Note on a result of Siegel.

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Siegel[1]) has recently given a proof that, if $L_1, \dots, L_n$ are n real homogeneous linear forms in $\mathfrak{x} = (x_1, \dots, x_n)$ with determinant 1, and $c_1, \dots, c_n$ are n real numbers, then there exist integral values of $x_1, \dots, x_n$ for which

$$\prod_i |L_i + c_i| < \gamma_n, \tag{1}$$

where γ_n depends only on n[2]). In this note, which I publish at Prof. Siegel's suggestion, I give a slightly different proof, using his ideas but expressing them in another form.

If $Q(\mathfrak{x}) = Q(x_1, \dots, x_n)$ is a positive definite quadratic form, the successive minima $S_1^2, \dots, S_n^2$ of Q are defined as follows. S_1^2 is the minimum for all (integral) $\mathfrak{x} \neq 0$, attained say for $\mathfrak{x}_1$, S_2^2 is the minimum for all $\mathfrak{x}$ not multiples of $\mathfrak{x}_1$, attained say for $\mathfrak{x}_2$, S_3^2 is the minimum for all $\mathfrak{x}$ not linear integral combinations of $\mathfrak{x}_1, \mathfrak{x}_2$, and so on. It is known[3]) that

$$\sqrt{D} \leq S_1 \dots S_n \leq \frac{2^n \Gamma\left(1 + \frac{1}{2} n\right)}{\Gamma\left(\frac{1}{2}\right)^n} \sqrt{D}, \tag{2}$$

where D is the determinant of Q.

[1]) In a letter of 10 October 1937 to Prof. Mordell.

[2]) Minkowski conjectured that this holds with $\gamma_n = 2^{-n}$.

[3]) Minkowski, *Geometrie der Zahlen* (1910), 198.

Take $Q = L_1^2 + \dots + L_n^2$. It is plain from the definition of $S_1^2, \dots, S_n^2$ that the inequality $Q < S_i^2$ implies that $L_1, \dots, L_n$ satisfy $n - i + 1$ independent linear conditions, the coefficients in which depend only on $\mathfrak{x}_1, \dots, \mathfrak{x}_n$ (which we suppose chosen once for all).

We order $L_1, \dots, L_n$ in the following way. In the linear condition

$$A_1 L_1 + \dots + A_n L_n = 0 \tag{3}$$

implied by $Q < S_n^2$, A_n is to be the largest coefficient in absolute value. In the additional linear relation implied by $Q < S_{n-1}^2$, which we can take in the form

$$B_1 L_1 + \dots + B_{n-1} L_{n-1} = 0, \tag{4}$$

B_{n-1} is to be the largest coefficient in absolute value, and so on.

Then, if $L_1, \dots, L_n$ satisfy (3), we have

$$(A_n L_n)^2 \leq (A_1^2 + \dots + A_{n-1}^2)(L_1^2 + \dots + L_{n-1}^2),$$

and so

$$L_n^2 \leq (n-1)(L_1^2 + \dots + L_{n-1}^2),$$

whence

$$L_1^2 + \dots + L_{n-1}^2 \geq \frac{1}{n}(L_1^2 + \dots + L_n^2).$$

If $L_1, \dots, L_n$ satisty both (3) and (4), we have, similarly,

$$L_1^2 + \dots + L_{n-2}^2 \geq \frac{1}{n-1}(L_1^2 + \dots + L_{n-1}^2)$$

$$\geq \frac{1}{n(n-1)}(L_1^2 + \dots + L_n^2),$$

and so on generally. It follows that for any $\mathfrak{x} \neq 0$ there is an i such that

$$L_1^2 + \dots + L_i^2 \geq \frac{1}{n(n-1)\dots(i+1)} S_i^2. \tag{5}$$

Now consider the quadratic form

$$R = \frac{L_1^2}{S_1^2} + \dots + \frac{L_n^2}{S_n^2},$$

and denote its successive minima by $T_1^2, \dots, T_n^2$. By (5),

$$T_1^2 \geqq \frac{1}{n!}.$$

Hence

(6) $$T_1 + \dots + T_n \leqq n T_n \leqq n (n!)^{\frac{n-1}{2}} T_1 \dots T_n.$$

By (2),

(7) $$T_1 \dots T_n \leqq \frac{2^n \Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(\frac{1}{2}\right)^n} \frac{1}{S_1 \dots S_n},$$

since the determinant of R is $(S_1 \dots S_n)^{-2}$.

Let $\mathfrak{x}^{(j)}$ be a point at which $R = T_j^2$. Then

$$| L_i(\mathfrak{x}^{(j)}) | \leqq S_i T_j.$$

Let $y_1, \dots, y_n$ be the integers nearest to $\eta_1, \dots, \eta_n$, the real solution of

$$\sum_j L_i(\mathfrak{x}^{(j)}) \eta_j + c_i = 0 \qquad (i = 1, \dots, n).$$

Then

(8) $$\left| \sum_j L_i(\mathfrak{x}^{(j)}) y_j + c_i \right| \leqq \frac{1}{2} \sum_j | L_i(\mathfrak{x}^{(j)}) |$$

$$\leqq \frac{1}{2} (T_1 + \dots + T_n) S_i.$$

Let $\mathfrak{x}$ be defined by

$$\mathfrak{x} = \sum_j y_j \mathfrak{x}^{(j)},$$

then $x_1, \dots, x_n$ are integers, and

(9) $$L_i(\mathfrak{x}) = \sum_j L_i(\mathfrak{x}^{(j)}) y_j.$$

By (6), (7), (8), (9),

$$\prod_i | L_i(\mathfrak{x}) + c_i | \leqq 2^{-n} (T_1 + \dots + T_n)^n S_1 \dots S_n$$

$$\leqq \left\{ \frac{n 2^{n-1} \Gamma\left(1+\frac{1}{2}n\right) (n!)^{\frac{n-1}{2}}}{\Gamma\left(\frac{1}{2}\right)^n} \right\}^n \frac{1}{(S_1 \dots S_n)^{n-1}}$$

$$\leqq \left\{ \frac{n 2^{n-1} \Gamma\left(1+\frac{1}{2}n\right) (n!)^{\frac{n-1}{2}}}{\Gamma\left(\frac{1}{2}\right)^n} \right\}^n,$$

on using the first half of (2). This proves the result.

Other convex forms than the quadratic forms Q, R may be used in the proof, e. g. instead of Q the form

$$\operatorname{Max} (|L_1|, \dots, |L_n|).$$

The consideration of the successive minima $M_1, \dots, M_n$ of this form leads to a simple proof that there exist numbers $N_1, \dots, N_n$ with $N_1 \dots N_n \geqq \delta_n$, where δ_n depends only on n, such that the domain

$$|L_i| < N_i \qquad (i = 1, \dots, n)$$

contains no lattice point other than the origin. After ordering the forms suitably one may take

$$N_i = \frac{M_i}{(n-1)(n-2)\dots i},$$

and then [1])

$$\delta_n = \frac{1}{1 \cdot 2^2 \cdot 3^3 \dots (n-1)^{n-1} \, n!}.$$

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[1]) $M_1 \dots M_n \geqq (n!)^{-1}$ (Minkowski, *loc. cit.* 192).