Demgegenüber lässt sich angesichts der neuen Fragestellung $\{\Phi(\Theta), \varphi(\Theta, \beta)\}$ leicht beweisen, dass der normale Fall $(\Phi(\Theta) > 0, \varphi(\Theta, \beta) < +\infty)$ bei jedem n für fast alle Θ vorliegt.

Bei der früheren Fragestellung bestand kein wesentlicher Unterschied zwischen dem ein- und dem mehrdimensionalen Fall; bei der neuen Fragestellung ist ein solcher bekanntlich vorhanden; indem nämlich im eindimensionalen Fall für *alle irrationalen* Θ $\Phi(\Theta) > 0$ und folglich $\varphi(\Theta, \beta) < +\infty$ gilt[7]) („normales" Verhalten), gibt es bei jedem $n > 1$ ausser den trivialen Θ mit linear abhängigen $\Theta_1, \Theta_2, \ldots, \Theta_n$ auch solche mit linear unabhängigen $\Theta_1, \Theta_2, \ldots, \Theta_n$ und mit $\Phi(\Theta) = 0$[8]).

Endlich sei bemerkt, dass ich bei der früheren Fragestellung als Zusatz beweisen konnte: *ist* $\varphi(\Theta) = 0$, *so ist für fast alle* β $\Phi(\Theta, \beta) = \infty$[9]). Ob ein analoger Zusatz auch bei der neuen Fragestellung gilt, weiss ich nicht; ich vermute, dass eine solche Behauptung im allgemeinen falsch wäre; auf jeden Fall scheint die hier entwickelte Methode keinen Anhaltspunkt für die Begründung eines derartigen Zusatzes zu liefern.

(Eingegangen am 14. Mai 1936.)

7) K. Kap. III Satz 24 und Kap. VI Satz I.

8) K. Kap. V Satz 8.

9) K. Kap. VII Satz 2.

An arithmetical theorem on linear forms.

By

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Van der Corput[1]) has recently stated and proved the following exceedingly simple and general theorem:

„Ist M eine im n-dimensionalen Raum liegende Menge vom Volumen $V > k_1 k_2 \ldots k_n$ $(k_1 > 0, \ldots, k_n > 0)$, und hat jedes zu M gehörige Punktepaar $(x_1, \ldots x_n)$ und $(y_1, \ldots y_n)$ die Eigenschaft, dass der Punkt $\left(\frac{x_1 - y_1}{k_1}, \ldots \frac{x_n - y_n}{k_n}\right)$ einer gewissen Menge N angehört, dann enthält N ausser dem Koordinatenursprung noch mindestens einen weiteren Gitterpunkt".

The proof tacitly assumes that M is a simple set of points, i. e. that no point is reckoned more than once in calculating the volume. If M is not a simple set, the theorem is not true as the origin may be the only lattice point in N. It then depends upon the nature of M whether or not the result is trivial. As an illustration, I give a result on linear forms not included in his theorem, and so also not in Minkowski's famous theorem on homogeneous linear forms which it includes.

Let

$$L_r(x) = \sum_{s=1}^{n} a_{rs} x_s \qquad (r = 1, 2, \ldots n)$$

1) Van der Corput. Verallgemeinerung einer Mordellschen Beweismethode in der Geometrie der Zahlen. Acta Arithmetica, 1. (1935), 62-66.

or say $L(x)$, be n linear forms with real coefficients in the n variables $x_1, x_2, \dots x_n$, or say x, with determinant $\Delta > 0$. Let $\mu_1, \mu_2, \dots, \mu_n$, $\nu_1, \nu_2, \dots \nu_n$ or say μ, ν be two sets of n non negative numbers for which

$$\mu_1 \mu_2 \dots \mu_n + \nu_1 \nu_2 \dots \nu_n \geqq \Delta$$

or say

$$\Pi \mu + \Pi \nu \geqq \Delta; \tag{1}$$

and let $c_1, c_2, \dots, c_n$, say c, be any set of n real numbers.

Then at least one of the three sets of inequalities

$$|L_r(x)| \leqq \mu_r \qquad (r = 1, 2, \dots, n),$$

or say

$$|L(x)| \leqq \mu, \tag{2}$$

$$|L(x)| \leqq \nu, \tag{3}$$

$$|L(x) + c| \leqq \frac{1}{2}(\mu + \nu), \tag{4}$$

has a solution in integers x besides the trivial solution $x = 0$ of (2) and (3).

A possible solution $x = 0$ of (4) is not trivial; and the theorem moreover is not true unless $x = 0$ is admitted as is clear from the case when the c's are small, the μ, ν are all $\sqrt[n]{\frac{1}{2}}$ and $L(x)$ is taken to be x.

It suffices to prove the theorem for integer values of the constants a, c, μ, ν. For it then holds for rational values of a, c, μ, ν on considering the linear forms $kL(x)$ and constants $k\mu$, $k\nu$, kc where k is the greatest common denominator of a, c, μ, ν. This shows also that we may assume all the μ and ν to be even integers. The case of irrational a, c, μ, ν follows by the usual easy limiting process on replacing a, c, μ, ν by sufficiently close rational approximations and noting that the sets of x's thus given are infinite in number and are bounded in value.

The idea in the proof is of the same kind as in my recent arithmetic proof of Minkowski's theorem [2]). It depends upon an old result of H. J. S. Smith stated here as follows:

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[2]) Minkowski's Theorem on Homogeneous Linear Forms. Journ. London Math. Soc., 8 (1933), 179—182.

If the a's are integers, there are exactly Δ^{n-1} sets of residues, say possible residues $p_1, p_2, \dots, p_n$, mod Δ, such that the system of congruences $L(x) \equiv p$ mod Δ admits of a solution. Hence there are exactly $\Delta^n - \Delta^{n-1}$ sets of residues, say impossible residues $i_1, i_2, \dots, i_n$, mod Δ such that the system $L(x) \equiv i$ mod Δ has no solution.

It is obvious that if p, p' are two possible sets, so are $p \pm p'$, and hence $p + i$ is an impossible set. It is also clear that a set of integers x exist such that $L(x) = p$ since the system $L(x) = k\Delta$, where k is any given set of integers, is obviously satisfied by an integer set x.

We may suppose that the μ, ν are all even, and that the inequalities $|L(x)| \leqq \frac{1}{2}\mu$ have only the solution $x = 0$, and that the system $|L(x) + c| \leqq \frac{1}{2}\nu$ has no solution, as otherwise there is nothing to prove. The first set of inequalities shows the existence of $(\mu_1 + 1)(\mu_2 + 1) \dots (\mu_n + 1) - 1$ different sets of integers i, where $-\frac{1}{2}\mu \leqq i \leqq \frac{1}{2}\mu$, which are non residues of the system $L(x)$, and hence from $i + p$ we have $\Delta^{n-1}(\Pi(\mu + 1) - 1)$ sets of impossible residues. The second inequality shows that there are no sets of integers x satisfying the inequalities

$$-c - \frac{1}{2}\nu \leqq L(x) \leqq -c + \frac{1}{2}\nu,$$

and so there exist $\Pi(\nu + 1)$ sets of impossible residues, say j, for which

$$-c - \frac{1}{2}\nu \leqq j \leqq -c + \frac{1}{2}\nu, \tag{5}$$

and hence $\Delta^{n-1}\Pi(\nu + 1)$ sets of impossible residues $p + j$.

But from (1)

$$\Delta^{n-1}(\Pi(\mu + 1) - 1) + \Delta^{n-1}\Pi(\nu + 1) > \Delta^n - \Delta^{n-1},$$

and so two of these impossible sets of residues must be congruent mod Δ. These two sets may arise in three different ways, i. e. from two i's, two j's, or an i and a j.

The first gives $p + i \equiv p' + i'$, or $p - p' \equiv i' - i$. Hence $i' - i$ is a possible set and so there is a set of integers x for which $|L(x)| \leqq \mu$ since $-\mu \leqq i' - i \leqq \mu$. The x's cannot all be zero since then $i' - i = 0$ and $p - p' \equiv 0 \pmod{\Delta}$, i. e. $p = p'$.

The second gives $p+j=p'+j'$ or $p-p'=j'-j$, and so the set of possible residues $j'-j$ gives rise to a set of integers x for which $|L(x)| \leqq \nu$ from (5). As before all the x's are not zero.

The third case gives $p+i=p'+j$ or $p-p'+c=j-i+c$. Hence the possible set $p-p'$ gives rise to a set of integers x such that $|L(x)+c| \leqq \frac{1}{2}\mu+\frac{1}{2}\nu$ since $|j+c| \leqq \frac{1}{2}\nu$, $|i| \leqq \frac{1}{2}\mu$. We note now that the x's may all be zero when $p=p'$, $i=j$.

This proves the theorem.

A sharper form of the theorem can be deduced by applying the theorem with μ_s (s arbitrary) replaced by $\mu_s(1+\varepsilon)^n$, μ_r by $\mu_r/(1+\varepsilon)$, $r \neq s$, ν by $\nu/(1+\eta)$ and making $\varepsilon>0$, $\eta>0$ tend to zero in such a way that

$$(1+\varepsilon)\Pi\mu+(1+\eta)^{-n}\Pi\nu \geqq \Pi\mu+\Pi\nu$$

or

$$\varepsilon\Pi\mu \geqq (1-(1+\eta)^{-n})\Pi\nu$$

We see then that if $\Pi\mu \neq 0$, all the $\leqq$ signs in (2), (3), (4) can be replaced by $<$ signs except the one corresponding to μ_s in (2) and the corresponding $\frac{1}{2}(\mu_s+\nu_s)$ of (4).

On making the $\mu \to 0$, we see that if $\Pi\nu=\Delta$ and the inequalities $|L(x)|<\nu$ have no solutions in integers x except $x=0$, then there is* for all c a solution of $|L(x)+c| \leqq \frac{1}{2}\nu$, in integers x. There cannot be two solutions unless for at least one s these make $L_s(x)+c_s=\pm\frac{1}{2}\nu_s$ respectively, for otherwise, their difference would give a solution of $|L(x)|<\nu$. This is of course known in connection with Minkowski's limiting case.

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* This is a known result. See Radò, Journal of the London Math. Soc. 10 (1935) 115—116; Perron, Ibid. 275—277.

Universal forms $\sum a_i x_i^n$ and Waring's problem.

By

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1. Introduction and summary. A form F is called *universal* if every positive integer is represented by F with integral values $\geqq 0$ of all the variables. Write

$$3^n=2^n q+r,\quad 0<r<2^n,\quad I=2^n+q-2.$$

THEOREM 1. *If $n>6$ and $r \leqq 2^n-q-3$, every positive integer is a sum of I n-th powers. Technically, $g(n)=I$.*

The inequality holds when $4 \leqq n \leqq 400$. The theorem was recently proved by the writer.[1]) For $n>8$, it is a corollary to the new theorem proved here:

THEOREM 2. *Let $d=1$ or 2 according as q is odd or even. If $9 \leqq n \leqq 400$, every positive integer is a sum of $4n+2-d$ n-th powers and the doubles of $P=\frac{1}{2}(2^n+q-4n+d)-2$ n-th powers. Here $4n+$ $+2-d+2P=I$.*

Expressed otherwise, in the ideal Waring Theorem 1, we may take $2P$ of the powers equal in pairs. While Theorem 1 states that $x_1^n+\ldots+x_I^n$ is universal, Theorem 2 yields a universal form (with

[1]) Amer. Jour. Math., vol. 58 (1936), pp. 521—35. In case the inequality fails hen $g(n)=I+f$ or $I+f-1$, according as $2^n=$ or $<fq+f+q$, where $f=[(4/3)^n]$. Announced March 13 in Bull. Amer. Math. Soc., 1936, p. 341.

If the inequality fails for any $n>400$, the dicimal $r/2^n$ begins with fifty figures 9. But 157 and 163 are the only values $\leqq 400$ of n for which it begins with two figures 9 (neither with three figures 9).