Teilerprobleme. Fünfte Abhandlung.

Von

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Es sei $\sigma(n)$ die Summe der reziproken Teiler einer natürlichen Zahl n

$$\sigma(n)=\sum_{d|n}\frac{1}{d},$$

also $n\sigma(n)$ die Teilersumme

$$n\sigma(n)=\sum_{d|n}d.$$

Ich betrachte ferner die beiden summatorischen Funktionen

$$S_0(x)=\sum_{n\leqq x}\sigma(n), \qquad S(x)=\sum_{n\leqq x}n\sigma(n)$$

(falls die untere Summationsgrenze nicht ausdrücklich angegeben wird, ist sie stets Eins) und setze

$$S_0(x)=\frac{\pi^2}{6}x-\frac{1}{2}\log x+T_0(x),$$

$$S(x)=\frac{\pi^2}{12}x^2+T(x).$$

Es sei ν eine der Zahlen 0, 1, 2, 3, 4;

(1) $$V_\nu(x)=\int_0^x T(2^\nu u)\,T(u)\,du.$$

Mit Q_0, Q_1, Q_2 bezeichne ich die quaternären quadratischen Formen

$$Q_0=n_1^2+n_2^2+n_3^2+n_4^2,$$
$$Q_1=n_1^2+n_2^2+2n_3^2+2n_4^2,$$
$$Q_2=n_1^2+2n_2^2+2n_3^2+4n_4^2.$$

Dann sind $0\leqq Q_0\leqq x$, $0\leqq Q_1\leqq x$, $0\leqq Q_2\leqq x$ vierdimensionale Ellipsoide vom Volumen $W_0(x)=\frac{\pi^2}{2}x^2$, $W_1(x)=\frac{\pi^2}{4}x^2$, $W_2(x)=\frac{\pi^2}{8}x^2$. Es seien $P_0(x)$, $P_1(x)$, $P_2(x)$ die entsprechenden Gitterreste

$$P_m(x)=\sum_{Q_m\leqq x}1-W_m(x) \qquad (m=0,1,2).$$

Im folgenden sei stets $x\geqq 3$. Mit B bezeichne ich unterschiedslos Zahlen, deren absolute Beträge unterhalb numerischer Schranken liegen. C sei die Eulersche Konstante.

Das Hauptziel dieser Arbeit besteht in dem Nachweis der folgenden Abschätzungen:

(I) $$\int_0^x T_0^2(u)\,du=\left(\left(\frac{C+\log 2\pi}{2}\right)^2+\frac{5\pi^2}{144}\right)x+Bx^{\frac{1}{2}},$$

(II) $$\int_0^x T^2(u)\,du=\frac{36+5\pi^2}{432}x^3+Bx^{\frac{5}{2}},$$

(III) $$\int_0^x P_0^2(u)\,du=\frac{2}{3}\pi^2x^3+Bx^{\frac{5}{2}},$$

(IV) $$\int_0^x P_1^2(u)\,du=\frac{\pi^2}{6}x^3+Bx^{\frac{5}{2}},$$

(V) $$\int_0^x P_2^2(u)\,du=\frac{\pi^2}{24}x^3+Bx^{\frac{5}{2}}.$$

Gemeinsame Quelle für (I)—(V) ist

(VI) $$V_\nu(x)=\left(\frac{2^\nu}{12}+\frac{(5+3\nu)\pi^2}{432}\right)x^3+Bx^{\frac{5}{2}}.$$

Zum Beweise von (VI) benutze ich im wesentlichen die Methode der dritten gleichnamigen Abhandlung (Journal für die reine und angewandte Mathematik **169** (1933), S. 111—130). Dabei mache ich auch von der zweiten und vierten Abhandlung (Mathematische Zeitschrift **34** (1931) S. 448—472 und Annali della R. Scuola Normale Superiore di Pisa, serie 2,**5** (1936) S. 289—298) Gebrauch, sowie von der fünften Abhandlung „Über Gitterpunkte in mehrdimensionalen Ellipsoiden" (Acta Arithmetica **1** (1936), S. 222—283). Ich nenne diese Arbeiten kurz T_{III}, T_{II}, T_{IV} und E_{V}.

Ich möchte noch erwähnen, dass (I)—(VI), mit den etwas ungünstigeren Restgliedern $Bx^{\frac{1}{2}}\log x$ und $Bx^{\frac{5}{2}}\log x$ in T_{II} ((I)—(III)) und E_{V} ((II)—(VI)) stehen.

Bevor ich zur Sache übergehe, schalte ich die folgende Bemerkung zu E_{V} ein:

Es sei

$$Q=\sum_{g,h=1}^{4} a_{gh} n_g n_h \qquad (a_{gh}=a_{hg} \text{ rational})$$

eine positiv definite quadratische Form der Determinante D,

$$P_Q(x)=\sum_{Q\leq x} 1-\frac{\pi^2}{2\sqrt{D}}x^2$$

der Gitterrest des vierdimensionalen Ellipsoides $0\leqq Q\leqq x$.

Ich brauchte in E_{V} die Abschätzung

$$\int_0^x P_Q^2(u)\,du=\Omega(x^3)$$

und gab dafür einen langen und sehr umständlichen Beweis. Einer freundlichen Mitteilung Prof. Jarníks entnehme ich, dass diese Abschätzung bereits bekannt ist und auf ganz einfache Weise in wenigen Zeilen erledigt werden kann. Vgl. V. Jarník „Über die Mittelwertsätze der Gitterpunktlehre. I." (Mathematische Zeitschrift 33 (1931), S. 62—84), S. 83.

Die folgenden Festsetzungen mögen für die genze vorliegende Arbeit gelten:

Für reelle u sei

$$(2)\qquad R(u)=u-[u],\ \psi(u)=\psi_1(u)=R(u)-\frac{1}{2},\qquad \psi_2(u)=R^2(u)-R(u)+\frac{1}{6}.$$

Die nicht kleinere der beiden Zahlen v und w heisse Max $(v;w)$, die nicht grössere Min $(v;w)$; dagegen bedeute (v,w) den grössten gemeinsamen Teiler der beiden ganzen Zahlen v und w.

Ferner sei

$$(3)\qquad E_1=0 \text{ oder } 1,\ E_2=0 \text{ oder } 1,\ \delta=\frac{1}{100}.$$

Die fetten Zahlen geben Nummern von Formeln an, die an den betreffenden Stellen benutzt werden, sind also als Hinweise zu verstehen.

Hilfssatz 1:

$$(4)\qquad T(u)-uT_0(u)=\frac{1}{2}(C+\log 2\pi-1)u+Bu^{\frac{1}{3}} \qquad (u\geqq 1).$$

Beweis: E. Landau „Über einige zahlentheoretische Funktionen" (Göttinger Nachrichten 1924, S. 116—134) und etwas einfacher in T_{IV}.

Hilfssatz 2:

$$(5)\qquad -T_0(u)=\frac{1}{2}(C+\log 2\pi)+\sum_{n\leqq\sqrt{u}}\frac{1}{n}\psi\left(\frac{u}{n}\right)+\frac{1}{u}\sum_{n\leqq\sqrt{u}} n\psi\left(\frac{u}{n}\right)$$
$$+\frac{1}{2u^2}\sum_{n\leqq\sqrt{u}} n^2\psi_2\left(\frac{u}{n}\right)+\frac{2}{\sqrt{u}}\psi_2(\sqrt{u})+\frac{1}{12\sqrt{u}}+Bu^{-1} \qquad (u\geqq 1).$$

Beweis: Das ist T_{IV} (4).

Hilfssatz 3:

$$(6)\qquad -T(u)=\frac{u}{2}+u\sum_{n\leqq\sqrt{u}}\frac{1}{n}\psi\left(\frac{u}{n}\right)+\sum_{n\leqq\sqrt{u}} n\psi\left(\frac{u}{n}\right)+2\sqrt{u}\,\psi_2(\sqrt{u})$$
$$+\frac{\sqrt{u}}{12}+Bu^{\frac{1}{3}} \qquad (u\geqq 1).$$

Beweis: Es sei $u\geqq 1$,

(7) $$-T(u)=\frac{u}{2}+u\sum_{n\leqq\sqrt{u}}\frac{1}{n}\,\psi\left(\frac{u}{n}\right)+\sum_{n\leqq\sqrt{u}}n\,\psi\left(\frac{u}{n}\right)+2\sqrt{u}\,\psi_2(\sqrt{u})$$

$$+\frac{\sqrt{u}}{12}+\frac{1}{2u}\sum_{n\leqq\sqrt{u}}n^2\,\psi_2\left(\frac{u}{n}\right)+B\,u^{\frac{1}{3}}\qquad(4,\ 5).$$

(8) $(1\leqq v\leqq\sqrt{u})\sum_{n\leqq v}\psi_2\left(\frac{u}{n}\right)=B\,u^{\frac{1}{3}}$ (T_{IV}, Schluss des Beweises von (2)).

(9) $$\sum_{n\leqq\sqrt{u}}n^2\,\psi_2\left(\frac{u}{n}\right)=B\,u^{\frac{4}{3}}\qquad(8).$$

(6) folgt aus (7) und (9).

Hilfssatz 4: *Mit*

(10) $$\mathfrak{c}_{E_1,E_2}=\mathfrak{c}_{E_1,E_2,\nu}=\begin{cases}2^\nu\ \text{für}\ E_1=0,\ E_2=0\ \text{oder für}\ E_1=0,\ E_2=1\\ 1\ \text{für}\ E_1=1,\ E_2=0\ \text{oder für}\ E_1=1,\ E_2=1,\end{cases}$$

(11) $$r_\nu=\mathrm{Max}\left(\frac{m^2}{2^\nu};\,n^2\right)$$

ist

(12) $$V_\nu(x)=\frac{2^\nu}{12}x^3+\sum_{E_1,E_2=0}^{1}\mathfrak{c}_{E_1,E_2}\sum_{\substack{m\leqq\sqrt{2^\nu x}\\ n\leqq\sqrt{x}}}m^{2E_1-1}\,n^{2E_2-1}\int_{r_\nu}^{x}\psi\left(\frac{2^\nu u}{m}\right)\psi\left(\frac{u}{n}\right)u^{2-E_1-E_2}\,du$$

$$+2^{\nu+1}\sum_{m\leqq\sqrt{2^\nu x}}\frac{1}{m}\int_{\mathrm{Max}\left(\frac{m^2}{2^\nu};1\right)}^{x}\psi\left(\frac{2^\nu u}{m}\right)\psi_2(\sqrt{u})\,u^{\frac{3}{2}}\,du$$

$$+2^{\frac{\nu}{2}+1}\sum_{n\leqq\sqrt{x}}\frac{1}{n}\int_{n^2}^{x}\psi\left(\frac{u}{n}\right)\psi_2(\sqrt{2^\nu u})\,u^{\frac{3}{2}}\,du+B\,x^{\frac{5}{2}}.$$

Beweis: Es sei $u\geqq 1$.

(13) $$-T(2^\nu u)=2^{\nu-1}u+2^\nu u\sum_{m\leqq\sqrt{2^\nu u}}\frac{1}{m}\,\psi\left(\frac{2^\nu u}{m}\right)+\sum_{m\leqq\sqrt{2^\nu u}}m\,\psi\left(\frac{2^\nu u}{m}\right)$$

$$+2^{\frac{\nu}{2}+1}\sqrt{u}\,\psi_2(\sqrt{2^\nu u})+2^{\frac{\nu}{2}}\frac{\sqrt{u}}{12}+B\,u^{\frac{1}{3}}\qquad(6).$$

(14)
$$T(2^\nu u)\,T(u)=2^{\nu-2}u^2+\sum_{E_1,E_2=0}^{1}\mathfrak{c}_{E_1,E_2}\sum_{\substack{m\leqq\sqrt{2^\nu u}\\ n\leqq\sqrt{u}}}m^{2E_1-1}\,n^{2E_2-1}\,\psi\left(\frac{2^\nu u}{m}\right)\psi\left(\frac{u}{n}\right)u^{2-E_1-E_2}$$

$$+2^{\nu+1}\sum_{m\leqq\sqrt{2^\nu u}}\frac{1}{m}\,\psi\left(\frac{2^\nu u}{m}\right)\psi_2(\sqrt{u})\,u^{\frac{3}{2}}+2^{\frac{\nu}{2}+1}\sum_{n\leqq\sqrt{u}}\frac{1}{n}\,\psi\left(\frac{u}{n}\right)\psi_2(\sqrt{2^\nu u})\,u^{\frac{3}{2}}$$

$$+2^{\nu-1}\sum_{n\leqq\sqrt{u}}\frac{1}{n}\,\psi\left(\frac{u}{n}\right)u^2+2^{\nu-1}\sum_{n\leqq\sqrt{u}}n\,\psi\left(\frac{u}{n}\right)u+2^{\nu-1}\sum_{m\leqq\sqrt{2^\nu u}}\frac{1}{m}\,\psi\left(\frac{2^\nu u}{m}\right)u^2$$

$$+\frac{2^\nu}{12}\sum_{m\leqq\sqrt{2^\nu u}}\frac{1}{m}\,\psi\left(\frac{2^\nu u}{m}\right)u^{\frac{3}{2}}+\frac{1}{2}\sum_{m\leqq\sqrt{2^\nu u}}m\,\psi\left(\frac{2^\nu u}{m}\right)u+\frac{2^{\frac{\nu}{2}}}{12}\sum_{n\leqq\sqrt{u}}\frac{1}{n}\,\psi\left(\frac{u}{n}\right)u^{\frac{3}{2}}$$

$$+B\,u^{\frac{3}{2}}\qquad(13,\ 6,\ 10).$$

$$V_\nu(x)=\int_1^x T(2^\nu u)\,T(u)\,du+B\qquad(1)$$

$$=\frac{2^\nu}{12}x^3+\sum_{E_1,E_2=0}^{1}\mathfrak{c}_{E_1,E_2}\sum_{\substack{m\leqq\sqrt{2^\nu x}\\ n\leqq\sqrt{x}}}m^{2F_1-1}\,n^{2E_2-1}\int_{r_\nu}^{x}\psi\left(\frac{2^\nu u}{m}\right)\psi\left(\frac{u}{n}\right)u^{2-E_1-E_2}\,du$$

$$+2^{\nu+1}\sum_{m\leqq\sqrt{2^\nu x}}\frac{1}{m}\int_{\mathrm{Max}\left(\frac{m^2}{2^\nu};1\right)}^{x}\psi\left(\frac{2^\nu u}{m}\right)\psi_2(\sqrt{u})\,u^{\frac{3}{2}}\,du$$

$$+2^{\frac{\nu}{2}+1}\sum_{n\leqq\sqrt{x}}\frac{1}{n}\int_{n^2}^{x}\psi\left(\frac{u}{n}\right)\psi_2(\sqrt{2^\nu u})\,u^{\frac{3}{2}}\,du$$

$$+2^{\nu-1}\sum_{n\leqq\sqrt{x}}\frac{1}{n}\int_{n^2}^{x}\psi\left(\frac{u}{n}\right)u^2\,du+2^{\nu-1}\sum_{n\leqq\sqrt{x}}n\int_{n^2}^{x}\psi\left(\frac{u}{n}\right)u\,du$$

$$+2^{\nu-1}\sum_{m\leqq\sqrt{2^\nu x}}\frac{1}{m}\int\limits_{\mathrm{Max}\left(\frac{m^2}{2^\nu};1\right)}^{x}\psi\left(\frac{2^\nu u}{m}\right)u^2\,du+\frac{2^\nu}{12}\sum_{m\leqq\sqrt{2^\nu x}}\frac{1}{m}\int\limits_{\mathrm{Max}\left(\frac{m^2}{2^\nu};1\right)}^{x}\psi\left(\frac{2^\nu u}{m}\right)u^{\frac{3}{2}}du$$

$$+\frac{1}{2}\sum_{m\leqq\sqrt{2^\nu x}}m\int\limits_{\mathrm{Max}\left(\frac{m^2}{2^\nu};1\right)}^{x}\psi\left(\frac{2^\nu u}{m}\right)u\,du+\frac{2^{\frac{\nu}{2}}}{12}\sum_{n\leqq\sqrt{x}}\frac{1}{n}\int\limits_{n^2}^{x}\psi\left(\frac{u}{n}\right)u^{\frac{3}{2}}\,du$$

$$+Bx^{\frac{5}{2}}\qquad(14,\ 11),$$

und hierin sind nach (2) die sechs letzten Summen, wenn über geeignete Strecken integriert wird,

$$=Bx^2\sum_{n\leqq\sqrt{x}}\frac{1}{n}\left|\int\psi\left(\frac{u}{n}\right)du\right|+Bx\sum_{n\leqq\sqrt{x}}n\left|\int\psi\left(\frac{u}{n}\right)du\right|$$

$$+Bx^2\sum_{m\leqq\sqrt{2^\nu x}}\frac{1}{m}\left|\int\psi\left(\frac{2^\nu u}{m}\right)du\right|+Bx^{\frac{3}{2}}\sum_{m\leqq\sqrt{2^\nu x}}\frac{1}{m}\left|\int\psi\left(\frac{2^\nu u}{m}\right)du\right|$$

$$+Bx\sum_{m\leqq\sqrt{2^\nu x}}m\left|\int\psi\left(\frac{2^\nu u}{m}\right)du\right|+Bx^{\frac{3}{2}}\sum_{n\leqq\sqrt{x}}\frac{1}{n}\left|\int\psi\left(\frac{u}{n}\right)du\right|$$

$$=Bx^2\sum_{n\leqq\sqrt{x}}1+Bx\sum_{n\leqq\sqrt{x}}n^2+Bx^2\sum_{m\leqq\sqrt{2^\nu x}}1+Bx^{\frac{3}{2}}\sum_{m\leqq\sqrt{2^\nu x}}1$$

$$+Bx\sum_{m\leqq\sqrt{2^\nu x}}m^2+Bx^{\frac{3}{2}}\sum_{n\leqq\sqrt{x}}1=Bx^{\frac{5}{2}}.$$

Hilfssatz 5: *Sind in dem Intervall $u_1\leqq u\leqq u_2$ die quadratisch integrierbaren Funktionen $g(u)$, $h(u)$, $s_k(u)$, $t_k(u)$ $(k\geqq 1)$ gegeben, für die bei wachsendem k*

$$\int\limits_{u_1}^{u_2}s_k^2(u)\,du\,,\ \int\limits_{u_1}^{u_2}t_k^2(u)\,du\quad\textit{beschränkt},$$

$$\int\limits_{u_1}^{u_2}(g(u)-s_k(u))^2\,du\to 0\,,\ \int\limits_{u_1}^{u_2}(h(u)-t_k(u))^2\,du\to 0\,,$$

so ist

$$\int\limits_{u_1}^{u_2}g(u)\,h(u)\,du=\lim_{k=\infty}\int\limits_{u_1}^{u_2}s_k(u)\,t_k(u)\,du. \tag{15}$$

Beweis: Das ist Hilfssatz 10 von T_{II}.

Hilfssatz 6:

$$\sum_{m\leqq\sqrt{2^\nu x}}\frac{1}{m}\int\limits_{\mathrm{Max}\left(\frac{m^2}{2^\nu};1\right)}^{x}\psi\left(\frac{2^\nu u}{m}\right)\psi_2(\sqrt{u})\,u^{\frac{3}{2}}\,du \tag{16}$$

$$=-\frac{1}{2\pi^3}\lim_{k=\infty}\sum_{m\leqq\sqrt{2^\nu x}}\frac{1}{m}\sum_{a,\,b=1}^{k}\frac{1}{ab^2}\left(\int\limits_{\mathrm{Max}\left(\frac{m^2}{2^\nu};1\right)}^{x}u^{\frac{3}{2}}\sin\left\{2\pi\left(\frac{2^\nu a u}{m}+b\sqrt{u}\right)\right\}du\right.$$

$$\left.+\int\limits_{\mathrm{Max}\left(\frac{m^2}{2^\nu};1\right)}^{x}u^{\frac{3}{2}}\sin\left\{2\pi\left(\frac{2^\nu a u}{m}-b\sqrt{u}\right)\right\}du\right),$$

$$\sum_{n\leqq\sqrt{x}}\frac{1}{n}\int\limits_{n^2}^{x}\psi\left(\frac{u}{n}\right)\psi_2(\sqrt{2^\nu u})\,u^{\frac{3}{2}}\,du \tag{17}$$

$$=-\frac{1}{2\pi^3}\lim_{k=\infty}\sum_{n\leqq\sqrt{x}}\frac{1}{n}\sum_{a,\,b=1}^{k}\frac{1}{ab^2}\left(\int\limits_{n^2}^{x}u^{\frac{3}{2}}\sin\left\{2\pi\left(\frac{au}{n}+b\sqrt{2^\nu u}\right)\right\}du\right.$$

$$\left.+\int\limits_{n^2}^{x}u^{\frac{3}{2}}\sin\left\{2\pi\left(\frac{au}{n}-b\sqrt{2^\nu u}\right)\right\}du\right).$$

Beweis: 1) Mit $u_1=\mathrm{Max}\left(\frac{m^2}{2^\nu};1\right)$, $u_2=x$, $g(u)=\psi\left(\frac{2^\nu u}{m}\right)$,

$h(u)=\psi_2(\sqrt{u})\,u^{\frac{3}{2}}$,

$$(18)\qquad s_k(u) = -\frac{1}{\pi}\sum_{a=1}^{k}\frac{1}{a}\sin\frac{2\pi a 2^{\nu} u}{m},\quad t_k(u) = \frac{u^{\frac{3}{2}}}{\pi^2}\sum_{b=1}^{k}\frac{1}{b^2}\cos\{2\pi b\sqrt{u}\}$$

sind wegen (2) die Bedingungen von Hilfssatz 5 erfüllt. Daher liefert (15) für die linke Seite von (16)

$$-\frac{1}{\pi^3}\lim_{k=\infty}\sum_{m\leqq\sqrt{2^{\nu}x}}\frac{1}{m}\sum_{a,\,b=1}^{k}\frac{1}{a b^2}\int\limits_{\mathrm{Max}\left(\frac{m^2}{2^{\nu}};\,1\right)}^{x} u^{\frac{3}{2}}\sin\frac{2\pi a 2^{\nu} u}{m}\cos\{2\pi b\sqrt{u}\}\,du,$$

was der rechten Seite von (16) gleich ist.

2) Mit $u_1 = n^2$, $u_2 = x$, $g(u) = \phi\left(\frac{u}{n}\right)$, $h(u) = \phi_2(\sqrt{2^{\nu} u})\, u^{\frac{3}{2}}$,

$$(19)\qquad s_k(u) = -\frac{1}{\pi}\sum_{a=1}^{k}\frac{1}{a}\sin\frac{2\pi a u}{n},\quad t_k(u) = \frac{u^{\frac{3}{2}}}{\pi^2}\sum_{b=1}^{k}\frac{1}{b^2}\cos\{2\pi b\sqrt{2^{\nu} u}\}$$

sind wegen (2) die Bedingungen von Hilfssatz 4 erfüllt. Daher liefert (15) für die linke Seite von (17)

$$-\frac{1}{\pi^3}\lim_{k=\infty}\sum_{n\leqq\sqrt{x}}\frac{1}{n}\sum_{a,\,b=1}^{k}\frac{1}{a b^2}\int\limits_{n^2}^{x} u^{\frac{3}{2}}\sin\frac{2\pi a u}{n}\cos\{2\pi b\sqrt{2^{\nu} u}\}\,du,$$

was der rechten Seite von (17) gleich ist.

Hilfssatz 7: *Für* $\nu = 0, 1, 2, 3, 4$ *und* $\nu' = 1, 2, 3, 4$ *ist*

$$(20)\qquad S^{\pm}_{\nu,\nu'} = \sum_{m\leqq\sqrt{2^{\nu}x}}\frac{1}{m}\sum_{a,\,b=1}^{\infty}\frac{1}{a b^2}\left|\int\limits_{\mathrm{Max}\left(\frac{m^2}{2^{\nu}};\,1\right)}^{x} u^{\frac{3}{2}}\sin\left\{2\pi\left(\frac{2^{\nu} a u}{m}\pm b\sqrt{2^{\nu'} u}\right)\right\}du\right|$$

$$(21)\qquad = B x^{\frac{9}{4}}.$$

Beweis: Es sei zur Abkürzung (nur für den vorliegenden Beweis)

$$(22)\qquad M^2 = \mathrm{Max}\left(\frac{m^2}{2^{\nu}};\,1\right),\quad M>0;\quad \rho = \frac{2^{\nu+1}\pi a}{m},\quad \omega = \frac{2^{\frac{\nu'}{2}-\nu-1} b m}{a}.$$

$$S^{\pm}_{\nu,\nu'} = 2\sum_{m\leqq\sqrt{2^{\nu}x}}\frac{1}{m}\sum_{a,\,b=1}^{\infty}\frac{1}{a b^2}\left|\int\limits_{M}^{\sqrt{x}} u^4\sin\left\{2\pi\left(\frac{2^{\nu} a u^2}{m}\pm b 2^{\frac{\nu'}{2}} u\right)\right\}du\right| \qquad (20,\,22)$$

$$\leqq 2\sum_{m\leqq\sqrt{2^{\nu}x}}\frac{1}{m}\sum_{a,\,b=1}^{\infty}\frac{1}{a b^2}\left|\int\limits_{M}^{\sqrt{x}} u^4\exp\left\{2\pi i\left(\frac{2^{\nu} a u^2}{m}\pm b 2^{\frac{\nu'}{2}} u\right)\right\}du\right|$$

$$= 2\sum_{m\leqq\sqrt{2^{\nu}x}}\frac{1}{m}\sum_{a,\,b=1}^{\infty}\frac{1}{a b^2}\left|\int\limits_{M}^{\sqrt{x}} u^4 e^{i\rho(u\pm\omega)^2}\,du\right| \qquad (22)$$

$$(23)\qquad = 2\sum_{m\leqq\sqrt{2^{\nu}x}}\frac{1}{m}\left(\sum_{\substack{a,\,b=1\\ \omega\leqq\frac{M}{2}}}^{\infty}+\sum_{\substack{a,\,b=1\\ \frac{M}{2}<\omega<\sqrt{2x}}}^{\infty}+\sum_{\substack{a,\,b=1\\ \omega\geqq\sqrt{2x}}}^{\infty}\right)\frac{1}{a b^2}\left|\int\limits_{M}^{\sqrt{x}} u^4 e^{i\rho(u\pm\omega)^2}\,du\right|$$

$$(24)\qquad = S' + S'' + S'''.$$

$$\left(\omega\leqq\frac{M}{2}\right)\qquad \int\limits_{M}^{\sqrt{x}} u^4 e^{i\rho(u\pm\omega)^2}\,du = \frac{1}{2i\rho}\int\limits_{M}^{\sqrt{x}}\frac{u^4}{u\pm\omega}\,d\,e^{i\rho(u\pm\omega)^2}$$

$$(25)\qquad = \frac{B x^2}{\rho M} = \frac{B x^2}{\rho m} = \frac{B x^2}{a} \qquad (22).$$

$$(26)\qquad S' = B x^2\sum_{m\leqq\sqrt{2^{\nu}x}}\frac{1}{m}\sum_{a,\,b=1}^{\infty}\frac{1}{a^2 b^2} = B x^2\log x \qquad (23\text{—}25).$$

$$\left(\frac{M}{2}<\omega<\sqrt{2x}\right)\qquad \int\limits_{M}^{\sqrt{x}} u^4 e^{i\rho(u\pm\omega)^2}\,du = \int\limits_{\substack{M\\ |u\pm\omega|\leqq\sqrt[4]{x}}}^{\sqrt{x}} + \int\limits_{\substack{M\\ |u\pm\omega|\geqq\sqrt[4]{x}}}^{\sqrt{x}}$$

$$= B(\omega^4 + x^2)\left(\left|\int\cos(\rho u^2)\,du\right| + \left|\sin(\rho u^2)\,du\right|\right) + \frac{1}{2i\rho}\int\limits_{\substack{M\\ |u\pm\omega|\geqq\sqrt[4]{x}}}^{\sqrt{x}}\frac{u^4}{u\pm\omega}\,d\,e^{i\rho(u\pm\omega)^2}$$

$$(27)\qquad = B(\omega^4 + x^2)\rho^{-\frac{1}{2}} + B\rho^{-1}x^{2-\frac{1}{4}} = B x^2 m^{\frac{1}{2}} a^{-\frac{1}{2}} + B x^{\frac{7}{4}} m a^{-1} \quad (22).$$

$$S''=B\,x^2\sum_{m\leqq\sqrt{2^\nu x}} m^{-\frac{1}{2}}\sum_{a=1}^{\infty} a^{-\frac{3}{2}}\sum_{b=1}^{\infty} b^{-2}+B\,x^{\frac{7}{4}}\sum_{m\leqq\sqrt{2^\nu x}} 1\cdot\sum_{a,\,b=1}^{\infty}(a\,b)^{-2}$$

(28) $\quad =B\,x^{\frac{9}{4}}$ (23, 24, 27).

$$(\omega\geqq\sqrt{2\,x})\qquad \int_M^{\sqrt{x}} u^4\, e^{i\rho(u\pm\omega)^2}\,d\,u=\frac{1}{2\,i\,\rho}\int_M^{\sqrt{x}}\frac{u^4}{u\pm\omega}\,d\,e^{i\rho(u\pm\omega)^2}$$

(29) $\quad =B\,\rho^{-1}x^{2-\frac{1}{2}}=B\,x^{\frac{3}{2}}\,m\,a^{-1}$ (22).

(30) $\quad S'''=B\,x^{\frac{3}{2}}\sum_{m\leqq\sqrt{2^\nu x}} 1\cdot\sum_{a,\,b=1}^{\infty}(a\,b)^{-2}=B\,x^2$ (23, 24, 29).

(21) folgt aus (24), (26), (28) und (30).

Hilfssatz 8:

(31)
$$V_\nu(x)=\frac{2^\nu}{12}x^3+\sum_{E_1,\,E_2=0}^{1} c_{E_1,\,E_2}\sum_{\substack{m\leqq\sqrt{2^\nu x}\\ n\leqq\sqrt{x}}} m^{2E_1-1}\,n^{2E_2-1}\int_{r_\nu}^{x}\psi\left(\frac{2^\nu u}{m}\right)\psi\left(\frac{u}{n}\right)u^{2-E_1-E_2}\,du$$
$$+B\,x^{\frac{5}{2}}.$$

Beweis: (12), (16), (17), (21) mit $\nu'=0$, (21) mit $\nu=0$, $\nu'=\nu$, $m=n$.

Hilfssatz 9:

(32) $$\sum_{\substack{m,\,n=1\\(m,\,n)=1}}^{\infty}\frac{1}{m^2 n^2}=\frac{5}{2},$$

(33) $$\sum_{\substack{m,\,n=1\\(m,\,n)=1,\;m\equiv1\ (\mathrm{mod}\ 2)}}^{\infty}\frac{1}{m^2 n^2}=2,$$

(34) $$\sum_{\substack{m=1\\(m,\,n)=1}}^{\infty}\frac{1}{m^2}=\frac{\pi^2}{6}\sum_{s\mid n}\frac{\mu(s)}{s^2},$$

(35) $$\sum_{\substack{m=1\\(m,\,n)=1,\;m\equiv1\ (\mathrm{mod}\ 2)}}^{\infty}\frac{1}{m^2}=\begin{cases}\dfrac{\pi^2}{8}\displaystyle\sum_{s\mid n}\frac{\mu(s)}{s^2} & \text{für } n\equiv1\ (\mathrm{mod}\ 2)\\[2ex] \dfrac{\pi^2}{6}\displaystyle\sum_{s\mid n}\frac{\mu(s)}{s^2} & \text{für } n\equiv0\ (\mathrm{mod}\ 2),\end{cases}$$

(36) $$\sum_{\substack{s=1\\s\equiv1\ (\mathrm{mod}\ 2)}}^{\infty}\frac{\mu(s)}{s^3}=\frac{8}{7\,\zeta(3)},$$

(37) $$\sum_{\substack{s=1\\s\equiv0\ (\mathrm{mod}\ 2)}}^{\infty}\frac{\mu(s)}{s^3}=-\frac{1}{7\,\zeta(3)},$$

(38) $$\sum_{m=1}^{\infty}\frac{\varphi(m)}{m^3}=\frac{\pi^2}{6\,\zeta(3)},$$

(39) $$\sum_{\substack{m=1\\m\equiv1\ (\mathrm{mod}\ 2)}}^{\infty}\frac{\varphi(m)}{m^3}=\frac{\pi^2}{7\,\zeta(3)},$$

(40) $$\sum_{n=1}^{\infty}\frac{\varphi(2\,n)}{n^3}=\frac{4\,\pi^2}{21\,\zeta(3)}.$$

Beweis: (32) ist T_{II} (39), (33) ist T_{II} (115).

Alle Kongruenzen im folgenden seien mod 2 gemeint, p möge über Primzahlen laufen.

$$\sum_{\substack{m=1\\(m,\,n)=1}}^{\infty}\frac{1}{m^2}=\prod_{p\nmid n}\left(1-\frac{1}{p^2}\right)^{-1}=\prod_{p}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p\mid n}\left(1-\frac{1}{p^2}\right)=\frac{\pi^2}{6}\sum_{s\mid n}\frac{\mu(s)}{s^2},$$

$$(n\equiv1)\quad \sum_{\substack{m=1\\(m,\,n)=1,\;m\equiv1}}^{\infty}\frac{1}{m^2}=\prod_{\substack{p>2\\p\nmid n}}\left(1-\frac{1}{p^2}\right)^{-1}=\left(1-\frac{1}{2^2}\right)\prod_{p}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p\mid n}\left(1-\frac{1}{p^2}\right)$$
$$=\frac{\pi^2}{8}\sum_{s\mid n}\frac{\mu(s)}{s^2},$$

$$(n\equiv0)\quad \sum_{\substack{m=1\\(m,\,n)=1,\;m\equiv1}}^{\infty}\frac{1}{m^2}=\prod_{p\nmid n}\left(1-\frac{1}{p^2}\right)^{-1}=\prod_{p}\left(1-\frac{1}{p^2}\right)^{-1}\prod_{p\mid n}\left(1-\frac{1}{p^2}\right)$$

$$= \frac{\pi^2}{6} \sum_{s \mid n} \frac{\mu(s)}{s^2},$$

$$\sum_{s=1}^{\infty} \frac{\mu(s)}{s^3} = \prod_{p>2} \left(1 - \frac{1}{p^3}\right) = \left(1 - \frac{1}{2^3}\right)^{-1} \prod_{p} \left(1 - \frac{1}{p^3}\right) = \frac{8}{7\,\zeta(3)},$$

$$\sum_{s=1}^{\infty} \frac{\mu(s)}{s^3} = \frac{1}{\zeta(3)} - \frac{8}{7\,\zeta(3)} = -\frac{1}{7\,\zeta(3)} \qquad (36),$$

$$\sum_{m=1}^{\infty} \frac{\varphi(m)}{m^3} = \prod_{p} \left(1 - \frac{1}{p^2}\right)^{-1} \left(1 - \frac{1}{p^3}\right) = \frac{\pi^2}{8\,\zeta(3)},$$

$$\sum_{m=1}^{\infty} \frac{\varphi(m)}{m^3} = \prod_{p>2} \left(1 - \frac{1}{p^2}\right)^{-1} \left(1 - \frac{1}{p^3}\right)$$

$$= \frac{6}{7} \prod_{p} \left(1 - \frac{1}{p^2}\right)^{-1} \left(1 - \frac{1}{p^3}\right) = \frac{\pi^2}{7\,\zeta(3)},$$

$$\sum_{n=1}^{\infty} \frac{\varphi(2n)}{n^3} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^3} + 2 \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^3} = 2 \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^3} - \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^3} = \frac{4\pi^2}{21\,\zeta(3)}$$

(38, 39).

Hilfssatz 10:

$$(41) \qquad \sum_{\substack{m \leqq \sqrt{2^\nu x} \\ n \leqq \sqrt{x}}} \frac{(m, 2^\nu n)^2}{m^2 n^2} =$$

$$= \frac{5\pi^2}{12} + \frac{\nu \pi^2}{4} - \frac{\pi^2}{\zeta(3)} \left(\frac{1 + 2^{\frac{\nu}{2}}}{12} + \frac{\nu}{28} + \frac{2^{\frac{\nu}{2}} - 2^{-\frac{\nu}{2}}}{42} \right) \frac{\log x}{\sqrt{x}} + \frac{B}{\sqrt{x}}.$$

Beweis: Ich bezeichne mit $m_0, m_1, \ldots, m_\nu$ die Restklassen

$$(42) \qquad m_0 \equiv 0 \pmod{2^\nu}; \quad m_\rho \equiv 2^{\rho - 1} \pmod{2^\rho} \text{ für } 1 \leqq \rho \leqq \nu.$$

Kongruenzen, deren Modul nicht angegeben ist, sind mod 2 gemeint. ρ sei eine ganze Zahl mit $1 \leqq \rho \leqq \nu$. Dieselben Festsetzungen mögen auch beim Beweise des nächsten Hilfssatzes gelten.

$$\sum_{\substack{m \leqq \sqrt{2^\nu x} \\ n \leqq \sqrt{x}}} \frac{(m, 2^\nu n)^2}{m^2 n^2} = \sum_{r \leqq \sqrt{2^\nu x}} r^2 \sum_{\substack{m \leqq \sqrt{2^\nu x},\ n \leqq \sqrt{x} \\ (m, 2^\nu n) = r}} \frac{1}{m^2 n^2}$$

$$(43) \qquad = \sum_{\mu=0}^{\nu} \sum_{r \leqq \sqrt{2^\nu x}} r^2 \sum_{\substack{m_\mu \leqq \sqrt{2^\nu x},\ n \leqq \sqrt{x} \\ (m_\mu, 2^\nu n) = r}} \frac{1}{m_\mu^2 n^2} = \sum_{\mu=0}^{\nu} S_\mu = S_0 + \sum_{\rho=1}^{\nu} S_\rho .$$

$$S_0 = \sum_{\substack{r \leqq \sqrt{2^\nu x} \\ r \equiv 0 \pmod{2^\nu}}} r^2 \sum_{\substack{m \leqq \sqrt{2^\nu x},\ n \leqq \sqrt{x} \\ (m, 2^\nu n) = r,\ m \equiv 0 \pmod{2^\nu}}} \frac{1}{m^2 n^2} \qquad (42, 43)$$

$$= \sum_{r \leqq \sqrt{x/2^\nu}} 2^{2\nu} r^2 \sum_{\substack{m \leqq \sqrt{x/2^\nu},\ n \leqq \sqrt{x} \\ (m, n) = r}} \frac{1}{2^{2\nu} m^2 n^2}$$

(ich habe m, n, r durch $2^\nu m, n, 2^\nu r$ ersetzt)

$$= \sum_{r \leqq \sqrt{x/2^\nu}} r^2 \sum_{\substack{m \leqq \frac{1}{r}\sqrt{x/2^\nu},\ n \leqq \frac{\sqrt{x}}{r} \\ (m, n) = 1}} \frac{1}{r^4 m^2 n^2}$$

(ich habe m, n, r durch rm, rn, r ersetzt)

$$= \sum_{\substack{m \leqq \sqrt{x/2^\nu},\ n \leqq \sqrt{x} \\ (m, n) = 1}} \frac{1}{m^2 n^2} \sum_{r \leqq \mathrm{Min}\left(\frac{1}{m}\sqrt{x/2^\nu};\ \frac{\sqrt{x}}{n}\right)} \frac{1}{r^2}$$

$$(44) \qquad = \sum_{\substack{m \leqq \sqrt{x/2^\nu},\ n \leqq \sqrt{x} \\ (m, n) = 1}} \frac{1}{m^2 n^2} \sum_{r=1}^{\infty} \frac{1}{r^2} - \sum_{\substack{m \leqq \sqrt{x/2^\nu},\ n \leqq \sqrt{x} \\ (m, n) = 1}} \frac{1}{m^2 n^2} \sum_{r > \sqrt{x}\, \mathrm{Min}\left(\frac{1}{m\sqrt{2^\nu}};\ \frac{1}{n}\right)} \frac{1}{r^2}$$

$$(45) \qquad = S_{01} - S_{02} .$$

$$(46) \qquad S_{01} = \frac{\pi^2}{6} \sum_{\substack{m, n = 1 \\ (m, n) = 1}}^{\infty} \frac{1}{m^2 n^2} + B \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{m > \sqrt{x/2^\nu}} \frac{1}{m^2} = \frac{5\pi^2}{12} + \frac{B}{\sqrt{x}} \qquad (44, 45, 32).$$

$$S_{02} = \sum_{\substack{m \leqq \sqrt{x/2^\nu},\ n \leqq \sqrt{x} \\ (m, n) = 1}} \frac{1}{m^2 n^2} \frac{\mathrm{Max}\,(m\sqrt{2^\nu};\ n)}{\sqrt{x}} + B \sum_{m,\ n \leqq \sqrt{x}} \frac{1}{m^2 n^2} \frac{\mathrm{Max}\,(m^2;\ n^2)}{x}$$

(44, 45

$$=\frac{1}{\sqrt{x}}\sum_{\substack{m\sqrt{2^\nu}\leqq n\leqq\sqrt{x}\\(m,\,n=1)}}\sum\frac{1}{m^2 n}+\sqrt{\frac{2^\nu}{x}}\sum_{\substack{n\leqq m\sqrt{2^\nu}\leqq\sqrt{x}\\(m,\,n)=1}}\sum\frac{1}{m\,n^2}$$

$$+\frac{B}{\sqrt{x}}\sum_{\substack{m\sqrt{2^\nu}=n\leqq\sqrt{x}\\(m,\,n)=1}}\sum\frac{1}{m^3}+\frac{B}{x}\sum_{m=1}^{\infty}\frac{1}{m^2}\sum_{n\leqq\sqrt{x}}1$$

$$=\frac{1}{\sqrt{x}}\sum_{n\leqq\sqrt{x}}\frac{1}{n}\sum_{\substack{m\leqq\frac{n}{\sqrt{2^\nu}}\\(m,\,n)=1}}\frac{1}{m^2}+\sqrt{\frac{2^\nu}{x}}\sum_{n\leqq\sqrt{x/2^\nu}}\frac{1}{n}\sum_{\substack{m\leqq n\sqrt{2^\nu}\\(m,\,n)=1}}\frac{1}{m^2}+\frac{B}{\sqrt{x}}$$

(im zweiten Glied habe ich m, n durch n, m ersetzt)

$$=\frac{1}{\sqrt{x}}\sum_{n\leqq\sqrt{x}}\frac{1}{n}\left(\sum_{\substack{m=1\\(m,\,n)=1}}^{\infty}\frac{1}{m^2}+\frac{B}{n}\right)+\sqrt{\frac{2^\nu}{x}}\sum_{n\leqq\sqrt{x/2^\nu}}\frac{1}{n}\left(\sum_{\substack{m=1\\(m,\,n)=1}}^{\infty}\frac{1}{m^2}+\frac{B}{n}\right)+\frac{B}{\sqrt{x}}$$

$$=\frac{\pi^2}{6\sqrt{x}}\sum_{n\leqq\sqrt{x}}\frac{1}{n}\sum_{s\mid n}\frac{\mu(s)}{s^2}+\frac{\pi^2}{6}\sqrt{\frac{2^\nu}{x}}\sum_{n\leqq\sqrt{x/2^\nu}}\frac{1}{n}\sum_{s\mid n}\frac{\mu(s)}{s^2}+\frac{B}{\sqrt{x}} \qquad (34)$$

$$=\frac{\pi^2}{6\sqrt{x}}\sum_{sn\leqq\sqrt{x}}\frac{1}{n}\frac{\mu(s)}{s^3}+\frac{\pi^2}{6}\sqrt{\frac{2^\nu}{x}}\sum_{sn\leqq\sqrt{x/2^\nu}}\frac{1}{n}\frac{\mu(s)}{s^3}+\frac{B}{\sqrt{x}}$$

(ich habe n, s durch sn, s ersetzt)

$$=\frac{\pi^2}{6\sqrt{x}}\sum_{s\leqq\sqrt{x}}\frac{\mu(s)}{s^3}\sum_{n\leqq\sqrt{x/s}}\frac{1}{n}+\frac{\pi^2}{6}\sqrt{\frac{2^\nu}{x}}\sum_{s\leqq\sqrt{x/2^\nu}}\frac{\mu(s)}{s^3}\sum_{n\leqq\frac{1}{s}\sqrt{x/2^\nu}}\frac{1}{n}+\frac{B}{\sqrt{x}}$$

$$=\frac{\pi^2}{6\sqrt{x}}\sum_{s\leqq\sqrt{x}}\frac{\mu(s)}{s^3}\left(\frac{1}{2}\log x+B\log 2s\right)+$$

$$+\frac{\pi^2}{6}\sqrt{\frac{2^\nu}{x}}\sum_{s\leqq\sqrt{x/2^\nu}}\frac{\mu(s)}{s^3}\left(\frac{1}{2}\log x+B\log 2s\right)+\frac{B}{\sqrt{x}}$$

$$=\frac{\pi^2}{12}\frac{\log x}{\sqrt{x}}\left(\sum_{s\leqq\sqrt{x}}\frac{\mu(s)}{s^3}+\sqrt{2^\nu}\sum_{s\leqq\sqrt{x/2^\nu}}\frac{\mu(s)}{s^3}\right)+\frac{B}{\sqrt{x}}$$

$$=\frac{\pi^2}{12}\frac{\log x}{\sqrt{x}}\left(1+\sqrt{2^\nu}\right)\left(\sum_{s=1}^{\infty}\frac{\mu(s)}{s^3}+\frac{B}{x}\right)+\frac{B}{\sqrt{x}}$$

$$(47) \qquad =\frac{\pi^2}{12\,\zeta(3)}\left(1+2^{\frac{\nu}{2}}\right)\frac{\log x}{\sqrt{x}}+\frac{B}{\sqrt{x}}.$$

$$(48) \qquad S_0=\frac{5\,\pi^2}{12}-\frac{\pi^2}{12\,\zeta(3)}\left(1+2^{\frac{\nu}{2}}\right)\frac{\log x}{\sqrt{x}}+\frac{B}{\sqrt{x}} \qquad (45\text{—}47).$$

$$S_\rho=\sum_{r\leqq\sqrt{2^\nu x}}r^2\sum_{\substack{m_\rho\leqq\sqrt{2^\nu x},\ n\leqq\sqrt{x}\\(m_\rho,\ 2^\nu n)=r}}\frac{1}{m_\rho^2\,n^2} \qquad (43)$$

$$=\sum_{\substack{r\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x}\\ r\equiv 1}}r^2\sum_{\substack{m\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x},\ n\leqq\sqrt{x}\\(m,\,n)=r,\ m\equiv 1}}\frac{1}{m^2\,n^2} \qquad (42)$$

(ich habe m_ρ, n, r durch $2^{\rho-1}m$, n, $2^{\rho-1}r$ ersetzt)

$$=\sum_{\substack{r\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x}\\ r\equiv 1}}\frac{1}{r^2}\sum_{\substack{m\leqq 2^{\frac{\nu}{2}-\rho+1}\frac{\sqrt{x}}{r},\ n\leqq\frac{\sqrt{x}}{r}\\(m,\,n)=1,\ m\equiv 1}}\frac{1}{m^2\,n^2}$$

$$=\sum_{\substack{m\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x},\ n\leqq\sqrt{x}\\(m,\,n)=1,\ m\equiv 1}}\frac{1}{m^2\,n^2}\sum_{\substack{r\leqq\mathrm{Min}\left(2^{\frac{\nu}{2}-\rho+1}\frac{\sqrt{x}}{m};\ \frac{\sqrt{x}}{n}\right)\\ r\equiv 1}}\frac{1}{r^2}$$

$$(49) \qquad =\sum_{\substack{m\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x},\ n\leqq\sqrt{x}\\(m,\,n)=1,\ m\equiv 1}}\frac{1}{m^2\,n^2}\sum_{\substack{r=1\\ r\equiv 1}}^{\infty}\frac{1}{r^2}$$

$$-\sum_{\substack{m\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x},\ n\leqq\sqrt{x}\\(m,\,n)=1,\ m\equiv 1}}\frac{1}{m^2\,n^2}\sum_{\substack{r>\sqrt{x}\,\mathrm{Min}\left(\frac{2^{\frac{\nu}{2}-\rho+1}}{m};\ \frac{1}{n}\right)\\ r\equiv 1}}\frac{1}{r^2}$$

$$(50)\qquad = S_{\rho 1} - S_{\rho 2}.$$

$$S_{\rho 1} = \frac{\pi^2}{8} \sum_{\substack{m \leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x},\ n \leqq \sqrt{x} \\ (m,\, n)=1,\ m\equiv 1}} \frac{1}{m^2 n^2} \qquad (49,\ 50)$$

$$(51)\qquad = \frac{\pi^2}{8} \sum_{\substack{m,\, n=1 \\ (m,\, n)=1,\ m\equiv 1}}^{\infty} \frac{1}{m^2 n^2} + \frac{B}{\sqrt{x}} = \frac{\pi^2}{4} + \frac{B}{\sqrt{x}} \qquad (33).$$

$$S_{\rho 2} = \sum_{\substack{m \leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x},\ n \leqq \sqrt{x} \\ (m,\, n)=1,\ m\equiv 1}} \frac{1}{m^2 n^2}\, \frac{\mathrm{Max}\left(2^{\rho-\frac{\nu}{2}-1} m;\ n\right)}{2\sqrt{x}}$$

$$+ \frac{B}{x} \sum_{m,\, n \leqq \sqrt{2^{\nu} x}} \frac{1}{m^2 n^2}\, \mathrm{Max}\,(m^2;\ n^2) \qquad (49,\ 50)$$

$$= \frac{1}{2\sqrt{x}} \sum\sum_{\substack{2^{\rho-\frac{\nu}{2}-1} m \leqq n \leqq \sqrt{x} \\ (m,\, n)=1,\ m\equiv 1}} \frac{1}{m^2 n} + \frac{1}{2\sqrt{x}} \sum\sum_{\substack{n \leqq 2^{\rho-\frac{\nu}{2}-1} m \leqq \sqrt{x} \\ (m,\, n)=1,\ m\equiv 1}} \frac{2^{\rho-\frac{\nu}{2}-1}}{m n^2} + \frac{B}{\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} \sum_{n \leqq \sqrt{x}} \frac{1}{n} \sum_{\substack{m \leqq 2^{\frac{\nu}{2}-\rho+1} n \\ (m,\, n)=1,\ m\equiv 1}} \frac{1}{m^2} + \frac{2^{\rho-\frac{\nu}{2}-2}}{\sqrt{x}} \sum_{\substack{n \leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x} \\ m\equiv 1}} \frac{1}{n} \sum_{\substack{m \leqq 2^{\rho-\frac{\nu}{2}-1} n \\ (m,\, n)=1}} \frac{1}{m^2}$$

$$+ \frac{B}{\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}} \sum_{n \leqq \sqrt{x}} \frac{1}{n} \sum_{\substack{m=1 \\ (m,\, n)=1,\ m\equiv 1}}^{\infty} \frac{1}{m^2} + \frac{2^{\rho-\frac{\nu}{2}-2}}{\sqrt{x}} \sum_{\substack{n \leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x} \\ n\equiv 1}} \frac{1}{n} \sum_{\substack{m=1 \\ (m,\, n)=1}}^{\infty} \frac{1}{m^2} + \frac{B}{\sqrt{x}}$$

$$= \frac{\pi^2}{12\sqrt{x}} \sum_{\substack{n \leqq \sqrt{x} \\ n\equiv 0}} \frac{1}{n} \sum_{s \mid n} \frac{\mu(s)}{s^2} + \frac{\pi^2}{16\sqrt{x}} \sum_{\substack{n \leqq \sqrt{x} \\ n\equiv 1}} \frac{1}{n} \sum_{s \mid n} \frac{\mu(s)}{s^2}$$

$$+ \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{24\sqrt{x}} \sum_{\substack{n \leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x} \\ n\equiv 1}} \frac{1}{n} \sum_{s \mid n} \frac{\mu(s)}{s^2} + \frac{B}{\sqrt{x}} \qquad (35,\ 34)$$

$$= \frac{\pi^2}{12\sqrt{x}} \sum_{\substack{sn \leqq \sqrt{x} \\ sn\equiv 0}} \frac{1}{n} \frac{\mu(s)}{s^3} + \frac{\pi^2}{16\sqrt{x}} \sum_{\substack{sn \leqq \sqrt{x} \\ sn\equiv 1}} \frac{1}{n} \frac{\mu(s)}{s^3}$$

$$+ \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{24\sqrt{x}} \sum_{\substack{sn \leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x} \\ sn\equiv 1}} \frac{1}{n} \frac{\mu(s)}{s^3} + \frac{B}{\sqrt{x}}$$

$$= \frac{\pi^2}{12\sqrt{x}} \sum_{\substack{s \leqq \sqrt{x} \\ s\equiv 1}} \frac{\mu(s)}{s^3} \sum_{\substack{n \leqq \frac{\sqrt{x}}{s} \\ n\equiv 0}} \frac{1}{n} + \frac{\pi^2}{12\sqrt{x}} \sum_{\substack{s \leqq \sqrt{x} \\ s\equiv 0}} \frac{\mu(s)}{s^3} \sum_{n \leqq \frac{\sqrt{x}}{s}} \frac{1}{n}$$

$$+ \frac{\pi^2}{16\sqrt{x}} \sum_{\substack{s \leqq \sqrt{x} \\ s\equiv 1}} \frac{\mu(s)}{s^3} \sum_{\substack{n \leqq \frac{\sqrt{x}}{s} \\ n\equiv 1}} \frac{1}{n} + \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{24\sqrt{x}} \sum_{\substack{s \leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x} \\ s\equiv 1}} \frac{\mu(s)}{s^3} \sum_{\substack{n \leqq 2^{\frac{\nu}{2}-\rho+1}\frac{\sqrt{x}}{s} \\ n\equiv 1}} \frac{1}{n}$$

$$+ \frac{B}{\sqrt{x}}$$

$$= \frac{\pi^2}{12\sqrt{x}} \sum_{\substack{s \leqq \sqrt{x} \\ s\equiv 1}} \frac{\mu(s)}{s^3} \left(\frac{1}{2}\log\sqrt{x} + B\log 2s\right)$$

$$+ \frac{\pi^2}{12\sqrt{x}} \sum_{\substack{s \leqq \sqrt{x} \\ s\equiv 0}} \frac{\mu(s)}{s^3} \left(\log\sqrt{x} + B\log 2s\right)$$

$$+ \frac{\pi^2}{16\sqrt{x}} \sum_{\substack{s \leqq \sqrt{x} \\ s\equiv 1}} \frac{\mu(s)}{s^3} \left(\frac{1}{2}\log\sqrt{x} + B\log 2s\right)$$

$$+ \frac{2^{\rho-\frac{\nu}{2}} \pi^2}{24\sqrt{x}} \sum_{\substack{s \leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x} \\ s\equiv 1}} \frac{\mu(s)}{s^3} \left(\frac{1}{2}\log\sqrt{x} + B\log 2s\right) + \frac{B}{\sqrt{x}}$$

$$=\left(\frac{\pi^2}{48}\sum_{\substack{s=1\\ s\equiv1}}^{\infty}\frac{\mu(s)}{s^3}+\frac{\pi^2}{24}\sum_{\substack{s=1\\ s\equiv0}}^{\infty}\frac{\mu(s)}{s^3}+\frac{\pi^2}{64}\sum_{\substack{s=1\\ s\equiv1}}^{\infty}\frac{\mu(s)}{s^3}+\right.$$

$$\left.+\frac{2^{\rho-\frac{\nu}{2}}\pi^2}{96}\sum_{\substack{s=1\\ s\equiv1}}^{\infty}\frac{\mu(s)}{s^3}\right)\frac{\log x}{\sqrt{x}}+\frac{B}{\sqrt{x}}$$

$$(52)\qquad =\left(\frac{\pi^2}{28\,\zeta(3)}+\frac{2^{\rho-\frac{\nu}{2}}\pi^2}{84\,\zeta(3)}\right)\frac{\log x}{\sqrt{x}}+\frac{B}{\sqrt{x}}\qquad (36,\ 37).$$

$$S_\rho=\frac{\pi^2}{4}-\left(\frac{\pi^2}{28\,\zeta(3)}+\frac{2^{\rho-\frac{\nu}{2}}\pi^2}{84\,\zeta(3)}\right)\frac{\log x}{\sqrt{x}}+\frac{B}{\sqrt{x}}\qquad (50-52),$$

$$\sum_{\rho=1}^{\nu}S_\rho=\frac{\nu\pi^2}{4}-\left(\frac{\nu\pi^2}{28\,\zeta(3)}+\frac{2^{-\frac{\nu}{2}}\pi^2}{84\,\zeta(3)}(2^{\nu+1}-2)\right)\frac{\log x}{\sqrt{x}}+\frac{B}{\sqrt{x}}$$

$$(53)\qquad =\frac{\nu\pi^2}{4}-\frac{\pi^2}{\zeta(3)}\left(\frac{\nu}{28}+\frac{2^{\frac{\nu}{2}+1}-2^{1-\frac{\nu}{2}}}{84}\right)\frac{\log x}{\sqrt{x}}+\frac{B}{\sqrt{x}}.$$

(41) folgt aus (43), (48) und (53).

Hilfssatz 11:

$$(54)\qquad \sum_{\substack{m\leq\sqrt{2^\nu x}\\ n\leq\sqrt{x}}}\frac{(m,2^\nu n)^2}{m^2}=\left(\frac{\pi^2}{12\,\zeta(3)}+\frac{\nu\pi^2}{28\,\zeta(3)}\right)\sqrt{x}\log x+B\sqrt{x},$$

$$(55)\qquad \sum_{\substack{m\leq\sqrt{2^\nu x}\\ n\leq\sqrt{x}}}\frac{(m,2^\nu n)^2}{n^2}=\left(\frac{2^{\frac{3}{2}\nu}\cdot 3\pi^2}{28\,\zeta(3)}-\frac{2^{\frac{\nu}{2}}\pi^2}{42\,\zeta(3)}\right)\sqrt{x}\log x+B\sqrt{x}.$$

Beweis:

$$\sum_{\substack{m\leq\sqrt{2^\nu x}\\ n\leq\sqrt{x}}}\frac{(m,2^\nu n)^2}{m^2}=\sum_{r\leq\sqrt{2^\nu x}}r^2\sum_{\substack{m\leq\sqrt{2^\nu x},\ n\leq\sqrt{x}\\ (m,2^\nu n)=r}}\frac{1}{m^2}$$

$$(56)\qquad =\sum_{\mu=0}^{\nu}\sum_{r\leq\sqrt{2^\nu x}}r^2\sum_{\substack{m_\mu\leq\sqrt{2^\nu x},\ n\leq\sqrt{x}\\ (m_\mu,2^\nu n)=r}}\frac{1}{m_\mu{}^2}=\sum_{\mu=0}^{\nu}S'_\mu\qquad (42).$$

$$S_0'=\sum_{\substack{r\leq\sqrt{2^\nu x}\\ r\equiv0\ (\mathrm{mod}\ 2^\nu)}}r^2\sum_{\substack{m\leq\sqrt{2^\nu x},\ n\leq\sqrt{x}\\ (m,2^\nu n)=r,\ m\equiv0\ (\mathrm{mod}\ 2^\nu)}}\frac{1}{m^2}\qquad (56,\ 42)$$

$$=\sum_{r\leq\sqrt{x/2^\nu}}2^{2\nu}r^2\sum_{\substack{m\leq\sqrt{x/2^\nu},\ n\leq\sqrt{x}\\ (m,n)=r}}\frac{1}{2^{2\nu}m^2}=\sum_{r\leq\sqrt{x/2^\nu}}\ \sum_{\substack{m\leq\frac{1}{r}\sqrt{x/2^\nu},\ n\leq\frac{\sqrt{x}}{r}\\ (m,n)=r}}\frac{1}{m^2}$$

$$=\sum_{r\leq\sqrt{x/2^\nu}}\ \sum_{m\leq\frac{1}{r}\sqrt{x/2^\nu}}\frac{1}{m^2}\sum_{\substack{n\leq\frac{\sqrt{x}}{r}\\ (n,m)=1}}1$$

$$=\sum_{r\leq\sqrt{x/2^\nu}}\ \sum_{m\leq\frac{1}{r}\sqrt{x/2^\nu}}\frac{1}{m^2}\frac{\sqrt{x}}{r\,m}\varphi(m)+B\sum_{r\leq\sqrt{x}}\ \sum_{m\leq\frac{\sqrt{x}}{r}}\frac{1}{m^2}m$$

(jedes Intervall der Länge m enthält $\varphi(m)$ zu m teilerfremde n)

$$=\sqrt{x}\sum_{r\leq\sqrt{x/2^\nu}}\frac{1}{r}\sum_{m=1}^{\infty}\frac{\varphi(m)}{m^3}+B\sqrt{x}\sum_{r\leq\sqrt{x}}\frac{1}{r}\sum_{m>\frac{1}{r}\sqrt{x/2^\nu}}\frac{\varphi(m)}{m^3}$$

$$+B\sum_{m\leq\sqrt{x}}\frac{1}{m}\sum_{r\leq\frac{\sqrt{x}}{m}}1$$

$$=\frac{\pi^2}{6\,\zeta(3)}\sqrt{x}\sum_{r\leq\sqrt{x/2^\nu}}\frac{1}{r}+B\sqrt{x}\sum_{r\leq\sqrt{x}}\frac{1}{r}\frac{r}{\sqrt{x}}+B\sqrt{x}\qquad (38)$$

$$(57)\qquad =\frac{\pi^2}{12\,\zeta(3)}\sqrt{x}\log x+B\sqrt{x}.$$

$$S_\rho'=\sum_{\substack{r\leq2^{\frac{\nu}{2}-\rho+1}\sqrt{x}\\ r\equiv1}}r^2\sum_{\substack{m\leq2^{\frac{\nu}{2}-\rho+1}\sqrt{x},\ n\leq\sqrt{x}\\ (m,n)=r,\ m\equiv1}}\frac{1}{m^2}\qquad (56,\ 42)$$

$$=\sum_{\substack{r\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x}\\ r\equiv 1}}\ \sum_{\substack{m\leqq 2^{\frac{\nu}{2}-\rho+1}\frac{\sqrt{x}}{r},\ n\leqq\frac{\sqrt{x}}{r}\\ (m,n)=1,\,m\equiv 1}}\frac{1}{m^2}$$

$$=\sum_{\substack{r\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x}\\ r\equiv 1}}\ \sum_{\substack{m\leqq 2^{\frac{\nu}{2}-\rho+1}\frac{\sqrt{x}}{r}\\ m\equiv 1}}\frac{1}{m^2}\sum_{\substack{n\leqq\frac{\sqrt{x}}{r}\\ (n,m)=1}}1$$

$$=\sum_{\substack{r\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x}\\ r\equiv 1}}\ \sum_{\substack{m\leqq 2^{\frac{\nu}{2}-\rho+1}\frac{\sqrt{x}}{r}\\ m\equiv 1}}\frac{1}{m^2}\left(\frac{\sqrt{x}}{r\,m}\varphi(m)+B\,m\right)$$

$$=\sqrt{x}\sum_{\substack{r\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x}\\ r\equiv 1}}\frac{1}{r}\sum_{\substack{m=1\\ m\equiv 1}}^{\infty}\frac{\varphi(m)}{m^3}+B\sqrt{x}\sum_{r\leqq\sqrt{2^\nu x}}\frac{1}{r}\frac{r}{\sqrt{x}}$$

$$+B\sum_{r\leqq\sqrt{2^\nu x}}\ \sum_{m\leqq\frac{\sqrt{2^\nu x}}{r}}\frac{1}{m}$$

$$=\frac{\pi^2}{7\,\zeta(3)}\sqrt{x}\sum_{\substack{r\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x}\\ r\equiv 1}}\frac{1}{r}+B\sqrt{x}\qquad (39)$$

$$(58)\qquad =\frac{\pi^2}{28\,\zeta(3)}\sqrt{x}\log x+B\sqrt{x}.$$

(54) folgt aus (56) — (58).

$$\sum_{\substack{m\leqq\sqrt{2^\nu x}\\ n\leqq\sqrt{x}}}\frac{(m,2^\nu n)^2}{n^2}=\sum_{r\leqq\sqrt{2^\nu x}}r^2\sum_{\substack{m\leqq\sqrt{2^\nu x},\ n\leqq\sqrt{x}\\ (m,2^\nu n)=r}}\frac{1}{n^2}$$

$$(59)\qquad =\sum_{\mu=0}^{\nu}\sum_{r\leqq\sqrt{2^\nu x}}r^2\sum_{\substack{m_\mu\leqq\sqrt{2^\nu x},\ n\leqq\sqrt{x}\\ (m_\mu,2^\nu n)=r}}\frac{1}{n^2}=\sum_{\mu=0}^{\nu}S_\mu''\qquad (42).$$

$$S_0''=\sum_{\substack{r\leqq\sqrt{2^\nu x}\\ r\equiv 0(\mathrm{mod}\,2^\nu)}}r^2\sum_{\substack{m\leqq\sqrt{2^\nu x},\ n\leqq\sqrt{x}\\ (m,2^\nu n)=r,\,m\equiv 0(\mathrm{mod}\,2^\nu)}}\frac{1}{n^2}\qquad (59,\ 42)$$

$$=\sum_{r\leqq\sqrt{x/2^\nu}}2^{2\nu}r^2\sum_{\substack{m\leqq\sqrt{x/2^\nu},\ n\leqq\sqrt{x}\\ (m,n)=r}}\frac{1}{n^2}=2^{2\nu}\sum_{r\leqq\sqrt{x/2^\nu}}\ \sum_{\substack{m\leqq\frac{1}{r}\sqrt{x/2^\nu},\ n\leqq\frac{\sqrt{x}}{r}\\ (m,n)=1}}\frac{1}{n^2}$$

$$=2^{2\nu}\sum_{r\leqq\sqrt{x/2^\nu}}\ \sum_{n\leqq\frac{\sqrt{x}}{r}}\frac{1}{n^2}\sum_{\substack{m\leqq\frac{1}{r}\sqrt{x/2^\nu}\\ (m,n)=1}}1$$

$$=2^{2\nu}\sum_{r\leqq\sqrt{x/2^\nu}}\ \sum_{n\leqq\frac{\sqrt{x}}{r}}\frac{1}{n^2}\left(\frac{1}{r\,n}\sqrt{\frac{x}{2^\nu}}\,\varphi(n)+B\,n\right)$$

$$=2^{\frac{3}{2}\nu}\sqrt{x}\sum_{r\leqq\sqrt{x/2^\nu}}\frac{1}{r}\sum_{n=1}^{\infty}\frac{\varphi(n)}{n^3}+B\sqrt{x}\sum_{r\leqq\sqrt{x}}\frac{1}{r}\frac{r}{\sqrt{x}}+B\sqrt{x}$$

$$=\frac{2^{\frac{3}{2}\nu}\pi^2}{6\,\zeta(3)}\sqrt{x}\sum_{r\leqq\sqrt{x/2^\nu}}\frac{1}{r}+B\sqrt{x}\qquad (38)$$

$$(60)\qquad =\frac{2^{\frac{3}{2}\nu}\pi^2}{12\,\zeta(3)}\sqrt{x}\log x+B\sqrt{x}.$$

$$S_\rho''=\sum_{\substack{r\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x}\\ r\equiv 1}}2^{2\rho-2}r^2\sum_{\substack{m\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x},\ n\leqq\sqrt{x}\\ (m,n)=r,\,m\equiv 1}}\frac{1}{n^2}\qquad (59,\ 42)$$

$$=2^{2\rho-2}\sum_{\substack{r\leqq 2^{\frac{\nu}{2}-\rho+1}\sqrt{x}\\ r\equiv 1}}\ \sum_{\substack{m\leqq 2^{\frac{\nu}{2}-\rho+1}\frac{\sqrt{x}}{r},\ n\leqq\frac{\sqrt{x}}{r}\\ (m,n)=1,\,m\equiv 1}}\frac{1}{n^2}$$

$$= 2^{2\rho-2} \sum_{\substack{r \le 2^{\frac{\nu}{2}-\rho+1}\sqrt{x} \\ r \equiv 1}} \sum_{n \le \frac{\sqrt{x}}{r}} \frac{1}{n^2} \sum_{\substack{m \le 2^{\frac{\nu}{2}-\rho+1}\frac{\sqrt{x}}{r} \\ (m,2n)=1}} 1$$

$$= 2^{2\rho-2} \sum_{\substack{r \le 2^{\frac{\nu}{2}-\rho+1}\sqrt{x} \\ r \equiv 1}} \sum_{n \le \frac{\sqrt{x}}{r}} \frac{1}{n^2}\left(2^{\frac{\nu}{2}-\rho+1}\frac{\sqrt{x}}{2rn}\varphi(2n) + Bn\right)$$

$$= 2^{\rho+\frac{\nu}{2}-2}\sqrt{x}\sum_{\substack{r \le 2^{\frac{\nu}{2}-\rho+1}\sqrt{x} \\ r \equiv 1}} \frac{1}{r}\sum_{n=1}^{\infty} \frac{\varphi(2n)}{n^3} + B\sqrt{x}$$

$$= \frac{2^{\rho+\frac{\nu}{2}}\pi^2}{21\,\zeta(3)}\sqrt{x}\sum_{\substack{r \le 2^{\frac{\nu}{2}-\rho+1}\sqrt{x} \\ r \equiv 1}} \frac{1}{r} + B\sqrt{x} \qquad (40)$$

(61) $$= \frac{2^{\rho+\frac{\nu}{2}}\pi^2}{84\,\zeta(3)}\sqrt{x}\log x + B\sqrt{x}.$$

$$\sum_{\substack{m \le \sqrt{2^\nu x} \\ n \le \sqrt{x}}} \frac{(m, 2^\nu n)^2}{n^2} = \left(\frac{2^{\frac{3}{2}\nu}\pi^2}{12\,\zeta(3)} + \frac{2^{\frac{\nu}{2}}\pi^2}{84\,\zeta(3)}(2^{\nu+1}-2)\right)\sqrt{x}\log x + B\sqrt{x} \qquad (59—61)$$

(55) $$= \left(\frac{2^{\frac{3}{2}\nu}\cdot 3\pi^2}{28\,\zeta(3)} - \frac{2^{\frac{\nu}{2}}\pi^2}{42\,\zeta(3)}\right)\sqrt{x}\log x + B\sqrt{x}.$$

Hilfssatz 12:

(62) $$\sum_{\substack{a,b \le x^2 \\ an=bm}} \frac{1}{ab} = \frac{\pi^2}{6}\,\frac{(m,n)^2}{m\,n} + \frac{B}{x^2}.$$

Beweis: Das ist T_{III} (18).

Hilfssatz 13:

(63) $$\frac{2^\nu}{2\pi^2}\sum_{\substack{m \le \sqrt{2^\nu x} \\ n \le \sqrt{x}}} \frac{1}{m\,n}\sum_{\substack{a,b \le x^2 \\ 2^\nu an=bm}} \frac{1}{ab}\int_{r_\nu}^{x} u^2\,du$$

(64) $$= \frac{(5+3\nu)\pi^2}{432}x^3 - \frac{\pi^2}{30\,\zeta(3)}\left(\frac{1+2^{\frac{\nu}{2}}}{12} + \frac{\nu}{28} + \frac{2^{\frac{\nu}{2}}-2^{-\frac{\nu}{2}}}{42}\right)x^{\frac{5}{2}}\log x + Bx^{\frac{5}{2}},$$

Beweis: Der Ausdruck (63) ist

$$= \frac{2^\nu}{2\pi^2}\sum_{\substack{m \le \sqrt{2^\nu x} \\ n \le \sqrt{x}}} \frac{1}{m\,n}\left(\frac{\pi^2}{6}\,\frac{(m, 2^\nu n)^2}{2^\nu m\,n} + \frac{B}{x^2}\right)\int_{r_\nu}^{x} u^2\,du \qquad (62,\ 11)$$

$$= \frac{1}{12}\sum_{\substack{m \le \sqrt{2^\nu x} \\ n \le \sqrt{x}}} \frac{(m, 2^\nu n)^2}{m^2 n^2}\int_{r_\nu}^{x} u^2\,du + \frac{B}{x^2}\sum_{m,n \le \sqrt{2^\nu x}} \frac{1}{m\,n}x^3$$

$$= \frac{1}{12}\int_{1}^{x} u^2 \sum_{\substack{m \le \sqrt{2^\nu u} \\ n \le \sqrt{u}}} \frac{(m, 2^\nu n)^2}{m^2 n^2} + Bx\log^2 x \qquad (11)$$

$$= \frac{1}{12}\left(\frac{5\pi^2}{12} + \frac{\nu\pi^2}{4}\right)\int_{3}^{x} u^2\,du - \frac{\pi^2}{12\,\zeta(3)}\left(\frac{1+2^{\frac{\nu}{2}}}{12} + \frac{\nu}{28} + \frac{2^{\frac{\nu}{2}}-2^{-\frac{\nu}{2}}}{42}\right)$$

$$\times \int_{3}^{x} u^{\frac{3}{2}}\log u\,du + Bx^{\frac{5}{2}} \qquad (41),$$

genügt also der Abschätzung (64).

Hilfssatz 14:

(65) $$\frac{2^\nu}{2\pi^2}\sum_{\substack{m \le \sqrt{2^\nu x} \\ n \le \sqrt{x}}} \frac{n}{m}\sum_{\substack{a,b \le x^2 \\ 2^\nu an=bm}} \frac{1}{ab}\int_{r_\nu}^{x} u\,du$$

(66) $$= \frac{\pi^2}{30\,\zeta(3)}\left(\frac{1}{12} + \frac{\nu}{28}\right)x^{\frac{5}{2}}\log x + Bx^{\frac{5}{2}}.$$

Beweis: Der Ausdruck (65) ist

$$= \frac{2^\nu}{2\pi^2} \sum_{\substack{m \leqq \sqrt{2^\nu x} \\ n \leqq \sqrt{x}}} \frac{n}{m}\left(\frac{\pi^2}{6}\frac{(m, 2^\nu n)^2}{2^\nu m n} + \frac{B}{x^2}\right)\int_{r_\nu}^{x} u\,d\,u \qquad (62,\ 11)$$

$$= \frac{1}{12}\sum_{\substack{m \leqq \sqrt{2^\nu x} \\ n \leqq \sqrt{x}}} \frac{(m, 2^\nu n)^2}{m^2}\int_{r_\nu}^{x} u\,d\,u + \frac{B}{x^2}\sum_{m,n \leqq \sqrt{2^\nu x}} \frac{n}{m}\, x^2$$

$$= \frac{1}{12}\int_1^x u \sum_{\substack{m \leqq \sqrt{2^\nu u} \\ n \leqq \sqrt{u}}} \frac{(m, 2^\nu n)^2}{m^2}\, d\,u + B\,x \log x \qquad (11)$$

$$= \frac{\pi^2}{12\,\zeta(3)}\left(\frac{1}{12} + \frac{\nu}{28}\right)\int_3^x u^{\frac{3}{2}} \log u\, d\,u + B\,x^{\frac{5}{2}} \qquad (54),$$

genügt also der Abschätzung (66).

Hilfssatz 15:

$$(67)\qquad \frac{1}{2\pi^2}\sum_{\substack{m \leqq \sqrt{2^\nu x} \\ n \leqq \sqrt{x}}} \frac{m}{n} \sum_{\substack{a,b \leqq x^2 \\ 2^\nu a n = b m}} \frac{1}{a\,b}\int_{r_\nu}^{x} u\,d\,u$$

$$(68)\qquad = \frac{\pi^2}{30\,\zeta(3)}\left(\frac{2^{\frac{\nu}{2}}\cdot 3}{28} - \frac{2^{-\frac{\nu}{2}}}{42}\right) x^{\frac{5}{2}} \log x + B\,x^{\frac{5}{2}}.$$

Beweis: Der Ausdruck (67) ist

$$= \frac{1}{2\pi^2}\sum_{\substack{m \leqq \sqrt{2^\nu x} \\ n \leqq \sqrt{x}}} \frac{m}{n}\left(\frac{\pi^2}{6}\frac{(m, 2^\nu n)^2}{2^\nu m n} + \frac{B}{x^2}\right)\int_{r_\nu}^{x} u\,d\,u \qquad (62,\ 11)$$

$$= \frac{2^{-\nu}}{12}\sum_{\substack{m \leqq \sqrt{2^\nu x} \\ n \leqq \sqrt{x}}} \frac{(m, 2^\nu n)^2}{n^2}\int_{r_\nu}^{x} u\,d\,u + \frac{B}{x^2}\sum_{m,\,n \leqq \sqrt{2^\nu x}} \frac{m}{n}\, x^2$$

$$= \frac{2^{-\nu}}{12}\int_1^x u \sum_{\substack{m \leqq \sqrt{2^\nu u} \\ n \leqq \sqrt{u}}} \frac{(m, 2^\nu n)^2}{n^2}\, d\,u + B\,x \log x \qquad (11)$$

$$= \frac{\pi^2}{12\,\zeta(3)}\left(\frac{2^{\frac{\nu}{2}}\cdot 3}{28} - \frac{2^{-\frac{\nu}{2}}}{42}\right)\int_3^x u^{\frac{3}{2}} \log u\, d\,u + B\,x^{\frac{5}{2}} \qquad (55),$$

genügt also der Abschätzung (68).

Hilfssatz 16:

$$(69)\qquad \frac{1}{2\pi^2}\sum_{\substack{m \leqq \sqrt{2^\nu x} \\ n \leqq \sqrt{x}}} m\,n \sum_{\substack{a,b \leqq x^2 \\ 2^\nu a n = b m}} \frac{1}{a\,b}\int_{r_\nu}^{x} d\,u = B x^{\frac{5}{2}}.$$

Beweis: Die linke Seite von (69) ist

$$= B \sum_{m,n \leqq \sqrt{2^\nu x}} m\,n\left(\frac{(m, n)^2}{m\,n} + \frac{1}{x^2}\right) x = B\,x \sum_{m,n \leqq \sqrt{2^\nu x}} (m, n)^2 \qquad (62,\ 11)$$

$$= B\,x \sum_{r \leqq \sqrt{2^\nu x}} r^2 \sum_{\substack{m,n \leqq \sqrt{2^\nu x} \\ (m,n) = r}} 1 = B\,x \sum_{r \leqq \sqrt{2^\nu x}} r^2 \sum_{m,n \leqq \frac{\sqrt{2^\nu x}}{r}} 1$$

$$= B\,x \sum_{r \leqq \sqrt{2^\nu x}} r^2 \left(\frac{\sqrt{x}}{r}\right)^2 = B\,x^{\frac{5}{2}}.$$

Hilfssatz 17:

$$(70)\qquad \sum_{E_1,E_2=0}^{1} \mathfrak{c}_{E_1,E_2} \sum_{\substack{m \leqq \sqrt{2^\nu x} \\ n \leqq \sqrt{x}}} m^{2E_1-1}\, n^{2E_2-1} \sum_{a,b \leqq x^2} \frac{1}{a\,b}\int_{r_\nu}^{x} u^{2-E_1-E_2} \cos\left\{2\pi u\left(\frac{2^\nu a}{m} + \frac{b}{n}\right)\right\} du$$

$$(71)\qquad = B\,x^{\frac{5}{2}}.$$

Beweis: Der Ausdruck (70) ist

$$= B \sum_{E_1,E_2=0}^{1} \sum_{m,n \leqq \sqrt{2^\nu x}} m^{2E_1-1}\, n^{2E_2-1} \sum_{a,b \leqq x^2} \frac{1}{a\,b}\, x^{2-E_1-E_2} \frac{1}{\frac{a}{m} + \frac{b}{n}}$$

$$= B \sum_{E_1,E_2=0}^{1} x^{2-E_1-E_2} \sum_{m,n\leq\sqrt{2^\nu x}} m^{2E_1-1} n^{2E_2-1} \sum_{a,b\leq x^2} \frac{1}{a\,b} \frac{1}{\sqrt{\frac{a}{m}\,\frac{b}{n}}}$$

$$= B \sum_{E_1,E_2=0}^{1} x^{2-E_1-E_2} \sum_{m,n\leq\sqrt{2^\nu x}} m^{2E_1-\frac{1}{2}} n^{2E_2-\frac{1}{2}} \sum_{a,b=1}^{\infty} (a\,b)^{-\frac{3}{2}} = B\,x^{\frac{5}{2}}.$$

Hilfssatz 18: *Mit*

(72) $$\mathfrak{S}(x) = \sum_{E_1,E_2=0}^{1} c_{E_1,E_2} \sum_{\substack{m\leq\sqrt{2^\nu x}\\ n\leq\sqrt{x}}} m^{2E_1-1} n^{2E_2-1}$$

$$\times \sum_{\substack{a,b\leq x^2\\ 2^\nu an\neq bm}} \frac{1}{a\,b} \int_{r_\nu}^{x} u^{2-E_1-E_2} \cos\left\{2\pi u\left(\frac{2^\nu a}{m}-\frac{b}{n}\right)\right\} d\,u$$

ist

(73) $$V_\nu(x) = \left(\frac{2^\nu}{12}+\frac{(5+3\nu)\pi^2}{432}\right)x^3 + \frac{1}{2\pi^2}\mathfrak{S}(x) + B\,x^{\frac{5}{2}}.$$

Beweis: Für reelle v sei (nur für den vorliegenden Beweis)

(74) $$(v) = \mathrm{Min}\,(v-[v];\ [v]+1-v)$$

(die positiv genommene Entfernung von v zur nächstliegenden ganzen Zahl). Dann gilt bekanntlich für jedes natürliche k, falls $\mathrm{Min}\left(1;\frac{1}{0}\right)$ Eins bedeutet,

(75) $$\sum_{q=k+1}^{\infty} \frac{\sin 2\pi q v}{q} = B\,\mathrm{Min}\left(1;\frac{1}{k(v)}\right),$$

(76) $$\psi(v) = -\frac{1}{\pi}\sum_{q=1}^{\infty}\frac{\sin 2\pi q v}{q} \quad (v \text{ nicht ganz}).$$

((76) wurde schon bei (18) und (19) benutzt.)

Es sei ferner

(77) $$k=[x^2],$$

(78) $$s_k(u) = -\frac{1}{\pi}\sum_{a=1}^{k}\frac{1}{a}\sin\frac{2\pi a 2^\nu u}{m},\quad t_k(u) = -\frac{1}{\pi}\sum_{b=1}^{k}\frac{1}{b}\sin\frac{2\pi b u}{n}.$$

(79) $$s_k(u) = B,\ t_k(u) = B \qquad (78,\ 75,\ 76,\ 2),$$

(80) $$\psi\left(\frac{2^\nu u}{m}\right) - s_k(u) = B\,\mathrm{Min}\left(1; x^{-2}\left(\frac{2^\nu u}{m}\right)^{-1}\right) \qquad (75-78),$$

(81) $$\psi\left(\frac{u}{n}\right) - t_k(u) = B\,\mathrm{Min}\left(1; x^{-2}\left(\frac{u}{n}\right)^{-1}\right) \qquad (75-78).$$

$$\int_{r_\nu}^{x} \psi\left(\frac{2^\nu u}{m}\right)\psi\left(\frac{u}{n}\right) u^{2-E_1-E_2}\,d\,u - \int_{r_\nu}^{x} s_k(u)\,t_k(u)\,u^{2-E_1-E_2}\,d\,u$$

$$= \int_{r_\nu}^{x}\left(\psi\left(\frac{2^\nu u}{m}\right) - s_k(u)\right) t_k(u)\,u^{2-E_1-E_2}\,d\,u + \int_{r_\nu}^{x}\left(\psi\left(\frac{u}{n}\right) - t_k(u)\right) s_k(u)\,u^{2-E_1-E_2}\,d\,u$$

$$+ \int_{r_\nu}^{x}\left(\psi\left(\frac{2^\nu u}{m}\right) - s_k(u)\right)\left(\psi\left(\frac{u}{n}\right) - t_k(u)\right) u^{2-E_1-E_2}\,d\,u$$

(82) $$= B\,x^{2-E_1-E_2}\left(\int_{r_\nu}^{x}\mathrm{Min}\left(1; x^{-2}\left(\frac{2^\nu u}{m}\right)^{-1}\right) d\,u + \int_{r_\nu}^{x}\mathrm{Min}\left(1; x^{-2}\left(\frac{u}{n}\right)^{-1}\right) d\,u\right) \qquad (79-81).$$

$$\int_{r_\nu}^{x}\mathrm{Min}\left(1; x^{-2}\left(\frac{2^\nu u}{m}\right)^{-1}\right) d\,u = \frac{m}{2^\nu}\int_{\frac{2^\nu r_\nu}{m}}^{\frac{2^\nu x}{m}}\mathrm{Min}\,(1; x^{-2}(v)^{-1})\,d\,v$$

(83) $$= B\,m\left(\int_{\substack{1\\ (v)\leq\frac{1}{x}}}^{\frac{2^\nu x}{m}} d\,v + x^{-2+1}\int_{\substack{1\\ (v)\geq\frac{1}{x}}}^{\frac{2^\nu x}{m}} d\,v\right) = B\,m\left(\frac{x}{m}\frac{1}{x}+\frac{1}{x}\frac{x}{m}\right) = B \quad (11,\ 74),$$

(84) $$\int_{r_\nu}^{x}\mathrm{Min}\left(1; x^{-2}\left(\frac{u}{n}\right)^{-1}\right) d\,u = B \qquad (11,\ 74).$$

$$(85)\qquad \int_{r_\nu}^{x} \psi\left(\frac{2^\nu u}{m}\right)\psi\left(\frac{u}{n}\right) u^{2-E_1-E_2}\,du - \int_{r_\nu}^{x} s_k(u)\,t_k(u)\,u^{2-E_1-E_2}\,du = B\, x^{2-E_1-E_2} \qquad (82-84).$$

$$\sum_{E_1,E_2=0}^{1} \mathfrak{c}_{E_1,E_2} \sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{x}}} m^{2E_1-1}\, n^{2E_2-1}\int_{r_\nu}^{x} \psi\left(\frac{2^\nu u}{m}\right)\psi\left(\frac{u}{n}\right) u^{2-E_1-E_2}\,du$$

$$-\sum_{E_1,E_2=0}^{1} \mathfrak{c}_{E_1,E_2} \sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{x}}} m^{2E_1-1}\, n^{2E_2-1}\int_{r_\nu}^{x} s_k(u)\,t_k(u)\,u^{2-E_1-E_2}\,du$$

$$(86)\qquad = B\sum_{E_1,E_2=0}^{1} x^{2-E_1-E_2}\sum_{m,\,n\le\sqrt{2^\nu x}} m^{2E_1-1}\, n^{2E_2-1} = B\, x^{\frac{5}{2}} \qquad (85).$$

$$\int_{r_\nu}^{x} s_k(u)\,t_k(u)\,u^{2-E_1-E_2}\,du = \frac{1}{\pi^2}\sum_{a,\,b\le x^2}\frac{1}{a\,b}\int_{r_\nu}^{x} u^{2-E_1-E_2}\sin\frac{2\pi a\, 2^\nu u}{m}\sin\frac{2\pi b u}{n}\,du \qquad (11,\ 77,\ 78)$$

$$(87)\qquad = \frac{1}{2\pi^2}\sum_{\substack{a,\,b\le x^2\\ 2^\nu an=bm}}\frac{1}{a\,b}\int_{r_\nu}^{x} u^{2-E_1-E_2}\,du +$$

$$+\frac{1}{2\pi^2}\sum_{\substack{a,\,b\le x^2\\ 2^\nu an\neq bm}}\frac{1}{a\,b}\int_{r_\nu}^{x} u^{2-E_1-E_2}\cos\left\{2\pi u\left(\frac{2^\nu a}{m}-\frac{b}{n}\right)\right\}du$$

$$-\frac{1}{2\pi^2}\sum_{a,\,b\le x^2}\frac{1}{a\,b}\int_{r_\nu}^{x} u^{2-E_1-E_2}\cos\left\{2\pi u\left(\frac{2^\nu a}{m}+\frac{b}{n}\right)\right\}du\,.$$

$$V_\nu(x) = \frac{2^\nu}{12}x^3 + \sum_{E_1,E_2=0}^{1} \mathfrak{c}_{E_1,E_2} \sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{x}}} m^{2E_1-1}\, n^{2E_2-1}\int_{r_\nu}^{x} s_k(u)\,t_k(u)\,u^{2-E_1-E_2}\,du$$

$$+B\, x^{\frac{5}{2}} \qquad (31,\ 86)$$

$$= \frac{2^\nu}{12}x^3 + \frac{1}{2\pi^2}\sum_{E_1,E_2=0}^{1} \mathfrak{c}_{E_1,E_2} \sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{x}}} m^{2E_1-1}\, n^{2E_2-1}\sum_{\substack{a,\,b\le x^2\\ 2^\nu an=bm}}\frac{1}{a\,b}\int_{r_\nu}^{x} u^{2-E_1-E_2}\,du$$

$$+\frac{1}{2\pi^2}\mathfrak{S}(x) - \frac{1}{2\pi^2}\sum_{E_1,E_2=0}^{1} \mathfrak{c}_{E_1,E_2} \sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{x}}} m^{2E_1-1}\, n^{2E_2-1}$$

$$\times\sum_{a,\,b\le x^2}\frac{1}{a\,b}\int_{r_\nu}^{x} u^{2-E_1-E_2}\cos\left\{2\pi u\left(\frac{2^\nu a}{m}+\frac{b}{n}\right)\right\}du + B\, x^{\frac{5}{2}} \qquad (87,\ 72)$$

$$= \left(\frac{2^\nu}{12}+\frac{(5+3\nu)\pi^2}{432}\right)x^3 - \frac{\pi^2}{30\,\zeta(3)}\left(\frac{1+2^{\frac{\nu}{2}}}{12}+\frac{\nu}{28}+\frac{2^{\frac{\nu}{2}}-2^{-\frac{\nu}{2}}}{42}-\frac{1}{12}-\frac{\nu}{28}\right.$$

$$\left.-\frac{2^{\frac{\nu}{2}}\cdot 3}{28}+\frac{2}{42}^{-\frac{\nu}{2}}\right)x^{\frac{5}{2}}\log x + \frac{1}{2\pi^2}\mathfrak{S}(x) + B\, x^{\frac{5}{2}} \qquad (10,\ 64,\ 66,\ 68,\ 69,\ 71)$$

$$(73)\qquad = \left(\frac{2^\nu}{12}+\frac{(5+3\nu)\pi^2}{432}\right)x^3 + \frac{1}{2\pi^2}\mathfrak{S}(x) + B\, x^{\frac{5}{2}}\,.$$

Es sei μ eine der Zahlen 0, 1, 2, 3, 4; ferner

$$(88)\qquad r = r_{\mu,\nu} = \mathrm{Max}\left(\frac{m^2}{2^\nu};\ \frac{n^2}{2^\mu}\right),$$

$$(89)\qquad \mathfrak{C}_{E_1,E_2} = \mathfrak{C}_{E_1,E_2,\mu,\nu} = \mathfrak{c}_{E_2,E_1,\mu}\,\mathfrak{c}_{E_1,E_2,\nu}\,,$$

$$(90)\qquad \mathfrak{S}_{\mu,\nu}(x) = \sum_{E_1,E_2=0}^{1} \mathfrak{C}_{E_1,E_2,\mu,\nu} \sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{2^\mu x}}} m^{2E_1-1}\, n^{2E_2-1}$$

$$\times\sum_{\substack{a,\,b\le x^2\\ 2^\nu an>2^\mu bm}}\frac{1}{a\,b}\int_{r_{\mu,\nu}}^{x} u^{2-E_1-E_2}\cos\left\{2\pi u\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)\right\}du\,.$$

Hilfssatz 19:

$$(91)\quad V_\nu(x) = \left(\frac{2^\nu}{12} + \frac{(5+3\nu)\pi^2}{432}\right) x^3 + \frac{1}{2\pi^2}\left(\mathfrak{S}_{\nu,0}(x) + \mathfrak{S}_{0,\nu}(x)\right) + B x^{\frac{5}{2}}.$$

Beweis: Ich ersetze in (90)

$$\mu, \nu, E_1, E_2, m, n, a, b \quad \text{durch} \quad \nu, \mu, E_2, E_1, n, m, b, a.$$

Dann folgt aus (90) und (88)

$$(92)\quad \mathfrak{S}_{\nu,\mu}(x) = \sum_{E_1,E_2=0}^{1} \mathfrak{C}_{E_2,E_1,\nu,\mu} \sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{2^\mu x}}} m^{2E_1-1} n^{2E_2-1}$$

$$\times \sum_{\substack{a,b\le x^2\\ 2^\nu an<2^\mu bm}} \frac{1}{ab}\int_{r_{\mu,\nu}}^{x} u^{2-E_1-E_2}\cos\left\{2\pi u\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)\right\} du.$$

$$(93)\quad \mathfrak{C}_{E_1,E_2,0,\nu} = \mathfrak{C}_{E_2,E_1,\nu,0} = \mathfrak{e}_{E_1,E_2} \qquad (89,\ 10),$$

$$(94)\quad r_{0,\nu} = r_\nu \qquad (88,\ 11).$$

(91) folgt aus (73), (72), (90) und (92)—(94).

Hilfssatz 20: *a, b, c, m, n seien natürliche Zahlen. Zu jedem vorgegebenen Tripel a, b, c gibt es höchstens*

$$B\left(\frac{(a,b)}{\sqrt{ab}}\sqrt{x}+1\right)$$

Paare m, n mit

$$m\le\sqrt{x},\ n\le\sqrt{x},\ an-bm=c.$$

Beweis: Das ist T_{III}, Hilfssatz 9.

Hilfssatz 21:

$$(95)\quad \mathfrak{S}_{\mu,\nu}(x) = \frac{1}{2\pi}\sum_{E_1,E_2=0}^{1} \mathfrak{C}_{E_1,E_2} \sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{x}}} m^{2E_1} n^{2E_2} \sum_{\substack{a,b\le x^2\\ 2^\nu an>2^\mu bm}} \left(ab(2^\nu an-2^\mu bm)\right)^{-1}$$

$$\times\left(x^{2-E_1-E_2}\sin\left\{2\pi x\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)\right\} - r^{2-E_1-E_2}\sin\left\{2\pi r\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)\right\}\right) + Bx^{\frac{5}{2}}.$$

Beweis:

$$(96)\quad \int_r^x u^{2-E_1-E_2}\cos\left\{2\pi u\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)\right\} du = \left.\frac{u^{2-E_1-E_2}\sin\left\{2\pi u\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)\right\}}{2\pi\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)}\right|_r^x$$

$$-\frac{2-E_1-E_2}{2\pi\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)}\int_r^x u^{1-E_1-E_2}\sin\left\{2\pi u\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)\right\} du$$

$$(97)\quad = \frac{1}{2\pi}\,\frac{mn}{(2^\nu an-2^\mu bm)}\left(x^{2-E_1-E_2}\sin\left\{2\pi x\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)\right\}\right.$$

$$\left.-r^{2-E_1-E_2}\sin\left\{2\pi r\left(\frac{2^\nu a}{m}-\frac{2^\mu b}{n}\right)\right\}\right) + Bx^{1-E_1-E_2}m^2n^2(2^\nu an-2^\mu bm)^{-2}$$

(für $E_1=E_2=1$ ist $2-E_1-E_2=0$ in (96)).

$$\sum_{E_1,E_2=0}^{1}\sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{2^\mu x}}} m^{2E_1-1}n^{2E_2-1}\sum_{\substack{a,b\le x^2\\ 2^\nu an>2^\mu bm}}(ab)^{-1}x^{1-E_1-E_2}m^2n^2(2^\nu an-2^\mu bm)^{-2}$$

$$=\sum_{E_1,E_2=0}^{1} x^{1-E_1-E_2}\sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{2^\mu x}}} m^{2E_1+1}n^{2E_2+1}\sum_{\substack{a,b\le x^2\\ 2^\nu an>2^\mu bm}}(ab)^{-1}(2^\nu an-2^\mu bm)^{-2}$$

$$=Bx^2\sum_{\substack{m\le\sqrt{2^\nu x}\\ n\le\sqrt{2^\mu x}}}\ \sum_{\substack{a,b\le x^2\\ 2^\nu an>2^\mu bm}}(ab)^{-1}(2^\nu an-2^\mu bm)^{-2}$$

$$(98)\quad = Bx^2\sum_{a,b\le x^2}\frac{1}{ab}\sum_{c=1}^{\infty}\frac{1}{c^2}\sum_{\substack{m\le\sqrt{2^\nu x},\ n\le\sqrt{2^\mu x}\\ 2^\nu an-2^\nu bm=c}} 1$$

$$(99)\quad = Bx^2\sum_{a,b\le x^2}\frac{1}{ab}\sum_{c=1}^{\infty}\frac{1}{c^2}\left(\frac{(a,b)}{\sqrt{ab}}\sqrt{x}+1\right)$$

(Hilfssatz 20 mit $2^{\nu+\mu}x,\ 2^\nu a,\ 2^\mu b,\ c,\ m,\ n$ statt $x,\ a,\ b,\ c,\ m,\ n$)

$$= B x^{\frac52} \sum_{a,b=1}^{\infty} \frac{(a,b)}{(ab)^{\frac32}} + B x^2 \sum_{a,b\le x^2} \frac{1}{ab}$$

$$= B x^{\frac52} \sum_{d=1}^{\infty} d \sum_{\substack{a,b=1\\(a,b)=d}}^{\infty} (ab)^{-\frac32} + B x^{\frac52} = B x^{\frac52} \sum_{d=1}^{\infty} d^{-2} \sum_{a,b=1}^{\infty} (ab)^{-\frac32}$$

(100) $$+ B x^{\frac52} = B x^{\frac52}.$$

(95) folgt aus (90), (89), (88), (97) und (100).

Hilfssatz 22:

(101) $$\mathfrak{S}_{\mu,\nu}(x) = \frac{1}{2\pi} \sum_{E_1,E_2=0}^{1} \mathfrak{C}_{E_1,E_2} \sum_{a,b\le x^\delta} \frac{1}{ab} \sum_{c\le 2^{\mu+\nu}(2^\nu a,\, 2^\mu b)\, x^{\frac12+\delta}} \frac1c$$

$$\times \sum_{\substack{m\le\sqrt{2^\nu x},\; n\le\sqrt{2^\mu x}\\ 2^\nu an - 2^\mu bm = c}} m^{2E_1} n^{2E_2} \left(x^{2-E_1-E_2} \sin\frac{2\pi xc}{mn} - r^{2-E_1-E_2}\sin\frac{2\pi rc}{mn}\right)$$

$$+ Bx^{\frac52}.$$

Beweis: In (95) ist

(102) $$2^\nu an - 2^\mu bm \le 2^\nu an \le 2^\nu x^2\sqrt{2^\mu x} \le 2^{\mu+\nu}x^3.$$

(103) $$\mathfrak{S}_{\mu,\nu}(x) = \frac{1}{2\pi} \sum_{E_1,E_2=0}^{1} \mathfrak{C}_{E_1,E_2} \sum_{a,b\le x^2} \frac{1}{ab} \sum_{c\le 2^{\mu+\nu}x^3} \frac1c$$

$$\times \sum_{\substack{m\le\sqrt{2^\nu x},\; n\le\sqrt{2^\mu x}\\ 2^\nu an - 2^\mu bm = c}} m^{2E_1} n^{2E_2} \left(x^{2-E_1-E_2} \sin\frac{2\pi xc}{mn} - r^{2-E_1-E_2}\sin\frac{2\pi rc}{mn}\right)$$

$$= Bx^{\frac52} \qquad (95, 102).$$

Behalte ich in (103) nur die $a, b\le x^\delta$ bei, so ist der Fehler

$$= B \sum_{E_1,E_2=0}^{1} \left(\sum_{\substack{x^\delta<a\le x^2\\ b\le x^2}} + \sum_{\substack{x^\delta<b\le x^2\\ a\le x^2}}\right) \frac{1}{ab} \sum_{c\le 2^{\mu+\nu}x^3} \sum_{\substack{m\le\sqrt{2^\nu x},\; n\le \sqrt{2^\mu x}\\ 2^\nu an-2^\mu bm=c}} m^{2E_1}n^{2E_2}x^{2-E_1-E_2} \qquad (88)$$

$$= Bx^2 \left(\sum_{\substack{x^\delta<a\le x^2\\ b\le x^2}} + \sum_{\substack{x^\delta<b\le x^2\\ a\le x^2}}\right) \frac1{ab} \sum_{c\le 2^{\mu+\nu}x^3} \frac1c \sum_{\substack{m\le\sqrt{2^\nu x},\; n\le\sqrt{2^\mu x}\\ 2^\nu an-2^\mu bm=c}} 1$$

$$= Bx^2 \left(\sum_{\substack{x^\delta<a\le x^2\\ b\le x^2}} + \sum_{\substack{x^\delta<b\le x^2\\ a\le x^2}}\right) \frac1{ab}\sum_{c\le 2^{\mu+\nu}x^3}\frac1c\left(\frac{(a,b)}{\sqrt{ab}}\sqrt x + 1\right)$$

(Hilfssatz 20, wie beim Übergang von (98) zu (99))

$$= Bx^{\frac52}\sum_{\substack{x^\delta<a\le x^2\\ b\le x^2}}\frac{(a,b)}{(ab)^{\frac32}}\sum_{c\le 2^{\mu+\nu}x^3}\frac1c + Bx^{2+\delta}$$

$$= Bx^{\frac{5+\delta}{2}}\sum_{\substack{x^\delta<a\le x^2\\ b\le x^2}}\frac{(a,b)}{(ab)^{\frac32}} + Bx^{\frac52} \qquad (3)$$

$$= Bx^{\frac{5+\delta}{2}}\sum_{d=1}^\infty d \sum_{a>\frac{x^\delta}{d},\; b\ge1} d^{-3}a^{-\frac32}b^{-\frac32} + Bx^{\frac52}$$

$$= Bx^{\frac{5+\delta}{2}}\sum_{d=1}^\infty d^{-2}\sum_{a>\frac{x^\delta}{d}} a^{-\frac32} + Bx^{\frac52} = Bx^{\frac52}.$$

Also ist

$$\mathfrak{S}_{\mu,\nu}(x) = \frac{1}{2\pi} \sum_{E_1,E_2=0}^{1} \mathfrak{C}_{E_1,E_2} \sum_{a,b\le x^\delta} \frac{1}{ab} \sum_{c\le 2^{\mu+\nu}x^3} \frac1c$$

$$\times \sum_{\substack{m\le\sqrt{2^\nu x},\; n\le\sqrt{2^\mu x}\\ 2^\nu an - 2^\mu bm = c}} m^{2E_1} n^{2E_2} \left(x^{2-E_1-E_2} \sin\frac{2\pi xc}{mn} - r^{2-E_1-E_2}\sin\frac{2\pi rc}{mn}\right) + Bx^{\frac52},$$

und hierin ist wegen

$$c = 2^\nu an - 2^\mu bm \le 2^\nu an \le 2^\nu x^\delta\sqrt{2^\mu x} \le 2^{\mu+\nu}x^{\frac12+\delta}$$

die m-n-Summe für $c > 2^{\mu+\nu}x^{\frac12+\delta}$ leer.

Bevor ich weitergehe, schicke ich einige einfache Bemerkungen voraus, die sich auf die in (101) auftretenden a, b, c, m, n und r beziehen.

Es sei

(104) $$(2^\nu a, 2^\mu b) = g,\quad 2^\nu a = \alpha g,\quad 2^\mu b = \beta g.$$

Ist die m-n-Summe in (101) nicht leer, so muss g in c aufgehen:

(105) $$c = \gamma g.$$

Die Lösungen von

(106) $$2^\nu a n - 2^\mu b m = c,$$

d. h. von

(107) $$\alpha n - \beta m = \gamma$$

sind

(108) $$m = m_h = m' + \alpha h,\quad n = n_h = n' + \beta h\quad (h = 0, 1, 2, \ldots),$$

wobei $m = m'$, $n = n'$ die Lösung mit kleinstem $m' > 0$ bedeutet. Es ist also

(109) $$\alpha n' - \beta m' = \gamma,$$

(110) $$0 < m' \le \alpha,\quad 0 < n' = \frac{\gamma + \beta m'}{\alpha} \le \frac{\gamma}{\alpha} + \beta.$$

Von jetzt ab bedeuten m, n die durch (108) gegebenen Funktionen von h; damit hängt auch r nach (88) von h ab.

Für das folgende wird es nötig sein, m und n als Funktionen einer stetigen reellen Veränderlichen zu betrachten. Um aber Missverständnissen vorzubeugen, will ich dem h seine Bedeutung als ganze Zahl lassen und lieber

(111) $$m_u = m' + \alpha u,\quad n_u = n' + \beta u$$

schreiben.

Hilfssatz 23: *Für $a \le 2^{\frac{\mu-\nu}{2}} b$ ist $m \le 2^{\frac{\nu-\mu}{2}} n$. Für $a > 2^{\frac{\mu-\nu}{2}} b$ ist $m \le 2^{\frac{\nu-\mu}{2}} n$ dann und nur dann, wenn $h \le \dfrac{2^{\frac{\nu-\mu}{2}} n' - m'}{\alpha - 2^{\frac{\nu-\mu}{2}}\beta}$ und nur dann, wenn $h \le 4^{\mu+\nu+1}(\gamma + ab)$ ist.*

Beweis: Für $a \le 2^{\frac{\mu-\nu}{2}} b$ ist $c \le 2^{\frac{\mu+\nu}{2}} b\left(n - 2^{\frac{\mu-\nu}{2}} m\right)$ (106), also sogar $m < 2^{\frac{\nu-\mu}{2}} n$. Für $a > 2^{\frac{\mu-\nu}{2}} b$, d. h. $\alpha > 2^{\frac{\nu-\mu}{2}} \beta$ (104), ist $m \le 2^{\frac{\nu-\mu}{2}} n$ mit

$h \le \dfrac{2^{\frac{\nu-\mu}{2}} n' - m'}{\alpha - 2^{\frac{\nu-\mu}{2}}\beta}$ gleichbedeutend (108), und hierin ist

$$\frac{2^{\frac{\nu-\mu}{2}} n' - m'}{\alpha - 2^{\frac{\nu-\mu}{2}}\beta} \le \frac{2^{\frac{\nu-\mu}{2}} n'\left(\alpha + 2^{\frac{\nu-\mu}{2}}\beta\right)}{\alpha^2 - 2^{\nu-\mu}\beta^2} = \frac{2^{\frac{\nu+\mu}{2}} n'\left(\alpha + 2^{\frac{\nu-\mu}{2}}\beta\right)}{2^\mu\alpha^2 - 2^\nu\beta^2}$$

$$\le 2^{\frac{\nu+\mu}{2}+1} n'\alpha \le 2^{\mu+\nu+1}(\gamma + \alpha\beta) \le 4^{\mu+\nu+1}(\gamma + ab) \qquad \textbf{(110, 104).}$$

Ich setze jetzt zur Abkürzung

(112) $$\mathrm{Min}\left(\frac{\sqrt{2^\nu x} - m'}{\alpha};\ \frac{\sqrt{2^\mu x} - n'}{\beta}\right) = X.$$

Dann ist

(113) $$X = B\frac{\sqrt{x}}{\alpha^j\beta^{1-j}}\ (j \text{ eine beliebige Zahl des Intervalls } 0 \le j \le 1).$$

Hilfssatz 24:

(114) $$\mathfrak{S}_{\mu,\nu}(x) = \frac{1}{2\pi}\sum_{E_1, E_2 = 0}^{1} \mathfrak{C}_{E_1,E_2} \sum_{a,b\le x^\delta}\frac{1}{abg}\sum_{\gamma \le 2^{\mu+\nu} x^{\frac12+\delta}}\frac{1}{\gamma}$$
$$\times \sum_{x^{\frac49}\le h\le X} m^{2E_1} n^{2E_2}\left(x^{2-E_1-E_2}\sin\frac{2\pi x\gamma g}{mn} - r^{2-E_1-E_2}\sin\frac{2\pi r\gamma g}{mn}\right) + Bx^{\frac52}.$$

Beweis:

(115) $$\mathfrak{S}_{\mu,\nu}(x) = \frac{1}{2\pi}\sum_{E_1, E_2 = 0}^{1}{}' \mathfrak{C}_{E_1,E_2} \sum_{a,b\le x^\delta}\frac{1}{abg}\sum_{\gamma \le 2^{\mu+\nu} x^{\frac12+\delta}}\frac{1}{\gamma}$$
$$\times \sum_{0\le h\le X} m^{2E_1} n^{2E_2}\left(x^{2-E_1-E_2}\sin\frac{2\pi x\gamma g}{mn} - r^{2-E_1-E_2}\sin\frac{2\pi r\gamma g}{mn}\right)$$
$$+ Bx^{\frac52} \qquad \textbf{(101, 105, 104, 108, 112).}$$

Lässt man in (115) die $h < x^{\frac49}$ weg, so ist der Fehler

$$= B\sum_{E_1, E_2 = 0}^{1}\ \sum_{a, b\le x^\delta}\frac{1}{ab}\sum_{\gamma\le 2^{\mu+\nu}x^{\frac12+\delta}}\frac{1}{\gamma}\sum_{0\le h< x^{\frac49}} m^{2E_1} n^{2E_2} x^{2-E_1-E_2} \qquad (88)$$

$$= B \sum_{E_1, E_2 = 0}^{1} \; \sum_{a,\, b,\, \gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}}^{1} \frac{1}{a b \gamma} \sum_{0 \leqq h < x^{\frac{4}{9}}} x^{E_1+E_2+2-E_1-E_2}$$

$$= B x^{2+\frac{4}{9}+\delta} = B x^{\frac{5}{2}} \qquad (3).$$

Hilfssatz 25: *Es sei $p-\frac{1}{2}$ und $q-\frac{1}{2}$ ganz, $0 \leqq p < q$; $f(u)$ im Intervall $p \leqq u \leqq q$ reell und zweimal differenzierbar; $f''(u)$ dort beständig $\geqq \omega$ und $= B\,\omega$, wo ω eine von u unabhängige positive Zahl bedeutet; $F(u)$ dort positiv, $F'(u)$ vorhanden und stetig.*

Dann ist

$$\sum_{p \leqq h \leqq q} F(h) \sin \{2\pi f(h)\} = B\left(F(p) + \int_p^q |F'(u)|\, du\right)\left(q\sqrt{\omega} + \frac{1}{\sqrt{\omega}}\right),$$

also, falls noch $F'(u) > 0$ vorausgesetzt wird,

$$\sum_{p \leqq h \leqq q} F(h) \sin \{2\pi f(h)\} = B\, F(q)\left(q\sqrt{\omega} + \frac{1}{\sqrt{\omega}}\right). \tag{116}$$

Beweis: Das ist T_{III}, Hilfssatz 15, der eine leichte Folgerung aus dem van der Corputschen Hilfssatz 14 von T_{III} ist.

Hilfssatz 26:

$$\mathfrak{S}_1(x) = \sum_{E_1, E_2 = 0}^{1} \mathfrak{F}_{E_1, E_2} \sum_{a,\, b \leqq x^{\delta}} \frac{1}{a b g} \sum_{\gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}} \frac{1}{\gamma} \tag{117}$$

$$\times \sum_{x^{\frac{4}{9}} \leqq h \leqq X} m^{2E_1} n^{2E_2} x^{2-E_1-E_2} \sin \frac{2\pi x \gamma g}{m n}$$

$$= B x^{\frac{5}{2}}. \tag{118}$$

Beweis: Es sei

$$x^{\frac{4}{9}} \leqq H \leqq H' \leqq 2H \leqq 2X, \; H \text{ und } H' \text{ ganz}, \tag{119}$$

$$p = H - \frac{1}{2}, \quad q = H' + \frac{1}{2}. \tag{120}$$

Aus (111), (120), (119), (112) und (110) folgt

$$m_p \leqq m_H \leqq m_q = m' + \alpha\left(H' + \frac{1}{2}\right) \leqq 3(m' + \alpha H) = 3\, m_H = B\sqrt{x}, \tag{121}$$

$$n_p \leqq n_H \leqq n_q \leqq 3(n' + \beta H) = 3\, n_H = B\sqrt{x}. \tag{122}$$

$$\begin{aligned} m_p &= m' + \alpha\left(H - \frac{1}{2}\right) \geqq \frac{1}{2}(m' + \alpha H) \geqq \frac{1}{4}(m' + \alpha H') \geqq \\ &\geqq \frac{1}{8}\left(m' + \alpha\left(H' + \frac{1}{2}\right)\right) = \frac{1}{8} m_q, \end{aligned} \tag{123}$$

$$n_p \geqq \frac{1}{2}(n' + \beta H) \geqq \frac{1}{8} n_q, \tag{124}$$

$$\frac{1}{m_q} = \frac{1}{m' + \alpha q} \leqq \frac{1}{\alpha H'} \leqq \frac{1}{\alpha x^{\frac{4}{9}}}, \tag{125}$$

$$\frac{1}{n_q} \leqq \frac{1}{\beta x^{\frac{4}{9}}}, \tag{126}$$

$$q \leqq m_q^{\frac{1}{2}} n_q^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}}. \tag{127}$$

Es sei u eine beliebige Zahl des Intervalls $p \leqq u \leqq q$,

$$f_1(u) = \frac{x \gamma g}{m_u n_u}, \quad F_1(u) = m_u^{2E_1} n_u^{2E_2}, \tag{128}$$

$$\omega_1 = f_1''(q). \tag{129}$$

Dann gilt

$$f_1'(u) = -\frac{x \gamma g}{(m_u n_u)^2}(\alpha n_u + \beta m_u) \qquad (128,\ 111), \tag{130}$$

$$f_1''(u) = \frac{2 x \gamma g}{(m_u n_u)^3}((\alpha n_u + \beta m_u)^2 - \alpha\beta m_u n_u) \qquad (130,\ 111)$$

$$= \frac{2 x \gamma g}{(m_u n_u)^3}((\alpha n_u - \beta m_u)^2 + 3\alpha\beta m_u n_u)$$

$$= \frac{2 x \gamma g}{(m_u n_u)^3}(\gamma^2 + 3\alpha\beta m_u n_u) \qquad (111,\ 109). \tag{131}$$

Mit wachsendem u nimmt $f_1''(u)$ ab **(131)**, da m_u und n_u zunehmen **(111)**. Also ist

$$\omega_1 = f_1''(q) \leqq f_1''(u) \leqq f_1''(p) = B f_1''(q) = B\,\omega_1 \qquad \textbf{(129, 131, 123, 124)}.$$

$F'_1(u)$ ist stetig und positiv **(128, 111, 3)**; $p-\frac{1}{2}$, $q-\frac{1}{2}$ sind ganz, $0 \leqq p < q$ **(120, 119)**. Also darf (116) auf die durch (120) (128), (129) gegebenen $p,\ q,\ f_1,\ F_1,\ \omega_1$ angewandt werden:

$$\text{(132)} \qquad \sum_{p \leqq h \leqq q} F_1(h) \sin\{2\pi f_1(h)\} = B F_1(q)\left(q\sqrt{\omega_1} + \frac{1}{\sqrt{\omega_1}}\right).$$

$$\text{(133)} \qquad \sum_{h=H}^{H'} m^{2E_1} n^{2E_2} \sin\frac{2\pi x\gamma g}{mn} = B\, m_q^{2E_1} n_q^{2E_2}\left(q\sqrt{\omega_1} + \frac{1}{\sqrt{\omega_1}}\right) \quad \textbf{(132, 120, 128, 108)}.$$

$$m_q^{2E_1} n_q^{2E_2} q\sqrt{\omega_1} = B m_q^{2E_1} n_q^{2E_2} m_q^{\frac{1}{2}} n_q^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} x^{\frac{1}{2}} \gamma^{\frac{1}{2}} g^{\frac{1}{2}} m_q^{-\frac{3}{2}} n_q^{-\frac{3}{2}}$$

$$\times\left(\gamma + a^{\frac{1}{2}} \beta^{\frac{1}{2}} m_q^{\frac{1}{2}} n_q^{\frac{1}{2}}\right) \quad \textbf{(127, 129, 131)}$$

$$= B x^{\frac{1}{2}} m_q^{2E_1-1} n_q^{2E_2-1} \gamma^{\frac{3}{2}} g^{\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} + B x^{\frac{1}{2}} m_q^{2E_1-\frac{1}{2}} n_q^{2E_2-\frac{1}{2}} \gamma^{\frac{1}{2}} g^{\frac{1}{2}}$$

$$= B x^{\frac{1}{2}+E_1-\frac{4}{9}+E_2-\frac{4}{9}} \gamma^{\frac{3}{2}} + B x^{\frac{1}{2}+E_1-\frac{2}{9}+E_2-\frac{2}{9}} \gamma^{\frac{1}{2}} g \qquad \textbf{(121, 122, 125, 126)}$$

$$\text{(134)} \qquad = B x^{E_1+E_2-\frac{7}{18}} g\gamma^{\frac{3}{2}} + B x^{E_1+E_2+\frac{1}{18}} g\gamma^{\frac{1}{2}}.$$

$$m_q^{2E_1} n_q^{2E_2} \frac{1}{\sqrt{\omega_1}} = B m_q^{2E_1} n_q^{2E_2} m_q^{\frac{3}{2}} n_q^{\frac{3}{2}} x^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} g^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} m_q^{-\frac{1}{2}} n_q^{-\frac{1}{2}} \qquad \textbf{(129, 131)}$$

$$= B x^{-\frac{1}{2}} m_q^{2E_1+1} n_q^{2E_2+1} \gamma^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}}$$

$$= B x^{-\frac{1}{2}} m_q x^{E_1} x^{E_2+\frac{1}{2}} \gamma^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} = B x^{E_1+E_2} \alpha H \gamma^{-\frac{1}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} \qquad \textbf{(121, 122, 110)}$$

$$\text{(135)} \qquad = B x^{E_1+E_2} \alpha^{\frac{1}{2}} \beta^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} H.$$

$$\text{(136)} \qquad \sum_{h=H}^{H'} m^{2E_1} n^{2E_2} x^{2-E_1-E_2} \sin\frac{2\pi x\gamma g}{mn} = B x^{2-\frac{7}{18}} g\gamma^{\frac{3}{2}}$$

$$+ B x^{2+\frac{1}{18}} g\gamma^{\frac{1}{2}} + B x^2 \alpha^{\frac{1}{2}} \beta^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} H \qquad \textbf{(133 — 135)}.$$

Ein bekannter Kungstgriff ergibt jetzt

$$\sum_{x^{\frac{4}{9}} \leqq h \leqq X} m^{2E_1} n^{2E_2} x^{2-E_1-E_2} \sin\frac{2\pi x\gamma g}{mn} = B x^{2-\frac{7}{18}+\delta} g\gamma^{\frac{3}{2}}$$

$$+ B x^{2+\frac{1}{18}+\delta} g\gamma^{\frac{1}{2}} + B x^2 \alpha^{\frac{1}{2}} \beta^{-\frac{1}{2}} \gamma^{-\frac{1}{2}} x^{\frac{1}{2}} \alpha^{-1} \qquad \textbf{(119, 136, 113)}$$

$$\text{(137)} = B x^{2-\frac{7}{18}+\delta} g\gamma^{\frac{3}{2}} + B x^{2+\frac{1}{18}+\delta} g\gamma^{\frac{1}{2}} + B x^{\frac{5}{2}} \alpha^{-\frac{1}{2}} \beta^{-\frac{1}{2}} \gamma^{-\frac{1}{2}}.$$

Setze ich (137) in (117) ein, so folgt

$$\mathfrak{S}_1(x) = B x^{2-\frac{7}{18}+\delta} \sum_{a,b \leqq x^{\delta}} (ab)^{-1} \sum_{\gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}} \gamma^{\frac{1}{2}}$$

$$+ B x^{2+\frac{1}{18}+\delta} \sum_{a,b \leqq x^{\delta}} (ab)^{-1} \sum_{\gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}} \gamma^{-\frac{1}{2}} + B x^{\frac{5}{2}} \sum_{a,b=1}^{\infty} (ab)^{-\frac{3}{2}} \sum_{\gamma=1}^{\infty} \gamma^{-\frac{3}{2}} \quad \textbf{(104)}$$

$$= B x^{2-\frac{7}{18}+\delta+\delta+\frac{3}{4}+2\delta} + B x^{2+\frac{1}{18}+\delta+\delta+\frac{1}{4}+\delta} + B x^{\frac{5}{2}}$$

$$\text{(118)} \qquad = B x^{\frac{5}{2}} \qquad \text{(3)}.$$

Hilfssatz 27:

$$\text{(138)} \qquad \mathfrak{S}_2(x) = \sum_{E_1, E_2 = 0}^{1} \mathfrak{C}_{E_1, E_2} \sum_{\substack{a,b \leqq x^{\delta} \\ a \leqq 2^{\frac{\mu-\nu}{2}} b}} \frac{1}{abg} \sum_{\gamma \leqq x^{\frac{1}{4}}} \frac{1}{\gamma}$$

$$\times \sum_{x^{\frac{4}{9}} \leqq h \leqq X} m^{2E_1} n^{2E_2} r^{2-E_1-E_2} \sin\frac{2\pi r\gamma g}{mn}$$

$$\text{(139)} \qquad = B x^{\frac{5}{2}}.$$

Beweis:

$$\text{(140)} \qquad \left(a \leqq 2^{\frac{\mu-\nu}{2}} b\right) m \leqq 2^{\frac{\nu-\mu}{2}} n,\ r = \frac{n^2}{2^{\mu}} \ \text{(Hilfssatz 23, 88)}.$$

(141) $$\mathfrak{S}_2(x)=\sum_{E_1,E_2=0}^{1} 2^{\mu(E_1+E_2-2)}\,\mathfrak{E}_{E_1,E_2}\sum_{\substack{a,b\le x^{\delta}\\ a\le 2^{\frac{\mu-\nu}{2}}b}}\frac{1}{a\,b\,g}\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}$$

$$\times\sum_{x^{\frac49}\le h\le X} n^4\left(\frac{m}{n}\right)^{2E_1}\sin\frac{2\pi n\gamma g}{2^{\mu}m}\qquad (138,\ 140).$$

(142) $$\left(a\le 2^{\frac{\mu-\nu}{2}}b\right)\qquad \alpha\le 2^{\frac{\nu-\mu}{2}}\beta\qquad (104).$$

(143) $$\frac{n}{m}-\frac{\beta}{\alpha}=\frac{\gamma}{\alpha m}=B\frac{\gamma}{\alpha^2 h},\quad \frac{m}{n}-\frac{\alpha}{\beta}=-\frac{\gamma}{\beta n}=B\frac{\gamma}{\beta^2 h}\qquad (107,\ 108,\ 110),$$

(144) $$\left(\frac{m}{n}\right)^{2E_1}-\left(\frac{\alpha}{\beta}\right)^{2E_1}=B\left(\left(\frac{m}{n}\right)^2-\left(\frac{\alpha}{\beta}\right)^2\right)=B\left(\frac{m}{n}-\frac{\alpha}{\beta}\right)=B\frac{\gamma}{\beta^2 h}$$
(3, 140, 142, 143),

(145) $$n^4-\beta^4h^4=B(n-\beta h)\,n^3=B\,n'\,x^{\frac32}=B(\gamma+\beta)\,x^{\frac32}\qquad (108,\ 101,\ 110),$$

$$\frac{2\pi n\gamma g}{2^{\mu}m}-\frac{2\pi\gamma b}{\alpha}=B\frac{\gamma^2 g}{\alpha^2 h}\qquad (143,\ 104),$$

(146) $$\sin\frac{2\pi n\gamma g}{2^{\mu}m}-\sin\frac{2\pi\gamma b}{\alpha}=B\,\mathrm{Min}\left(\frac{\gamma^2 g}{\alpha^2 h};1\right)=B\frac{\gamma}{\alpha}\sqrt{\frac{g}{h}}.$$

$$n^4\left(\frac{m}{n}\right)^{2E_1}\sin\frac{2\pi n\gamma g}{2^{\mu}m}-\beta^4h^4\left(\frac{\alpha}{\beta}\right)^{2E_1}\sin\frac{2\pi\gamma b}{\alpha}$$

$$=\beta^4h^4\left(\frac{\alpha}{\beta}\right)^{2E_1}\left(\sin\frac{2\pi n\gamma g}{2^{\mu}m}-\sin\frac{2\pi\gamma b}{\alpha}\right)$$

$$+(n^4-\beta^4h^4)\left(\frac{\alpha}{\beta}\right)^{2E_1}\sin\frac{2\pi n\gamma g}{2^{\mu}m}+n^4\left(\left(\frac{m}{n}\right)^{2E_1}-\left(\frac{\alpha}{\beta}\right)^{2E_1}\right)\sin\frac{2\pi n\gamma g}{2^{\mu}m}$$

(147) $$=B\beta^4h^{\frac72}\gamma g^{\frac12}\alpha^{-1}+B(\gamma+\beta)x^{\frac32}+Bx^2\frac{\gamma}{\beta^2h}\qquad (142,\ 146,\ 145,\ 101,\ 144).$$

Ersetzt man in der h-Summe von (141)

$$n^4\left(\frac{m}{n}\right)^{2E_1}\sin\frac{2\pi n\gamma g}{2^{\mu}m}\ \text{durch}\ \beta^4h^4\left(\frac{\alpha}{\beta}\right)^{2E_1}\sin\frac{2\pi\gamma b}{\alpha},$$

so ändert sich nach (147) $\mathfrak{S}_2(x)$ um

$$B\sum_{a,b\le x^{\delta}}\frac{1}{a\,b\,g}\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}\sum_{h\le X}\left(\beta^4h^{\frac72}\gamma g^{\frac12}\alpha^{-1}+(\gamma+\beta)x^{\frac32}+x^2\frac{\gamma}{\beta^2h}\right)$$

$$=B\sum_{a,b\le x^{\delta}}\frac{\beta^4}{a\,b\,\alpha\sqrt{g}}\sum_{\gamma\le x^{\frac14}}1\cdot\sum_{h\le\frac{\sqrt{2^{\mu}x}}{\beta}}h^{\frac72}+Bx^{\frac32}\sum_{a,b\le x^{\delta}}\frac{1}{a\,b}\sum_{\gamma\le x^{\frac14}}1\cdot\sum_{h\le\sqrt{2^{\mu}x}}1$$

$$+Bx^{\frac32}\sum_{a,b\le x^{\delta}}\frac{\beta}{a\,b}\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}\sum_{h\le\frac{\sqrt{2^{\mu}x}}{\beta}}1+Bx^2\sum_{a,b\le x^{\delta}}\frac{1}{a\,b}\sum_{\gamma\le x^{\frac14}}1\cdot\sum_{h\le\sqrt{2^{\mu}x}}\frac{1}{h}\qquad (112)$$

$$=Bx^{\frac52}\sum_{a,b=1}^{\infty}\left(a\,b\,\alpha^{\frac14}\beta^{\frac14}g^{\frac12}\right)^{-1}+Bx^{\frac74+\delta}+Bx^{2+\delta}+Bx^{\frac94+\delta}$$

(148) $$=Bx^{\frac52}\sum_{a,b=1}^{\infty}(a\,b)^{-\frac54}+Bx^{\frac52}=Bx^{\frac52}\qquad (104,\ 3).$$

$$\mathfrak{S}_2(x)=\sum_{E_1,E_2=0}^{1} 2^{\mu(E_1+E_2-2)}\,\mathfrak{E}_{E_1,E_2}\sum_{\substack{a,b\le x^{\delta}\\ a\le 2^{\frac{\mu-\nu}{2}}b}}\frac{1}{a\,b\,g}\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}$$

$$\times\sum_{x^{\frac49}\le h\le X}\beta^4h^4\left(\frac{\alpha}{\beta}\right)^{2E_1}\sin\frac{2\pi\gamma b}{\alpha}+Bx^{\frac52}\qquad (141,\ 148)$$

(149) $$=\sum_{E_1,E_2=0}^{1} 2^{\mu(E_1+E_2-2)}\,\mathfrak{E}_{E_1,E_2}\sum_{\substack{a,b\le x^{\delta}\\ a\le 2^{\frac{\mu-\nu}{2}}b}}\frac{1}{a\,b\,g}\beta^4\left(\frac{\alpha}{\beta}\right)^{2E_1}$$

$$\times\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}\sin\frac{2\pi\gamma b}{\alpha}\sum_{x^{\frac49}\le h\le X}h^4+Bx^{\frac52}.$$

(150) $$\frac{\sqrt{2^{\nu}x}-m'}{\alpha}-\frac{\sqrt{2^{\mu}x}-n'}{\beta}=\frac{(\sqrt{2^{\nu}}\beta-\sqrt{2^{\mu}}\alpha)\sqrt{x}+\gamma}{\alpha\beta}\ge\frac{\gamma}{\alpha\beta}>0$$
(109, 142).

(151) $$X=\frac{\sqrt{2^\mu x}-n'}{\beta} \qquad (151,\ 150),$$

(152) $$\sum_{x^{\frac{4}{9}}\le h\le X} h^4=\sum_{x^{\frac{4}{9}}\le h\le \frac{\sqrt{2^\mu x}}{\beta}} h^4+B\gamma\frac{x^2}{\beta^4} \qquad (151,\ 110).$$

$$\mathfrak{S}_2(x)=\sum_{E_1,E_2=0}^{1} 2^{\mu(E_1+E_2-2)}\mathfrak{C}_{E_1,E_2}\sum_{\substack{a,b\le x^\delta\\ a\le 2^{\frac{\mu-\nu}{2}}b}}\frac{1}{abg}\beta^4\left(\frac{\alpha}{\beta}\right)^{2E_1}$$

$$\times\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}\sin\frac{2\pi\gamma b}{\alpha}\cdot\sum_{x^{\frac49}\le h\le\frac{\sqrt{2^\mu x}}{\beta}} h^4$$

$$+Bx^2\sum_{a,b\le x^\delta}\frac{1}{abg}\left(\frac{\alpha}{\beta}\right)^{2E_1}\sum_{\gamma\le x^{\frac14}}1+Bx^{\frac52} \qquad (149,\ 152)$$

$$=B\sum_{\substack{a,b\le x^\delta\\ a\le 2^{\frac{\mu-\nu}{2}}b}}\frac{1}{abg}\beta^4\frac{x^{\frac52}}{\beta^5}+Bx^{\frac94}\sum_{a,b\le x^\delta}\frac{1}{ab}+Bx^{\frac52} \qquad (142)$$

(man beachte, dass die letzte h-Summe nicht von γ abhängt und

$$\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}\sin\frac{2\pi\gamma b}{\alpha}=B \qquad (\text{z. B. } 75, 76, 2))$$

$$=Bx^{\frac52}\sum_{\substack{a,b\le x^\delta\\ a\le 2^{\frac{\mu-\nu}{2}}b}}\frac{1}{ab^2}+Bx^{\frac52} \qquad (104,\ 3)$$

$$=Bx^{\frac52}\sum_{a,b=1}^{\infty}(ab)^{-\frac32}+Bx^{\frac52}=Bx^{\frac52}.$$

Hilfssatz 28:

(153) $$\mathfrak{S}_3(x)=\sum_{E_1,E_2=0}^{1}\mathfrak{C}_{E_1,E_2}\sum_{\substack{a,b\le x^\delta\\ b<2^{\frac{\nu-\mu}{2}}a}}\frac{1}{abg}\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}$$

$$\times\sum_{x^{\frac49}\le h\le X} m^{2E_1}n^{2E_2}r^{2-E_1-E_2}\sin\frac{2\pi rg}{mn}$$

(154) $$=Bx^{\frac52}.$$

Beweis: Für $b<2^{\frac{\nu-\mu}{2}}a$, $h>4^{\mu+\nu+1}(\gamma+ab)$ ist $m>2^{\frac{\nu-\mu}{2}}n$ (Hilfssatz 23), also $r=\frac{m^2}{2^\nu}$ (88). Ersetzt man daher in (153) r durch $\frac{m^2}{2^\nu}$, so ist der Fehler

$$=B\sum_{a,b\le x^\delta}\frac{1}{ab}\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}\sum_{h\le 4^{\mu+\nu+1}(\gamma+ab)}x^2 \qquad (101,\ 88)$$

$$=Bx^2\sum_{a,b\le x^\delta}\frac{1}{ab}\sum_{\gamma\le x^{\frac14}}1+Bx^2\sum_{a,b\le x^\delta}1\cdot\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}$$

(155) $$=Bx^{\frac94+\delta}+Bx^{2+3\delta}=Bx^{\frac52} \qquad (3).$$

(156) $$\mathfrak{S}_3(x)=\sum_{E_1,E_2=0}^{1}2^{\nu(E_1+E_2-2)}\mathfrak{C}_{E_1,E_2}\sum_{\substack{a,b\le x^\delta\\ b<2^{\frac{\nu-\mu}{2}}a}}\frac{1}{abg}\sum_{\gamma\le x^{\frac14}}\frac{1}{\gamma}$$

$$\times\sum_{x^{\frac49}\le h\le X}m^4\left(\frac{n}{m}\right)^{2E_2}\sin\frac{2\pi m\gamma g}{2^\nu n}+Bx^{\frac52} \qquad (153,\ 155).$$

(157) $$\left(b<2^{\frac{\nu-\mu}{2}}a\right) \qquad \beta<2^{\frac{\mu-\nu}{2}}\alpha \qquad (104).$$

(158) $$\frac{n}{m}=B\left(\frac{\beta}{\alpha}+\frac{\gamma}{\alpha^2h}\right)=B\left(\beta+\frac{\gamma}{h}\right)=B\left(\beta+x^{\frac14-\frac49}\right)=B\beta \qquad (143,\ 153).$$

(159) $$\left(\frac{n}{m}\right)^{2E_2}-\left(\frac{\beta}{\alpha}\right)^{2E_2}=B\left(\left(\frac{n}{m}\right)^{2}-\left(\frac{\beta}{\alpha}\right)^{2}\right)=B\frac{\beta\gamma}{\alpha^2 h}=B\frac{\gamma}{\alpha h}$$ (3, 143, 158, 157),

(160) $$m^4-\alpha^4 h^4=B(m-\alpha h)m^3=B\,m' x^{\frac{3}{2}}=B\,\alpha\, x^{\frac{3}{2}}$$ (108, 101, 110),

$$\frac{2\pi m\gamma g}{2^\nu n}-\frac{2\pi\gamma a}{\beta}=B\frac{\gamma^2 g}{\beta^2 h}$$ (143, 104),

(161) $$\sin\frac{2\pi m\gamma g}{2^\nu n}-\sin\frac{2\pi\gamma a}{\beta}=B\,\mathrm{Min}\left(\frac{\gamma^2 g}{\beta^2 h};1\right)=B\frac{\gamma}{\beta}\sqrt{\frac{g}{h}}\,.$$

$$m^4\left(\frac{n}{m}\right)^{2E_2}\sin\frac{2\pi m\gamma g}{2^\nu n}-\alpha^4 h^4\left(\frac{\beta}{\alpha}\right)^{2E_1}\sin\frac{2\pi\gamma a}{\beta}$$

$$=\alpha^4 h^4\left(\frac{\beta}{\alpha}\right)^{2E_3}\left(\sin\frac{2\pi m\gamma g}{2^\nu n}-\sin\frac{2\pi\gamma a}{\beta}\right)$$

$$+(m^4-\alpha^4 h^4)\left(\frac{\beta}{\alpha}\right)^{2E_2}\sin\frac{2\pi m\gamma g}{2^\nu n}+m^4\left(\left(\frac{n}{m}\right)^{2E_2}-\left(\frac{\beta}{\alpha}\right)^{2E_2}\right)\sin\frac{2\pi m\gamma g}{2^\nu n}$$

(162) $$=B\,\alpha^4 h^{\frac{7}{2}}\gamma\, g^{\frac{1}{2}}\beta^{-1}+B\,\alpha\, x^{\frac{3}{2}}+B\,x^2\frac{\gamma}{\alpha h}$$ (157, 161, 160, 101, 159).

Ersetzt man in der h-Summe von (156)

$$m^4\left(\frac{n}{m}\right)^{2E_2}\sin\frac{2\pi m\gamma g}{2^\nu n}\quad\text{durch}\quad \alpha^4 h^4\left(\frac{\beta}{\alpha}\right)^{2E_2}\sin\frac{2\pi\gamma a}{\beta},$$

so ändert sich $\mathfrak{S}_3(x)$ nach (162) um

$$B\sum_{a,\,b\le x^\delta}\frac{1}{abg}\sum_{\gamma\le x^{\frac{1}{4}}}\frac{1}{\gamma}\sum_{h\le X}\left(\alpha^4 h^{\frac{7}{2}}\gamma\, g^{\frac{1}{2}}\beta^{-1}+\alpha\, x^{\frac{3}{2}}+x^2\frac{\gamma}{\alpha h}\right)$$

$$=B\sum_{a,\,b\le x^\delta}\frac{\alpha^4}{ab\beta\sqrt{g}}\sum_{\gamma\le x^{\frac{1}{4}}}1\cdot\sum_{h\le\frac{\sqrt{2^\nu x}}{\alpha}}h^{\frac{7}{2}}+B\,x^{\frac{3}{2}}\sum_{a,\,b\le x^\delta}\frac{\alpha}{ab}\sum_{\gamma\le x^{\frac{1}{4}}}\frac{1}{\gamma}\sum_{h\le\frac{\sqrt{2^\nu x}}{\alpha}}1$$

$$+B\,x^2\sum_{a,\,b\le x^\delta}\frac{1}{ab}\sum_{\gamma\le x^{\frac{1}{4}}}1\cdot\sum_{h\le\sqrt{2^\nu x}}\frac{1}{h}$$ (112)

$$=B\,x^{\frac{5}{2}}\sum_{a,\,b=1}^{\infty}\left(ab\,\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}g^{\frac{1}{2}}\right)^{-1}+B\,x^{2+\delta}+B\,x^{\frac{9}{4}+\delta}$$

(163) $$=B\,x^{\frac{5}{2}}\sum_{a,\,b=1}^{\infty}(ab)^{-\frac{5}{4}}+B\,x^{\frac{5}{2}}=B\,x^{\frac{5}{2}}$$ (104, 3).

$$\mathfrak{S}_3(x)=\sum_{E_1,\,E_2=0}^{1}2^{\nu(E_1+E_2-2)}\mathfrak{C}_{E_1,\,E_2}\sum_{\substack{a,\,b\le x^\delta\\ b<2^{\frac{\nu-\mu}{2}}a}}\frac{1}{abg}\sum_{\gamma\le x^{\frac{1}{4}}}\frac{1}{\gamma}$$

$$\times\sum_{x^{\frac{4}{9}}\le h\le X}\alpha^4 h^4\left(\frac{\beta}{\alpha}\right)^{2E_2}\sin\frac{2\pi\gamma a}{\beta}+B\,x^{\frac{5}{2}}$$ (156, 163)

(164) $$=\sum_{E_1,\,E_2=0}^{1}2^{\nu(E_1+E_2-2)}\mathfrak{C}_{E_1,\,E_2}\sum_{\substack{a,\,b\le x^\delta\\ b<2^{\frac{\nu-\mu}{2}}a}}\frac{1}{abg}\alpha^4\left(\frac{\beta}{\alpha}\right)^{2E_2}$$

$$\times\sum_{\gamma\le x^{\frac{1}{4}}}\frac{1}{\gamma}\sin\frac{2\pi\gamma a}{\beta}\sum_{x^{\frac{4}{9}}\le h\le X}h^4+B\,x^{\frac{5}{2}}\,.$$

Im weiteren Verlauf des Beweises von (154) soll

(165) $$x>2^{\frac{\mu+\nu+1}{\frac{1}{4}-\delta}}+3$$

angenommen werden (rechts steht eine absolute Konstante).

$$\sqrt{2^\mu}\,\alpha-\sqrt{2^\nu}\,\beta=\frac{2^\mu\alpha^2-2^\nu\beta^2}{\sqrt{2^\mu}\,\alpha+\sqrt{2^\nu}\,\beta}\ge\frac{1}{\sqrt{2^\mu}\,\alpha+\sqrt{2^\nu}\,\beta}$$ (157)

(166) $$\ge\frac{1}{2^{\mu+\nu}(a+b)}\ge\frac{1}{2^{\mu+\nu+1}x^\delta}$$ (104, 164).

$$\left(\gamma\le x^{\frac{1}{4}}\right)\qquad\frac{\sqrt{2^\mu x}-n'}{\beta}-\frac{\sqrt{2^\nu x}-m'}{\alpha}=\frac{(\sqrt{2^\mu}\,\alpha-\sqrt{2^\nu}\,\beta)\sqrt{x}-\gamma}{\alpha\beta}$$

$$\geqq \frac{\left(\sqrt{2^{\mu}}\,\alpha - \sqrt{2^{\nu}}\,\beta\right)\sqrt{x} - \sqrt[4]{x}}{\alpha\,\beta} \qquad \textbf{(109)}$$

(167) $$\geqq \frac{1}{\alpha\,\beta}\left(2^{-\mu-\nu-1}\,x^{\frac{1}{2}-\delta} - x^{\frac{1}{4}}\right) = \frac{x^{\frac{1}{4}}}{\alpha\,\beta}\left(2^{-\mu-\nu-1}\,x^{\frac{1}{4}-\delta} - 1\right) > 0$$

(166, 165),

(168) $$X = \frac{\sqrt{2^{\nu}\,x} - m'}{\alpha} \qquad \textbf{(112, 167),}$$

(169) $$\sum_{x^{\frac{4}{9}} \leqq h \leqq X} h^4 = \sum_{x^{\frac{4}{9}} \leqq h \leqq \frac{\sqrt{2^{\nu} x}}{\alpha}} h^4 + B\,\frac{x^2}{\alpha^4} \qquad \textbf{(168, 110).}$$

$$\mathfrak{S}_3(x) = \sum_{E_1, E_2 = 0}^{1} 2^{\nu(E_1+E_2-2)}\,\mathfrak{F}_{E_1, E_2} \sum_{\substack{a,\, b \leqq x^{\delta} \\ b < 2^{\frac{\nu-\mu}{2}} a}} \frac{1}{a\,b\,g}\,\alpha^4 \left(\frac{\beta}{\alpha}\right)^{2E_2}$$

$$\times \sum_{\gamma \leqq x^{\frac{1}{4}}} \frac{1}{\gamma} \sin\frac{2\pi\gamma a}{\beta} \cdot \sum_{x^{\frac{4}{9}} \leqq h \leqq \frac{\sqrt{2^{\nu} x}}{\alpha}} h^4 + B\,x^2 \sum_{a,\, b \leqq x^{\delta}} \frac{1}{a\,b} \sum_{\gamma \leqq x^{\frac{1}{4}}} \frac{1}{\gamma}$$

$$+ B\,x^{\frac{5}{2}} \qquad \textbf{(164, 169, 157)}$$

$$= B \sum_{\substack{a,\, b \leqq x^{\delta} \\ b < 2^{\frac{\nu-\mu}{2}} a}} \frac{1}{a\,b\,g}\,\alpha^4\,\frac{x^{\frac{5}{2}}}{\alpha^5} + B\,x^{\frac{5}{2}} \qquad \textbf{(75, 76, 2)}$$

$$= B\,x^{\frac{5}{2}} \sum_{\substack{a,\, b \leqq x^{\delta} \\ b < 2^{\frac{\nu-\mu}{2}} a}} \frac{1}{a^2\,b} + B\,x^{\frac{5}{2}} = B\,x^{\frac{5}{2}} \sum_{a,\, b = 1}^{\infty} (a\,b)^{-\frac{3}{2}} + B\,x^{\frac{5}{2}} = B\,x^{\frac{5}{2}} \qquad \textbf{(104).}$$

Hilfssatz 29:

(170) $$\mathfrak{S}_4(x) = \sum_{E_1, E_2 = 0}^{1} \mathfrak{F}_{E_1, E_2} \sum_{a,\, b \leqq x^{\delta}} \frac{1}{a\,b\,g} \sum_{x^{\frac{1}{4}} < \gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}} \frac{1}{\gamma}$$

$$\times \sum_{\substack{x^{\frac{4}{9}} \leqq h \leqq X \\ m \leqq 2^{\frac{\nu-\mu}{2}} n}} m^{2E_1}\,n^{2E_2}\,r^{2-E_1-E_2} \sin\frac{2\pi r\gamma g}{m\,n}$$

(171) $$= B\,x^{\frac{5}{2}}.$$

Beweis: Für $m \leqq 2^{\frac{\nu-\mu}{2}} n$ ist $r = \frac{n^2}{2^{\mu}}$ **(88)**. Nach (170) ist also

(172) $$\mathfrak{S}_4(x) = \sum_{E_1, E_2 = 0}^{1} 2^{\mu(E_1+E_2-2)}\,\mathfrak{F}_{E_1, E_2} \sum_{a,\, b \leqq x^{\delta}} \frac{1}{a\,b\,g} \sum_{x^{\frac{1}{4}} < \gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}} \frac{1}{\gamma}$$

$$\times \sum_{\substack{x^{\frac{4}{9}} \leqq h \leqq X \\ m \leqq 2^{\frac{\nu-\mu}{2}} n}} n^4 \left(\frac{m}{n}\right)^{2E_1} \sin\frac{2\pi n\gamma g}{2^{\mu}\,m}.$$

Ich mache wieder die Annahmen (119) und (120), setze jetzt aber überdies voraus, dass

(173) $$m \leqq 2^{\frac{\nu-\mu}{2}} n \text{ für } H \leqq h \leqq H'$$

ist. Es sei u eine beliebige Zahl des Intervalls $p \leqq u \leqq q$.

(174) $$f_2(u) = \frac{n_u\,\gamma\,g}{2^{\mu}\,m_u}, \quad F_2(u) = m_u^{2E_1}\,n_u^{4-2E_1},$$

(175) $$\omega_2 = f_2''(q).$$

Dann gilt

(176) $$f_2'(u) = \frac{(\beta\,m_u - \alpha\,n_u)\,\gamma\,g}{2^{\mu}\,m_u^2} = -\frac{\gamma^2\,g}{2^{\mu}\,m_u^2} \qquad \textbf{(174, 111, 109),}$$

(177) $$f_2''(u) = \frac{2\,\gamma^2\,\alpha\,g}{2^{\mu}\,m_u^3} \qquad \textbf{(176, 111).}$$

Mit wachsendem u nimmt $f_2''(u)$ ab **(177)**, da m_u wächst **(111)**. Also ist

$$\omega_2 = f_2''(q) \leqq f_2''(u) \leqq f_2''(p) = B\,f_2''(q) = B\,\omega_2 \qquad \textbf{(175, 177, 123).}$$

$F_2'(u)$ ist stetig und positiv (174, 111, 3); $p-\frac{1}{2}$, $q-\frac{1}{2}$ sind ganz, $0 \leqq p < q$ (120, 119). Also darf (116) auf die durch (120), (174), (175) gegebenen p, q, f_2, F_2, ω angewandt werden:

(178) $$\sum_{p \leqq h \leqq q} F_2(h) \sin\{2\pi f_2(h)\} = B F_2(q)\left(q\sqrt{\omega_2} + \frac{1}{\sqrt{\omega_2}}\right),$$

$$\sum_{h=H}^{H'} n^4 \left(\frac{m}{n}\right)^{2E_1} \sin\frac{2\pi n\gamma g}{2^\mu m} = B n_q^4 \left(\frac{m_q}{n_q}\right)^{2E_1}\left(q\sqrt{\omega_2} + \frac{1}{\sqrt{\omega_2}}\right) \quad \textbf{(178, 120, 174, 108)}$$

$$= B x^2 \left(\frac{m_H}{n_H}\right)^{2E_1}\left(H\gamma\alpha^{\frac{1}{2}} g^{\frac{1}{2}} m_q^{-\frac{3}{2}} + m_q^{\frac{1}{2}} m_q \gamma^{-1} \alpha^{-\frac{1}{2}} g^{-\frac{1}{2}}\right) \quad \textbf{(121, 122, 119, 120, 175, 177)}$$

$$= B x^2 \left(H\gamma\alpha^{\frac{1}{2}} g\,\alpha^{-\frac{3}{2}} x^{-\frac{2}{3}} + x^{\frac{1}{4}} \alpha H \gamma^{-1} \alpha^{-\frac{1}{2}}\right) \quad \textbf{(173, 108, 125, 121, 110)}$$

(179) $$= B x^{\frac{4}{3}} g\gamma H + B x^{\frac{9}{4}} \alpha^{\frac{1}{2}} \gamma^{-1} H.$$

Ist für ein gegebenes Tripel a, b, γ die h-Summe in (172) nicht leer, so läuft h über ein Intervall (Hilfssatz 23). Zerlege ich dieses in Teilintervalle $H \leqq h \leqq H'$ mit (119), (173) und wende (179) an, so folgt

$$\sum_{\substack{x^{\frac{4}{9}} \leqq h \leqq X \\ m \leqq 2^{\frac{\nu-\mu}{2}} n}} n^4 \left(\frac{m}{n}\right)^{2E_1} \sin\frac{2\pi n\gamma g}{2^\mu m} = B x^{\frac{4}{3}} g\gamma X + B x^{\frac{9}{4}} \alpha^{\frac{1}{2}} \gamma^{-1} X$$

$$= B x^{\frac{4}{3}} g\gamma x^{\frac{1}{2}} + B x^{\frac{9}{4}} \alpha^{\frac{1}{2}} \gamma^{-1} x^{\frac{1}{2}} \alpha^{-\frac{3}{4}} \beta^{-\frac{1}{4}} \quad \textbf{(113)}$$

(180) $$= B x^{\frac{11}{6}} g\gamma + B x^{\frac{11}{4}} \alpha^{-\frac{1}{4}} \beta^{-\frac{1}{4}} \gamma^{-1}.$$

$$\mathfrak{S}_4(x) = B x^{\frac{11}{6}} \sum_{a, b \leqq x^\delta} \frac{1}{ab} \sum_{\gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}} 1 + B x^{\frac{11}{4}} \sum_{a, b=1}^{\infty} (ab)^{-\frac{5}{4}} \sum_{\gamma > x^{\frac{1}{4}}} \gamma^{-2} \quad \textbf{(172, 180, 104)}$$

$$= B x^{\frac{5}{2}-\frac{1}{6}+2\delta} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}} \quad \textbf{(3).}$$

Hilfssatz 30:

(181) $$\mathfrak{S}_5(x) = \sum_{E_1, E_2=0}^{1} \mathfrak{C}_{E_1, E_2} \sum_{a, b \leqq x^\delta} \frac{1}{abg} \sum_{x^{\frac{1}{4}} < \gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}} \frac{1}{\gamma}$$

$$\times \sum_{\substack{x^{\frac{4}{9}} \leqq h \leqq X \\ m > 2^{\frac{\nu-\mu}{2}} n}} m^{2E_1} n^{2E_2} r^{2-E_1-E_2} \sin\frac{2\pi r\gamma g}{mn}$$

182) $$= B x^{\frac{5}{2}}.$$

Beweis: Für $m > 2^{\frac{\nu-\mu}{2}} n$ ist $r = \frac{m^2}{2^\nu}$ (88). Nach (181) ist also

(183) $$\mathfrak{S}_5(x) = \sum_{E_1, E_2=0}^{1} 2^{\nu(E_1+E_2-2)} \mathfrak{C}_{E_1, E_2} \sum_{a, b \leqq x^\delta} \frac{1}{abg} \sum_{x^{\frac{1}{4}} < \gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}} \frac{1}{\gamma}$$

$$\times \sum_{\substack{x^{\frac{4}{9}} \leqq h \leqq X \\ m > 2^{\frac{\nu-\mu}{2}} n}} m^4 \left(\frac{n}{m}\right)^{2E_2} \sin\frac{2\pi m\gamma g}{2^\nu n}.$$

Ich mache wieder die Annahmen (119) und (120), setze jetzt aber überdies voraus, dass

(184) $$m > 2^{\frac{\nu-\mu}{2}} n \text{ für } H \leqq h \leqq H'$$

ist. Es sei u eine beliebige Zahl des Intervalls $p \leqq u \leqq q$,

(185) $$f_3(u) = -\frac{m_u \gamma g}{2^\nu n_u}, \quad F_3(u) = n_u^{2E_2} m_u^{4-2E_2},$$

(186) $$\omega_3 = f_3''(q).$$

Dann gilt

(187) $$f_3'(u) = \frac{(\beta m_u - \alpha n_u)\gamma g}{2^\nu n_u^2} = -\frac{\gamma^2 g}{2^\nu n_u^2} \quad \textbf{(185, 111, 109),}$$

(188) $$f_3''(u) = \frac{2\gamma^2 \beta g}{2^\nu n_u^3} \quad \textbf{(187, 111).}$$

Mit wachsendem u nimmt $f_3''(u)$ ab (188), da n_u wächst (111). Also ist

$$\omega_3 = f_3''(q) \leqq f_3''(u) \leqq f_3''(p) = B f_3''(q) = B\,\omega_2 \quad (186, 188, 124).$$

$F_3'(u)$ ist stetig und positiv (185, 111, 3); $p-\frac{1}{2}$, $q-\frac{1}{2}$ sind ganz, $0 \leqq p < q$ (120, 119). Also darf (116) auf die durch (120), (185), (186) gegebenen p, q, f_3, F_3, ω_3 angewandt werden:

$$(189) \qquad \sum_{p \leqq h \leqq q} F_3(h) \sin\{2\pi f_3(h)\} = B F_3(q)\left(q\sqrt{\omega_3} + \frac{1}{\sqrt{\omega_3}}\right),$$

$$\sum_{h=H}^{H'} m^4 \left(\frac{n}{m}\right)^{2E_2} \sin\frac{2\pi m\gamma g}{2^\nu n} = B\, m_q^4 \left(\frac{n_q}{m_q}\right)^{2E_2}\left(q\sqrt{\omega_3} + \frac{1}{\sqrt{\omega_3}}\right) \quad (189, 120, 185, 108)$$

$$= B x^2 \left(\frac{n_H}{m_H}\right)^{2E_2}\left(H\gamma\beta^{\frac{1}{2}} g^{\frac{1}{2}} n_q^{-\frac{3}{2}} + n_q^{\frac{1}{2}} n_q \gamma^{-1} \beta^{-\frac{1}{2}} g^{-\frac{1}{2}}\right) \quad (121, 122, 119, 120, 186, 188)$$

$$= B x^2 \left(H\gamma\beta^{\frac{1}{2}} g \beta^{-\frac{3}{2}} x^{-\frac{2}{3}} + x^{\frac{1}{4}}(\gamma + \beta H)\gamma^{-1}\beta^{-\frac{1}{2}}\right) (184, 108, 126, 122, 110)$$

$$(190) \qquad = B x^{\frac{4}{3}} g\gamma H + B x^{\frac{9}{4}} + B x^{\frac{9}{4}} \beta^{\frac{1}{2}} \gamma^{-1} H.$$

Ist für ein gegebenes Tripel a, b, γ die h-Summe in (183) nicht leer, so läuft h über e i n Intervall (Hilfssatz 23). Zerlege ich dieses in Teilintervalle $H \leqq h \leqq H'$ mit (119), (184) und wende (190) an, so folgt

$$\sum_{\substack{x^{\frac{4}{9}} \leqq h \leqq X \\ m > 2^{\frac{\nu-\mu}{2}} n}} m^4 \left(\frac{n}{m}\right)^{2E_2} \sin\frac{2\pi m\gamma g}{2^\nu n} = B x^{\frac{4}{3}} g\gamma X + B x^{\frac{9}{4}+\delta} + B x^{\frac{9}{4}} \beta^{\frac{1}{2}} \gamma^{-1} X$$

$$= B x^{\frac{4}{3}} g\gamma x^{\frac{1}{2}} + B x^{\frac{9}{4}+\delta} + B x^{\frac{9}{4}} \beta^{\frac{1}{2}} \gamma^{-1} x^{\frac{1}{2}} \;\; {}^{\frac{1}{4}}\, \beta^{-\frac{3}{4}} \qquad (113)$$

$$(191) \qquad = B x^{\frac{11}{6}} g\gamma + B x^{\frac{9}{4}+\delta} + B x^{\frac{11}{4}} \alpha^{-\frac{1}{4}} \beta^{-\frac{1}{4}} \gamma^{-1}.$$

$$\mathfrak{S}_5(x) = B x^{\frac{11}{6}} \sum_{a,b \leqq x^\delta} \frac{1}{ab} \sum_{\gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}} 1 + B x^{\frac{9}{4}+\delta} \sum_{a,b \leqq x^\delta} \frac{1}{ab} \sum_{\gamma \leqq 2^{\mu+\nu} x^{\frac{1}{2}+\delta}} \frac{1}{\gamma}$$

$$+ B x^{\frac{11}{4}} \sum_{a,b=1}^{\infty} (ab)^{-\frac{5}{4}} \sum_{\gamma > x^{\frac{1}{4}}} \gamma^{-2} \qquad (183, 191, 104)$$

$$= B x^{\frac{5}{2}-\frac{1}{6}+2\delta} + B x^{\frac{9}{4}+2\delta} + B x^{\frac{5}{2}} = B x^{\frac{5}{2}} \qquad (3).$$

B e w e i s von (VI): (91), (114), (118), (139), (154), (171), (182).

B e w e i s von (II): (VI) mit $\nu = 0$ und (1).

B e w e i s von (I): Zur Abkürzung werde gesetzt

$$(192) \qquad \lambda = \frac{1}{2}(C + \log 2\pi - 1), \; \mu = \frac{36 + 5\pi^2}{432}.$$

$$\int_0^x T_0^2(u)\,du = \int_1^x T_0^2(u)\,du + B$$

$$= \int_1^x \left(\frac{T(u)}{u} - \lambda\right)^2 du + \int_1^x \left(T_0^2(u) - \left(\frac{T(u)}{u} - \lambda\right)^2\right) du + B$$

$$(193) = \int_1^x \frac{T^2(u)}{u^2}\,du - 2\lambda \int_1^x \frac{T(u)}{u}\,du + \lambda^2 x +$$

$$+ B \int_1^x \left|T_0(u) - \frac{T(u)}{u} + \lambda\right| \left|T_0(u) + \frac{T(u)}{u} - \lambda\right| du + B.$$

$$(194) \qquad \int_1^x \frac{T^2(u)}{u^2}\,du = \frac{1}{x^2}\int_1^x T^2(u)\,du + 2\int_1^x \frac{dv}{v^3}\int_1^v T^2(u)\,du = 3\mu x + B x^{\frac{1}{2}}$$

(II, 193).

$$(u \geqq 1) \quad -\frac{T(u)}{u} = \frac{1}{2} + \sum_{n \leqq \sqrt{u}} \frac{1}{n}\psi\left(\frac{u}{n}\right) + \frac{1}{u}\sum_{n \leqq \sqrt{u}} n\,\psi\left(\frac{u}{n}\right) + B u^{-\frac{1}{2}} \qquad (6),$$

$$-\int_1^x \frac{T(u)}{u}\,du = \frac{x}{2} + \sum_{n \leqq \sqrt{x}} \frac{1}{n}\int_{n^2}^x \psi\left(\frac{u}{n}\right) du + \sum_{n \leqq \sqrt{x}} n \int_{n^2}^x \psi\left(\frac{u}{n}\right)\frac{du}{u} + B x^{\frac{1}{2}}$$

$$(195)\qquad = \frac{x}{2} + \sum_{n \leqq \sqrt{x}} \frac{1}{n}\, n + \sum_{n \leqq \sqrt{x}} n\, \frac{1}{n^2}\, n + B x^{\frac{1}{2}} = \frac{x}{2} + B x^{\frac{1}{2}} \qquad (2).$$

$$\int_1^x \left| T_0(u) - \frac{T(u)}{u} + \lambda \right| \left| T_0(u) + \frac{T(u)}{u} - \lambda \right| d\,u = B \int_1^x u^{-\frac{2}{3}} \log u \, d\,u =$$

$$(196)\qquad = B x^{\frac{1}{2}} \qquad (4,\ 192).$$

$$\int_0^x T_0^2(u)\, d\,u = (3\mu + \lambda + \lambda^2)\, x + B x^{\frac{1}{2}} \qquad (193-196)$$

$$(\mathrm{I})\qquad = \left(\left(\frac{C + \log 2\pi}{2} \right)^2 + \frac{5\pi^2}{144} \right) x + B x^{\frac{1}{2}} \qquad (192).$$

Hilfssatz 31:

$$(197)\qquad \mathrm{P}_0(x) = 8\, T(x) - 32\, T\left(\frac{x}{4}\right),$$

$$(198)\qquad \mathrm{P}_1(x) = 4\, T(x) - 4\, T\left(\frac{x}{2}\right) + 8\, T\left(\frac{x}{4}\right) - 32\, T\left(\frac{x}{8}\right),$$

$$(199)\qquad \mathrm{P}_2(x) = 2\, T(x) - 2\, T\left(\frac{x}{2}\right) + 8\, T\left(\frac{x}{8}\right) - 32\, T\left(\frac{x}{16}\right).$$

Beweis: (197) ist T_{II} (106), (198) ist E_{V} (219), (199) ist E_{V} (220).

Beweis von (III) — (V):

$$(200)\qquad \int_0^x \mathrm{P}_0^2(u)\, d\,u = 2^6 V_0(x) + 2^{12} V_0\left(\frac{x}{4}\right) - 2^{11} V_2\left(\frac{x}{4}\right) + B \qquad (197,\ 1),$$

$$(201)\qquad \int_0^x \mathrm{P}_1^2(u)\, d\,u = 2^4 V_0(x) + 2^5 V_0\left(\frac{x}{2}\right) + 2^8 V_0\left(\frac{x}{4}\right) + 2^{13} V_0\left(\frac{x}{8}\right)$$

$$- 2^6 V_1\left(\frac{x}{2}\right) - 2^8 V_1\left(\frac{x}{4}\right) - 2^{12} V_1\left(\frac{x}{8}\right) + 2^8 V_2\left(\frac{x}{4}\right)$$

$$+ 2^{11} V_2\left(\frac{x}{8}\right) - 2^{11} V_3\left(\frac{x}{8}\right) + B \qquad (198,\ 1),$$

$$(202)\qquad \int_0^x \mathrm{P}_2^2(u)\, d\,u = 2^2 V_0(x) + 2^3 V_0\left(\frac{x}{2}\right) + 2^9 V_0\left(\frac{x}{8}\right) + 2^{14} V_0\left(\frac{x}{16}\right)$$

$$- 2^4 V_1\left(\frac{x}{2}\right) - 2^{13} V_1\left(\frac{x}{16}\right) - 2^8 V_2\left(\frac{x}{8}\right) + 2^8 V_3\left(\frac{x}{8}\right)$$

$$+ 2^{11} V_3\left(\frac{x}{16}\right) - 2^{11} V_4\left(\frac{x}{16}\right) + B \qquad (199,\ 1).$$

(III) — (V) folgen aus (200) — (202) und (VI).

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