_E F_3(\theta)\, e^{-2\pi i\theta n}\, d\theta\right| \le B_6\, n^{\beta-\delta}. \tag{47}$$

**12.** Putting

$$J(h,q) = \int_{h/q-\vartheta_0}^{h/q+\vartheta_0} F_3(\theta)\, e^{-2\pi i \theta n}\, d\theta, \tag{48}$$

we have, by (44) and (47),

$$\left|a_n - \sum_{q\le t_0}\sum_h{}' J(h,q)\right| \le B_6\, n^{\beta-\delta}. \tag{49}$$

On the assumption that (20) and (43) hold, we have, by **8**,

$$|f(\theta, n^{1/k})| \le B_4\, q^{-1/k}(n^{-1}+|\vartheta|)^{-1/k}, \tag{50}$$

and it easily follows from **6**, (18), and (19) that

$$|f(\theta, n^{1/k-\delta})| \le B_7\, q^{-1/k}(n^{-1}+|\vartheta|)^{-1/k},$$

so that, by (29),

$$|f_0(\theta)| \le (B_4+B_7)\, q^{-1/k}(n^{-1}+|\vartheta|)^{-1/k}. \tag{51}$$

From this and (50) it follows that

$$|f_0^2(\theta)\, f^{s-2m_0-6}(\theta, n^{1/k})| \le B_8\{q(n^{-1}+|\vartheta|)\}^{-(s-2m_0-4)/k}. \tag{52}$$

Now, by (31), (37), (38), (9), and (40),

$$|F_1(\theta)| < u^2_{m_0+1}\prod_{m=1}^{m_0} u_m = n^{\alpha-1/k}, \tag{53}$$

and hence, by (33), (32), and (52),

$$|F_3(\theta)| < B_8\, n^{2\alpha-2/k}\{q(n^{-1}+|\vartheta|)\}^{-(s-2m_0-4)/k}. \tag{54}$$

**13.** By (2),

$$s-2m_0-4 > 2+2^{k-1}(k-2)(1-k^{-1})^{m_0+1},$$



and, by (1),

$$(1-k^{-1})^{m_0} \ge 2^{2-k}k/(k-2).$$

Hence

$$s-2m_0-4>2k. \tag{55}$$

**14.** By (48),

$$|J(h,q)| \le \int_{-\vartheta_0}^{\vartheta_0}\left|F_3\left(\frac{h}{q}+\vartheta\right)\right| d\vartheta.$$

Hence, by (20), (54), (55), and (46),

$$|J(h,q)| \le B_8\, n^{2\alpha-2/k}\, q^{-(2k+1)/k}\int_{-\infty}^{\infty}(n^{-1}+|\vartheta|)^{-(s-2m_0-4)/k}\, d\vartheta$$

$$= B_9\, n^\beta\, q^{-2-1/k},$$

and hence

$$\left|\sum_{n^{k\delta}<q\le t_0}\sum_h{}' J(h,q)\right| < B_9\, n^\beta \sum_{q>n^{k\delta}} q^{-1-1/k} < B_{10}\, n^{\beta-\delta}.$$

From this and (49) it follows that

$$\left|a_n - \sum_{q\le n^{k\delta}}\sum_h{}' J(h,q)\right| \le B_{11}\, n^{\beta-\delta}. \tag{56}$$

**15.** Until further notice let

$$q \le n^{k\delta}. \tag{57}$$

Putting

$$J_1(h,q) = \int_{h/q-\vartheta_1}^{h/q+\vartheta_1} F_3(\theta)\, e^{-2\pi i\theta n}\, d\theta, \tag{58}$$

where

$$\vartheta_1 = q^{-1}n^{-1+k\delta}, \tag{59}$$

we have, by (48),

$$|J(h,q)-J_1(h,q)| \le \int_{-\vartheta_0}^{-\vartheta_1}\left|F_3\left(\frac hq+\vartheta\right)\right|d\vartheta + \int_{\vartheta_1}^{\vartheta_0}\left|F_3\left(\frac hq+\vartheta\right)\right|d\vartheta,$$

and hence, by (20), (54), (46), and (55),

$$|J(h,q)-J_1(h,q)| \le 2B_8\, n^{2\alpha-2/k}\int_{\vartheta_1}^{\infty}(q\vartheta)^{-(s-2m_0-4)/k}\, d\vartheta$$

$$= B_{12} n^{\beta-(s-2m_0-4-k)\delta} q^{-1} \leqq B_{12} n^{\beta-(k+1)\delta} q^{-1},$$

so that

$$\Big| \sum_{q \leqq n^{k\delta}} \sum_h{}' \{ J(h, q) - J_1(h, q) \} \Big| \leqq B_{12} n^{\beta-\delta}.$$

From this and (56) it follows that

(60) $$\Big| a_n - \sum_{q \leqq n^{k\delta}} \sum_h{}' J_1(h, q) \Big| \leqq B_{13} n^{\beta-\delta}.$$

**16.** Until further notice suppose also

(61) $$|\vartheta| \leqq q^{-1} n^{-1+k\delta}.$$

Putting

(62) $$g_0(\vartheta) = S_{h,q} q^{-1} g(\vartheta, n^{1/k}),$$

we have, by (18) and (19),

(63) $$|g_0(\vartheta)| \leqq B_{14} q^{-1/k} (n^{-1} + |\vartheta|)^{-1/k},$$

and, by **6** and (57),

(64) $$|f(\theta, n^{1/k}) - g_0(\vartheta)| \leqq B_{15} n^{k\delta}.$$

Similarly

$$|f(\theta, n^{1/k-\delta}) - S_{h,q} q^{-1} g(\vartheta, n^{1/k-\delta})| \leqq B_{15} n^{k\delta},$$

and hence, by (18), (19), and (57),

(65) $$|f(\theta, n^{1/k-\delta})| \leqq B_{16} q^{-1/k} n^{1/k-\delta}.$$

By (50), (63), and (64),

$$|f^{s-2m_0-6}(\theta, n^{1/k}) - g_0^{s-2m_0-6}(\vartheta)| \leqq B_{17} n^{k\delta} \{ q(n^{-1} + |\vartheta|)\}^{-(s-2m_0-7)/k},$$

and hence, by (51),

(66) $$|f_0^2(\theta) f^{s-2m_0-6}(\theta, n^{1/k}) - f_0^2(\theta) g_0^{s-2m_0-6}(\vartheta)|$$
$$\leqq B_{18} n^{k\delta} \{ q(n^{-1} + |\vartheta|) \}^{-(s-2m_0-5)/k}.$$

By (29), (64), (65), and (57),

$$|f_0(\theta) - g_0(\vartheta)| \leqq B_{15} n^{k\delta} + B_{16} q^{-1/k} n^{1/k-\delta} \leqq B_{19} q^{-1/k} n^{1/k-\delta}.$$

Hence, by (51) and (63),

$$|f_0^2(\theta) - g_0^2(\vartheta)| \leqq B_{20} n^{1/k-\delta} q^{-2/k} (n^{-1} + |\vartheta|)^{-1/k};$$

from this and (63) it follows that



$$|f_0^2(\theta) g_0^{s-2m_0-6}(\vartheta) - g_0^{s-2m_0-4}(\vartheta)| \leqq B_{21} n^{1/k-\delta} q^{-(s-2m_0-4)/k} (n^{-1} + |\vartheta|)^{-(s-2m_0-5)/k},$$

and hence, by (66) and (57),

$$|f_0^2(\theta) f^{s-2m_0-6}(\theta, n^{1/k}) - g_0^{s-2m_0-4}(\vartheta)|$$
$$\leqq B_{22} n^{1/k-\delta} q^{-(s-2m_0-4)/k} (n^{-1} + |\vartheta|)^{-(s-2m_0-5)/k}.$$

From this and (32), (33), (53), and (55) we obtain

(67) $$|F_3(\theta) - F_1^2(\theta) g_0^{s-2m_0-4}(\vartheta)| \leqq B_{22} n^{2\alpha-1/k-\delta} q^{-(2k+1)/k} (n^{-1} + |\vartheta|)^{-(s-2m_0-5)/k}.$$

**17.** It easily follows from (31), (30), (12), and (9) that

$$F_1^2(\theta) = \sum_{r=1}^{r_0} c_r e^{2\pi i \theta r},$$

where $c_1, c_2, \ldots$ are certain integers, positive or zero, and

$$r_0 = (2m_0 + 4)[n^{1-1/k}].$$

Hence

$$\Big| F_1^2(\theta_1) - F_1^2(\theta_2) \Big| \leqq \sum_{r=1}^{r_0} c_r \Big| e^{2\pi i \theta_1 r} - e^{2\pi i \theta_2 r} \Big| \leqq 2\pi \Big| \theta_1 - \theta_2 \Big| r_0 \sum_{r=1}^{r_0} c_r$$
$$= 2\pi |\theta_1 - \theta_2| r_0 F_1^2(0) \leqq B_{23} |\theta_1 - \theta_2| n^{1-1/k} F_1^2(0),$$

and hence, by (20), (53), and (61),

$$|F_1^2(\theta) - F_1^2(h/q)| \leqq B_{23} |\vartheta| n^{1+2\alpha-3/k} \leqq B_{23} n^{2\alpha-3/k+k\delta} q^{-1}.$$

From this and (63) and (55) it follows that

$$|F_1^2(\theta) g_0^{s-2m_0-4}(\vartheta) - F_1^2(h/q) g_0^{s-2m_0-4}(\vartheta)|$$
$$\leqq B_{24} n^{2\alpha-3/k+k\delta} q^{-3} (n^{-1} + |\vartheta|)^{-(s-2m_0-4)/k}$$
$$\leqq B_{24} n^{2\alpha-2/k+k\delta} q^{-3} (n^{-1} + |\vartheta|)^{-(s-2m_0-5)/k},$$

and hence, by (67),

(68) $$|F_3(\theta) - F_1^2(h/q) g_0^{s-2m_0-4}(\vartheta)| \leqq B_{25} n^{2\alpha-1/k-\delta} q^{-2-1/k} (n^{-1} + |\vartheta|)^{-(s-2m_0-5)/k}.$$

**18.** Putting

(69) $$J_2(h, q) = \int_{-\vartheta_1}^{\vartheta_1} F_1^2\left(\frac{h}{q}\right) g_0^{s-2m_0-4}(\vartheta) e^{-2\pi i (h/q+\vartheta) n} d\vartheta,$$

we have, by (58), (20), (68), and (46),

$$(70)\quad |J_1(h,q)-J_2(h,q)|\leqq B_{25}\,n^{2\alpha-1/k-\delta}\,q^{-2-1/k}\int_{-\vartheta_1}^{\vartheta_1}(n^{-1}+|\vartheta|)^{-(s-2m_0-5)/k}\,d\vartheta$$

$$<2B_{25}\,n^{2\alpha-1/k-\delta}\,q^{-2-1/k}\int_{1/n}^{\infty}x^{-(s-2m_0-5)/k}\,dx=B_{26}\,n^{\beta-\delta}\,q^{-2-1/k}.$$

**19.** I now drop the assumption (61), and put

$$(71)\quad J_3(h,q)=F_1^2\left(\frac{h}{q}\right)\xi_q^{-hn}\int_{-\infty}^{\infty}g_0^{s-2m_0-4}(\vartheta)\,e^{-2\pi i\vartheta n}\,d\vartheta.$$

Then, by (11), (69), (53), (63), (59), and (46),

$$|J_3(h,q)-J_2(h,q)|\leqq B_{27}\,n^{2\alpha-2/k}\int_{\vartheta_1}^{\infty}(q\vartheta)^{-(s-2m_0-4)/k}\,d\vartheta$$

$$=B_{28}\,n^{\beta-(s-2m_0-4-k)\delta}\,q^{-1}.$$

Hence, by (55),

$$|J_3(h,q)-J_2(h,q)|\leqq B_{28}\,n^{\beta-(k+1)\delta}\,q^{-1},$$

and hence

$$(72)\quad \left|\sum_{q\leqq n^{k\delta}}\sum_h{}'\{J_3(h,q)-J_2(h,q)\}\right|\leqq B_{28}\,n^{\beta-\delta}.$$

By (70),

$$\left|\sum_{q\leqq n^{k\delta}}\sum_h{}'\{J_1(h,q)-J_2(h,q)\}\right|\leqq B_{29}\,n^{\beta-\delta}.$$

Hence, by (60) and (72),

$$(73)\quad \left|a_n-\sum_{q\leqq n^{k\delta}}\sum_h{}' J_3(h,q)\right|\leqq B_{30}\,n^{\beta-\delta}.$$

**20.** By (71) and (62),

$$(74)\quad J_3(h,q)=F_1^2\left(\frac{h}{q}\right)\xi_q^{-hn}\,(S_{h,q}\,q^{-1})^{s-2m_0-4}\int_{-\infty}^{\infty}g^{s-2m_0-4}(\vartheta,n^{1/k})\,e^{-2\pi i\vartheta n}\,d\vartheta.$$

Now

$$\int_{-\infty}^{\infty}g^{s-2m_0-4}(\vartheta,n^{1/k})\,e^{-2\pi i\vartheta n}\,d\vartheta$$



$$=n^{-1}\int_{-\infty}^{\infty}g^{s-2m_0-4}(x/n,n^{1/k})\,e^{-2\pi ix}\,dx$$

and, by (13),

$$g(x/n,n^{1/k})=\int_0^{n^{1/k}}e^{2\pi ix(n^{-1/k}y)^k}\,dy=n^{1/k}\,\Psi(x),$$

where

$$\Psi(x)=\int_0^1 e^{2\pi ixt^k}\,dt.$$

Hence

$$(75)\quad \int_{-\infty}^{\infty}g^{s-2m_0-4}(\vartheta,n^{1/k})\,e^{-2\pi i\vartheta n}\,d\vartheta=L\,n^{-1+(s-2m_0-4)/k},$$

where

$$L=\int_{-\infty}^{\infty}\Psi^{s-2m_0-4}(x)\,e^{-2\pi ix}\,dx.$$

**21.** It easily follows from (12), (14), and (11) that

$$(76)\quad |f(h/q,u)-u\,S_{h,q}\,q^{-1}|\leqq q,$$

and from (30), (12), (14), and (11) that

$$(77)\quad \left|f_m(h/q)-\frac{1}{2}u_m\,S_{h,q}\,q^{-1}\right|\leqq q\qquad(m\geqq 1).$$

Put

$$U_{2m-1}=U_{2m}=f_m(h/q),\quad V_{2m-1}=V_{2m}=\frac{1}{2}u_m\,S_{h,q}\,q^{-1}\qquad(m=1,2,\dots m_0)$$

and

$$U_r=f(h/q,u_{m_0+1}),\quad V_r=u_{m_0+1}\,S_{h,q}\,q^{-1}\qquad(r=2m_0+1,\ 2m_0+2,\ 2m_0+3,\ 2m_0+4),$$

Then, by (77) and (76),

$$(78)\quad |U_r-V_r|\leqq q\qquad(r=1,2,\dots 2m_0+4),$$

by (31),

$$(79)\quad F_1^2\left(\frac{h}{q}\right)=\prod_{r=1}^{2m_0+4}U_r,$$

and, by (9) and (40),

$$(80)\qquad \prod_{r=1}^{2m_0+4} V_r = 2^{-2m_0} n^{2\alpha-2/k} (S_{h,q} q^{-1})^{2m_0+4}.$$

Also, by (38) and (37),

$$(81)\qquad |U_{2m-1}| = |U_{2m}| < u_m \quad (m=1,2,\dots,m_0),$$
$$|U_r| < u_{m_0+1} \quad (r = 2m_0+1,\ 2m_0+2,\ 2m_0+3,\ 2m_0+4),$$

and it is trivial that

$$(82)\qquad |V_{2m-1}| = |V_{2m}| < u_m \quad (m=1,2,\dots,m_0),$$
$$|V_r| \leqq u_{m_0+1} \quad (r = 2m_0+1,\ 2m_0+2,\ 2m_0+3,\ 2m_0+4).$$

Now

$$\prod_{r=1}^{2m_0+4} U_r - \prod_{r=1}^{2m_0+4} V_r = \sum_{l=1}^{2m_0+4} \left\{ (U_l - V_l) \prod_{r=l+1}^{2m_0+4} U_r \prod_{r=1}^{l-1} V_r \right\},$$

any empty product $\left( \prod_{r=1}^{0} \text{ or } \prod_{r=2m_0+5}^{2m_0+4} \right)$ meaning 1, and it follows from (81), (82), (9), and (40) that

$$\left| \prod_{r=l+1}^{2m_0+4} U_r \prod_{r=1}^{l-1} V_r \right| \leqq u_{m_0+1}^3 \prod_{m=1}^{m_0} u_m^2 = n^{2\alpha-2/k-k^{-1}(1-k^{-1})^{m_0+1}}$$
$$\leqq n^{2\alpha-2/k-4k\delta}.$$

Hence, by (79), (80), and (78),

$$(83)\qquad |F_1^2(h/q) - 2^{-2m_0} n^{2\alpha-2/k} (S_{h,q} q^{-1})^{2m_0+4}|$$
$$\leqq (2m_0+4)\, q\, n^{2\alpha-2/k-4k\delta}.$$

**22.** Since

$$|\xi_q^{-hn} (S_{h,q} q^{-1})^{s-2m_0-4}| \leqq 1,$$

it follows from (74), (75), (83), and (46) that

$$|J_3(h,q) - 2^{-2m_0} L n^{\beta} \xi_q^{-hn} S_{h,q}^{s} q^{-s}| \leqq B_{31}\, q\, n^{\beta-4k\delta}.$$

Hence, by (15),

$$\left| \sum_h{}' J_3(h,q) - 2^{-2m_0} L n^{\beta} A_q \right| \leqq B_{31} q^2 n^{\beta-4k\delta},$$

and hence

$$(84)\qquad \left| \sum_{q \leqq n^{k\delta}} \sum_h{}' J_3(h,q) - 2^{-2m_0} L n^{\beta} \sum_{q \leqq n^{k\delta}} A_q \right| \leqq B_{31} n^{\beta-k\delta}.$$



**23.** I now drop the assumption (57).

By (15), (18), and (55),

$$|A_q| \leqq B_{32} q^{1-s/k} \leqq B_{32} q^{-1-1/k}.$$

Hence, by (17),

$$\left| S - \sum_{q \leqq n^{k\delta}} A_q \right| \leqq B_{33} n^{-\delta},$$

and hence, by (73) and (84),

$$(85)\qquad |a_n - 2^{-2m_0} L n^{\beta} S| \leqq B_{34} n^{\beta-\delta}.$$

**24.** It follows from (1) and (4) that $m_0 \geqq k-2$. Hence, by (55) $s > 4k$, and hence [6])

$$S \geqq B_{35}^{-1}.$$

Also [7])

$$L = \frac{\Gamma^{s-2m_0-4}(1+1/k)}{\Gamma\{(s-2m_0-4)/k\}},$$

so that $L > 0$. It therefore follows from (85) that $a_n \neq 0$ for any sufficiently large $n$, *q. e. d.*

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(Received 30 August, 1936.)

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[6]) Hardy and Littlewood, l. c. (footnote 4), Theorems 12 and 15.
[7]) Landau, Math. Zeitschrift, 31 (1930), 338