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On the representations of a number as a sum of squares.

By

T. Estermann (London).

Introduction.

If $r_s(n)$ denotes the number of solutions of the equation

$$x_1^2+x_2^2+\ldots+x_s^2=n$$

in integers $x_1, x_2, \ldots, x_s$, and [1])

$$\vartheta_3(\tau)=\sum_{m=-\infty}^{\infty} e^{\pi i m^2 \tau} \qquad (\Im\tau>0), \tag{1}$$

then

$$\{\vartheta_3(\tau)\}^s=\sum_{n=0}^{\infty} r_s(n)\, e^{\pi i n \tau} \qquad (\Im\tau>0). \tag{2}$$

The object of this paper is to use (2) for the evaluation of $r_s(n)$ in the cases $s=5, 6, 7, 8$ in a more elementary way than has been done before [2]). Thus I hope to make the subject accessible even to those

[1]) Readers familiar with elliptic functions will perhaps prefer the notation $\vartheta_3(0|\tau)$, but the simpler notation $\vartheta_3(\tau)$ is sufficient for the present purpose.

[2]) Hardy, Trans. American Math. Soc. 21 (1920), 255—284, and Proc. Nat. Acad. of Sciences 4 (1918), 189—193.

Mordell, Quart. J. of. Math. 48 (1917), 93—104 and Trans. Camb. Phil. Soc. 22 (1919), 361—372.

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who know nothing of the theories of modular functions, theta functions, and Gaussian sums.

The main result of Part 1 is this:

THEOREM 1. *Let*

$$\xi_m = e^{2\pi i/m}, \tag{3}$$

$$A_k = \sum_h \left\{ \frac{1}{2k} \sum_{q=1}^{2k} \xi_{2k}^{hq^2} \right\}^s \xi_{2k}^{-nh}, \tag{4}$$

where h runs through all positive integers $\leqq 2k$ and prime to k, and

$$S(n) = \sum_{k=1}^{\infty} A_k. \tag{5}$$

Then, for any positive integer n,

$$r_s(n) = cn^{\frac{1}{2}s-1} S(n) \qquad (s = 5, 6, 7, 8), \tag{6}$$

where c depends only on s.

In Part 2 I obtain expressions for $S(n)$ in the cases[3]) $s = 8$ and $s = 5$ which, when substituted in (6), lead to the following two theorems:

THEOREM 2. *Let $\sigma_3(x)$ denote the sum*[4]) *of the cubes of the positive divisors of x. Then, for any positive integer n,*

$$r_8(n) = 16\,\sigma_3(n) - 32\,\sigma_3\left(\frac{1}{2}n\right) + 256\,\sigma_3\left(\frac{1}{4}n\right).$$

THEOREM 3. *Let*

$$R(l) = C_l \pi^{-2} l^{\frac{3}{2}} \sum_{m=1}^{\infty} \left(\frac{l}{m}\right) m^{-2}, \tag{7}$$

where $\left(\frac{l}{m}\right)$ is Jacobi's residue symbol[5]) *if $(m, 2l) = 1$, $\left(\frac{l}{m}\right) = 0$ otherwise, $C_l = 80$ if $l \equiv 0 \pmod 4$ or $\equiv 1 \pmod 8$, $C_l = 160$ if $l \equiv 2$ or $3 \pmod 4$, and $C_l = 112$ if $l \equiv 5 \pmod 8$. Then, for any positive integer n,*

[3]) Following Hardy, I have chosen these as typical, but my method can also be applied when s is 6 or 7.

[4]) If x is not an integer, it has no divisors. The sum is then 'empty' and interpreted as 0.

[5]) Usually denoted by $\left(\frac{l}{m}\right)$. The dotted line is used here to prevent confusion with the quotient of l and m.

$$r_5(n) = \sum_q R\left(\frac{n}{q^2}\right), \tag{8}$$

where q runs through those positive integers whose squares are divisors of n.

It follows easily from (8) that $R(l)$ is the number of *primitive* representations of l as a sum of 5 squares, i. e. the number of solutions of the equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = l$$

in integers x_1, x_2, x_3, x_4, x_5 with greatest common divisor 1.

None of these results are new, and for the general ideas underlying my proof of Theorem 1 I am greatly indebted to the papers quoted, especially the first, but I hope the publication of Part 1 is justified by the simplifications obtained in it. The method used in Part 2 is my own.

Part 1.

1·1. *Notation.*

1·11. x and y are real numbers, and τ is a number whose imaginary part is positive.

1·12. r is a rational number.

1·13. In $\sum_r \ldots$, r runs through all rational numbers. Similarly, in $\sum_{r \neq 0} \ldots$, $\sum_{0 < r \leqq 2} \ldots$, etc., r runs through all rational numbers satisfying the condition stated.

These sums are said to *exist* only if they are absolutely convergent. It follows that, if $\sum_{r \neq 0} f(r)$ exists, then

$$\sum_{r \neq 0} f(r) = \sum_{r \neq 0} f\left(-\frac{1}{r}\right), \tag{9}$$

and if $\sum_r f(r)$ exists, then

$$\sum_r f(-r) = \sum_r f(r) = \sum_{0 < r \leqq 2} \sum_{m=-\infty}^{\infty} f(r + 2m). \tag{10}$$

1·14. $\log z$ is the principal value of the logarithm of z, so that $-\pi < \Im \log z \leqq \pi$ $(z \neq 0)$.

z^α means $\exp(\alpha \log z)$.

On this definition, the equation $(z_1 z_2)^\alpha = z_1^\alpha z_2^\alpha$ is not always true, but it is true if $\Re z_1 > 0$, $\Re z_2 \geqq 0$, and $z_2 \neq 0$.

1·15. „$\lim\limits_{\Im \tau \to \infty} f(\tau) = l$" means „$\lim\limits_{y \to \infty} f(x + iy) = l$ for every x".

1·151. It easily follows that, if $\lim\limits_{\Im \tau \to \infty} f(\tau) = l$, $a > 0$, and b is any number, then $\lim\limits_{\Im \tau \to \infty} f(a\tau + b) = l$.

1·16. $f^s(\tau)$ is an abbreviation for $\{f(\tau)\}^s$.

1·17. $\bar{z} = \Re z - i \Im z$ (i. e. $\bar{z}$ is the conjugate complex number to z).

1·2. *Proof of Theorem 1.*

1·201. Let [6])

$$\vartheta_0(\tau) = \sum_{m=-\infty}^{\infty} (-1)^m e^{\pi i m^2 \tau} \tag{11}$$

and

$$\vartheta_2(\tau) = \sum_{m=-\infty}^{\infty} e^{\pi i \left(m + \frac{1}{2}\right)^2 \tau}. \tag{12}$$

Then, by (1), (11), and (12),

$$\vartheta_0(\tau + 1) = \vartheta_3(\tau), \quad \vartheta_3(\tau + 1) = \vartheta_0(\tau), \quad \vartheta_2(\tau + 1) = e^{\frac{1}{4}\pi i} \vartheta_2(\tau). \tag{13}$$

1·202. We have

$$\vartheta_3(\tau) = (-i\tau)^{-\frac{1}{2}} \vartheta_3\left(-\frac{1}{\tau}\right). \tag{14}$$

Many proofs of this formula are known. Here is the outline of one: It is sufficient to prove (14) in the case $\tau = i\eta$, $\eta > 0$, when it reduces to

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2 \eta} = \eta^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/\eta}. \tag{15}$$

Now the residue of the function $f(z) = e^{-\pi z^2 \eta} \cot \pi z$ at $z = m$ is $\pi^{-1} e^{-\pi m^2 \eta}$. It easily follows that

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2 \eta} = \frac{1}{2i} \int_{-i-\infty}^{-i+\infty} f(z)\, dz + \frac{1}{2i} \int_{i+\infty}^{i-\infty} f(z)\, dz$$

$$= i \int_{i-\infty}^{i+\infty} f(z)\, dz = \int_{i-\infty}^{i+\infty} e^{-\pi z^2 \eta} \frac{e^{-\pi i z} + e^{\pi i z}}{e^{-\pi i z} - e^{\pi i z}}\, dz$$

[6]) Cf. footnote to formula (1).

$$= \int_{i-\infty}^{i+\infty} e^{-\pi z^2 \eta} \left\{1 + 2 \sum_{m=1}^{\infty} e^{2\pi i m z}\right\} dz = a_0 + 2 \sum_{m=1}^{\infty} a_m,$$

where

$$a_m = \int_{i-\infty}^{i+\infty} e^{-\pi(z^2 \eta - 2imz)}\, dz$$

$$= \int_{i-im/\eta-\infty}^{i-im/\eta+\infty} e^{-\pi w^2 \eta - \pi m^2/\eta}\, dw = \int_{-\infty}^{\infty} e^{-\pi w^2 \eta - \pi m^2/\eta}\, dw,$$

as is shown by the substitution $z = w + im/\eta$ and a subsequent application of Cauchy's theorem. Hence

$$a_m = e^{-\pi m^2/\eta} \int_{-\infty}^{\infty} e^{-\pi w^2 \eta}\, dw = c_0 \eta^{-\frac{1}{2}} e^{-\pi m^2/\eta},$$

where $c_0 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx$, and we obtain

$$\sum_{m=-\infty}^{\infty} e^{-\pi m^2 \eta} = a_0 + 2 \sum_{m=1}^{\infty} a_m = c_0 \eta^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/\eta}. \tag{16}$$

Since this holds, in particular, for $\eta = 1$, we have $c_0 = 1$, which, together with (16), proves (15). Incidentally, we have proved the well-known formula

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$

1·203. *We have*

$$\vartheta_2(\tau) = (-i\tau)^{-\frac{1}{2}} \vartheta_0\left(-\frac{1}{\tau}\right). \tag{17}$$

Proof. By (12) and (1),

$$\vartheta_2(\tau) = \sum_{m=-\infty}^{\infty} e^{\frac{1}{4}\pi i (2m+1)^2 \tau} = \sum_{n \text{ odd}} e^{\frac{1}{4}\pi i n^2 \tau}$$

$$= \sum_{n=-\infty}^{\infty} e^{\frac{1}{4}\pi i n^2 \tau} - \sum_{n\ \text{even}} e^{\frac{1}{4}\pi i n^2 \tau} = \vartheta_3\left(\frac{1}{4}\tau\right) - \sum_{m=-\infty}^{\infty} e^{\frac{1}{4}\pi i (2m)^2 \tau}$$

$$= \vartheta_3\left(\frac{1}{4}\tau\right) - \vartheta_3(\tau).$$

Hence, by (14), (1), and (11),

$$\vartheta_2(\tau) = \left(-\frac{1}{4}i\tau\right)^{-\frac{1}{2}} \vartheta_3\left(-\frac{4}{\tau}\right) - (-i\tau)^{-\frac{1}{2}} \vartheta_3\left(-\frac{1}{\tau}\right)$$

$$= (-i\tau)^{-\frac{1}{2}} \left\{2 \sum_{n=-\infty}^{\infty} e^{-\pi i (2n)^2/\tau} - \vartheta_3\left(-\frac{1}{\tau}\right)\right\}$$

$$= (-i\tau)^{-\frac{1}{2}} \left\{2 \sum_{m\ \text{even}} e^{-\pi i m^2/\tau} - \sum_{m=-\infty}^{\infty} e^{-\pi i m^2/\tau}\right\}$$

$$= (-i\tau)^{-\frac{1}{2}} \left\{\sum_{m\ \text{even}} e^{-\pi i m^2/\tau} - \sum_{m\ \text{odd}} e^{-\pi i m^2/\tau}\right\}$$

$$= (-i\tau)^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} (-1)^m e^{-\pi i m^2/\tau} = (-i\tau)^{-\frac{1}{2}} \vartheta_0\left(-\frac{1}{\tau}\right).$$

1 · 204. Let us call a function $\varphi(\tau)$ the comparison function of dimension $-\alpha$ or, more briefly, the c.f. $-\alpha$, of $f(\tau)$, if the following conditions hold:

(i) $\alpha > 0$.

(ii) $f(\tau)$ is regular for $\Im\tau > 0$.

(iii) There is a number L and a function $l(r)$, defined for every r (cf. 1 · 12), such that

(a) $\varphi(\tau) = L + \sum_r l(r)(ir - i\tau)^{-\alpha}$ for every τ (which implies the existence of the last sum as defined in 1 · 13),

(b) $\lim_{\Im\tau\to\infty} f(\tau) = L$, and

(c) $\lim_{\Im\tau\to\infty} \left\{(-i\tau)^{-\alpha} f\left(r - \frac{1}{\tau}\right)\right\} = l(r)$ for every r.

1 · 205. It is obvious that any function $f(\tau)$ cannot have more than one c.f. $-\alpha$ (for a given α).

1 · 206. Let $\varphi(\tau)$ be the c.f. $-\alpha$ of $f(\tau)$, and let a be a constant.

Then

(i) *$a\varphi(\tau)$ is the c. f. $-\alpha$ of $af(\tau)$,*

(ii) *$\varphi(\tau+1)$ is the c. f. $-\alpha$ of $f(\tau+1)$,*

and

(iii) *$(-i\tau)^{-\alpha}\varphi\left(-\frac{1}{\tau}\right)$ is the c. f. $-\alpha$ of $(-i\tau)^{-\alpha} f\left(-\frac{1}{\tau}\right)$.*

It may be left to the reader to prove (i) and (ii).

Proof of (iii). We are given that there is a number L and a function $l(r)$ such that

(18) $$L = \lim_{\Im\tau\to\infty} f(\tau),$$

(19) $$l(r) = \lim_{\Im\tau\to\infty} \left\{(-i\tau)^{-\alpha} f\left(r - \frac{1}{\tau}\right)\right\},$$

and

(20) $$\varphi(\tau) = L + \sum_r l(r)(ir - i\tau)^{-\alpha}$$

$$= L + l(0)(-i\tau)^{-\alpha} + \sum_{r\neq 0} l(r)(ir - i\tau)^{-\alpha}.$$

Putting

(21) $$f_1(\tau) = (-i\tau)^{-\alpha} f\left(-\frac{1}{\tau}\right),$$

we have to prove that there is a number L_1 and a function $l_1(r)$ such that

(22) $$L_1 = \lim_{\Im\tau\to\infty} f_1(\tau),$$

(23) $$l_1(r) = \lim_{\Im\tau\to\infty} \left\{(-i\tau)^{-\alpha} f_1\left(r - \frac{1}{\tau}\right)\right\},$$

and

(24) $$(-i\tau)^{-\alpha}\varphi\left(-\frac{1}{\tau}\right) = L_1 + \sum_r l_1(r)(ir - i\tau)^{-\alpha}.$$

Now, by (9),

$$\sum_{r\neq 0} l(r)(ir - i\tau)^{-\alpha} = \sum_{r\neq 0} l\left(-\frac{1}{r}\right)\left(-\frac{i}{r} - i\tau\right)^{-\alpha},$$

and hence, by (20),

(25) $$\varphi\left(-\frac{1}{\tau}\right) = l(0)\left(\frac{i}{\tau}\right)^{-\alpha} + L + \sum_{r\neq 0} l\left(-\frac{1}{r}\right)\left(-\frac{i}{r} + \frac{i}{\tau}\right)^{-\alpha}.$$

By 1 · 11 and 1 · 14,

$$(26)\qquad (-i\tau)^{-\alpha}\left(\frac{i}{\tau}\right)^{-\alpha}=1$$

and

$$(27)\qquad (-i\tau)^{-\alpha}\left(-\frac{i}{r}+\frac{i}{\tau}\right)^{-\alpha}=\left(-\frac{\tau}{r}+1\right)^{-\alpha}=\left(-\frac{i}{r}\right)^{-\alpha}(ir-i\tau)^{-\alpha}\quad(r\neq 0).$$

Hence, putting

$$(28)\qquad L_1=l(0),$$

$$(29)\qquad l_1(0)=L,$$

and

$$(30)\qquad l_1(r)=l\left(-\frac{1}{r}\right)\left(-\frac{i}{r}\right)^{-\alpha}\qquad (r\neq 0),$$

we have, by (25),

$$(-i\tau)^{-\alpha}\varphi\left(-\frac{1}{\tau}\right)=l(0)+L(-i\tau)^{-\alpha}+\sum_{r\neq 0} l\left(-\frac{1}{r}\right)\left(-\frac{i}{r}\right)^{-\alpha}(ir-i\tau)^{-\alpha}$$

$$=L_1+\sum_r l_1(r)(ir-i\tau)^{-\alpha},$$

which implies (24).

As to (22), it follows immediately from (28), (19), and (21).

It thus remains to prove (23). Now, by (21) and (26),

$$(-i\tau)^{-\alpha}f_1\left(-\frac{1}{\tau}\right)=(-i\tau)^{-\alpha}\left(\frac{i}{\tau}\right)^{-\alpha}f(\tau)=f(\tau),$$

and hence, by (29) and (18),

$$(31)\qquad l_1(0)=\lim_{\Im\tau\to\infty}\left\{(-i\tau)^{-\alpha}f_1\left(-\frac{1}{\tau}\right)\right\}.$$

Finally, if $r\neq 0$, by (21) and 1 · 14,

$$(32)\qquad (-i\tau)^{-\alpha}f_1\left(r-\frac{1}{\tau}\right)=(-i\tau)^{-\alpha}\left(-ir+\frac{i}{\tau}\right)^{-\alpha}f\left(\frac{-\tau}{r\tau-1}\right)$$

$$=(-r\tau+1)^{-\alpha}f\left(\frac{-\tau}{r\tau-1}\right)$$

$$=\left(-\frac{i}{r}\right)^{-\alpha}(-ir^2\tau+ir)^{-\alpha}f\left(-\frac{1}{r}-\frac{1}{r^2\tau-r}\right)$$

$$=\left(-\frac{i}{r}\right)^{-\alpha}g(r^2\tau-r),$$

where

$$(33)\qquad g(\tau)=(-i\tau)^{-\alpha}f\left(-\frac{1}{r}-\frac{1}{\tau}\right).$$

By (19) and (33),

$$l\left(-\frac{1}{r}\right)=\lim_{\Im\tau\to\infty} g(\tau),$$

and hence, by 1 · 151,

$$(34)\qquad l\left(-\frac{1}{r}\right)=\lim_{\Im\tau\to\infty} g(r^2\tau-r).$$

By (30), (34), and (32),

$$l_1(r)=\lim_{\Im\tau\to\infty}\left\{\left(-\frac{i}{r}\right)^{-\alpha}g(r^2\tau-r)\right\}=\lim_{\Im\tau\to\infty}\left\{(-i\tau)^{-\alpha}f_1\left(r-\frac{1}{\tau}\right)\right\}\qquad(r\neq 0),$$

which, together with (31), proves (23).

1 · 207. *Let $f(\tau)$ be such that*

$$(35)\qquad f(-\bar{\tau})=\overline{f(\tau)}$$

for every τ, and let $\varphi(\tau)$ be the c. f. $-\alpha$ of $f(\tau)$. Then

$$(36)\qquad \varphi(-\bar{\tau})=\overline{\varphi(\tau)}.$$

Proof. By 1 · 204,

$$(37)\qquad \varphi(\tau)=L+\sum_r l(r)(ir-i\tau)^{-\alpha},$$

where

$$(38)\qquad L=\lim_{\Im\tau\to\infty} f(\tau)$$

and

$$(39)\qquad l(r)=\lim_{\Im\tau\to\infty}\left\{(-i\tau)^{-\alpha}f\left(r-\frac{1}{\tau}\right)\right\}.$$

Now, by (37) and (10),

$$(40)\qquad \varphi(-\bar{\tau})=L+\sum_r l(-r)(-ir+i\bar{\tau})^{-\alpha}.$$

Also, by (38), (39), and 1 · 15,

$$(41)\qquad L=\lim_{y\to\infty} f(iy)$$

and

$$(42)\qquad l(r)=\lim_{y\to\infty}\left\{y^{-\alpha}f\left(r+\frac{i}{y}\right)\right\},$$

so that

$$(43)\qquad l(-r)=\lim_{y\to\infty}\left\{y^{-\alpha}f\left(-r+\frac{i}{y}\right)\right\}.$$

Using (35) with $\tau = r + i/y$ $(y > 0)$, we find that $f(r + i/y)$ and $f(-r + i/y)$ are conjugate complex numbers. Hence, by (42) and (43),

(44) $$l(-r) = \overline{l(r)}.$$

Similarly, using (35) with $\tau = iy$, we deduce from (41) that

(45) $$L = \overline{L}$$

(which, of course, means that L is real). Also $ir - i\tau$ and $-ir + i\overline{\tau}$ are conjugate complex numbers and, by 1 · 11, certainly not ≤ 0. Hence, by 1 · 14, $(ir - i\tau)^{-\alpha}$ and $(-ir + i\overline{\tau})^{-\alpha}$ are conjugate complex numbers. From this and (40), (37), (45), and (44) we obtain (36).

1 · 208. For any integers h, k, such that $k > 0$ and $(h, k) = 1$, let

(46) $$\lambda\left(\frac{h}{k}\right) = \frac{1}{2k}\sum_{q=1}^{2k} \xi_{2k}^{hq^2},$$

where ξ_{2k} is defined by (3). Then $\lambda(r)$ is defined for every r.

1 · 209. *We have*

(47) $$\left|\lambda\left(\frac{h}{k}\right)\right| \leq k^{-\frac{1}{2}} \qquad (k > 0,\ (h, k) = 1).$$

Proof. By (46), (3), and 1 · 17,

$$2k\lambda\left(\frac{h}{k}\right) = \sum_{q=1}^{2k} \xi_{2k}^{h(m+q)^2}$$

for any integer m, and

$$2k\overline{\lambda\left(\frac{h}{k}\right)} = \sum_{m=1}^{2k} \xi_{2k}^{-hm^2}.$$

Hence

$$4k^2\left|\lambda\left(\frac{h}{k}\right)\right|^2 = \sum_{m=1}^{2k} \xi_{2k}^{-hm^2} \sum_{q=1}^{2k} \xi_{2k}^{h(m+q)^2} = \sum_{q=1}^{2k} \xi_{2k}^{hq^2} \sum_{m=1}^{2k} \xi_k^{hmq}.$$

Observing that $\sum_{m=1}^{2k} \xi_k^{hmq}$ is equal to $2k$ or 0 according as q is or is not a multiple of k, we deduce from the last formula that

$$4k^2\left|\lambda\left(\frac{h}{k}\right)\right|^2 = 2k\left(\xi_{2k}^{hk^2} + \xi_{2k}^{h(2k)^2}\right) = 2k\left((-1)^{hk} + 1\right) \leq 4k,$$

which implies (47).

1.210. *We have*

(48) $$\lambda(r) = \lim_{\Im\tau \to \infty}\left\{(-i\tau)^{-\frac{1}{2}}\vartheta_3\left(r - \frac{1}{\tau}\right)\right\}.$$

Proof. Let

(49) $$r = \frac{h}{k}, \quad k > 0, \quad (h, k) = 1.$$

Then, by (1) and (3),

(50) $$(-i\tau)^{-\frac{1}{2}}\vartheta_3\left(r - \frac{1}{\tau}\right) = (-i\tau)^{-\frac{1}{2}} \sum_{q=1}^{2k} \sum_{m \equiv q \pmod{2k}} \xi_{2k}^{hm^2} e^{-\pi i m^2/\tau}$$

$$= \sum_{q=1}^{2k} \xi_{2k}^{hq^2} u_q,$$

where

(51) $$u_q = (-i\tau)^{-\frac{1}{2}} \sum_{m \equiv q \pmod{2k}} e^{-\pi i m^2/\tau}$$

$$= \sum_{m \equiv q \pmod{2k}} \int_{m^2}^{\infty} \pi(-i\tau)^{-\frac{3}{2}} e^{-\pi i v/\tau}\, dv$$

$$= \int_0^{\infty} \pi(-i\tau)^{-\frac{3}{2}} e^{-\pi i v/\tau}\, \Psi_q(v)\, dv,$$

and $\Psi_q(v)$ is the number of those integers m for which $m \equiv q \pmod{2k}$ and $m^2 \leq v$, so that

(52) $$\left|\Psi_q(v) - \frac{\sqrt{v}}{k}\right| \leq 1.$$

To evaluate the integral

$$\int_0^{\infty} \pi(-i\tau)^{-\frac{3}{2}} e^{-\pi i v/\tau} \sqrt{v}\, dv,$$

we put $v = -i\tau z$, and replace the new path of integration (a half-line in the half-plane $\Re z > 0$) by the positive real axis, which does not alter the value of the integral, as can be shown in a well-known way by means of Cauchy's theorem. Thus we obtain

(53) $$\int_0^{\infty} \pi(-i\tau)^{-\frac{3}{2}} e^{-\pi i v/\tau} \sqrt{v}\, dv = \int_0^{\infty} \pi e^{-\pi z} \sqrt{z}\, dz = \frac{1}{2}.$$

(Readers not familiar with the Γ-function may deduce the last equation from the formula at the end of $1\cdot202$.) By (51), (53), and (52),

$$\left|u_q-\frac{1}{2k}\right|=\left|\int_0^\infty \pi(-i\tau)^{-\frac{3}{2}}e^{-\pi i v/\tau}\left\{\Psi_q(v)-\frac{\sqrt{v}}{k}\right\}dv\right|$$

$$\leqq\int_0^\infty \pi|\tau|^{-\frac{3}{2}}\exp(-\pi v|\tau|^{-2}\Im\tau)\,dv=|\tau|^{\frac{1}{2}}(\Im\tau)^{-1}.$$

Hence, by $1\cdot15$,

$$\lim_{\Im\tau\to\infty}u_q=\frac{1}{2k}.$$

From this and (50), (46), and (49) we obtain (48).

$1\cdot211$. Henceforth let $s\geqq5$. Then it easily follows from (47) that

$$\sum_r\lambda^s(r)(ir-i\tau)^{-\frac{1}{2}s}$$

exists for any τ. Also, by (1),

(54) $$\lim_{\Im\tau\to\infty}\vartheta_3(\tau)=1.$$

Put

(55) $$\varphi_3(\tau)=1+\sum_r\lambda^s(r)(ir-i\tau)^{-\frac{1}{2}s}.$$

Then, by $1\cdot204$, (54), and (48), $\varphi_3(\tau)$ is the c. f. $-\frac{1}{2}s$ of $\vartheta_3^s(\tau)$.

Put

(56) $$\varphi_0(\tau)=\varphi_3(\tau+1),\qquad \varphi_2(\tau)=(-i\tau)^{-\frac{1}{2}s}\varphi_0\left(-\frac{1}{\tau}\right).$$

Then, by $1\cdot206$, (13), and (17) $\varphi_0(\tau)$ and $\varphi_2(\tau)$ are the c. f. $-\frac{1}{2}s$ of $\vartheta_0^s(\tau)$ and $\vartheta_2^s(\tau)$ respectively.

$1\cdot212$. Putting

(57) $$g_q(\tau)=\varphi_q(\tau)\vartheta_q^{-s}(\tau)\qquad(q=0,2,3),$$

we have, by (13), (17), and (56),

(58) $$g_0(\tau)=g_3(\tau+1)$$

and

(59) $$g_2(\tau)=g_0\left(-\frac{1}{\tau}\right).$$

Also, by $1\cdot206$, $1\cdot211$, and (13), $\varphi_0(\tau+1)$ is the c. f. $-\frac{1}{2}s$ of $\vartheta_3^s(\tau)$. Hence, by $1\cdot205$,

(60) $$\varphi_0(\tau+1)=\varphi_3(\tau).$$

Similarly, by $1\cdot206$, $1\cdot211$, and (14), $(-i\tau)^{-\frac{1}{2}s}\varphi_3\left(-\frac{1}{\tau}\right)$ is the c. f. $-\frac{1}{2}s$ of $\vartheta_3^s(\tau)$, and hence, by $1\cdot205$,

(61) $$(-i\tau)^{-\frac{1}{2}s}\varphi_3\left(-\frac{1}{\tau}\right)=\varphi_3(\tau).$$

By (57), (60), and (13),

(62) $$g_0(\tau+1)=g_3(\tau).$$

By (57), (61), and (14),

(63) $$g_3\left(-\frac{1}{\tau}\right)=g_3(\tau).$$

Also, by $1\cdot206$ and $1\cdot211$, $\varphi_2(\tau+1)$ is the c. f. $-\frac{1}{2}s$ of $\vartheta_2^s(\tau+1)$, and $e^{\frac{1}{4}\pi is}\varphi_2(\tau)$ is the c. f. $-\frac{1}{2}s$ of $e^{\frac{1}{4}\pi is}\vartheta_2^s(\tau)$. Hence, by (13) and 1.205,

(64) $$\varphi_2(\tau+1)=e^{\frac{1}{4}\pi is}\varphi_2(\tau).$$

By (57), (64), and (13),

(65) $$g_2(\tau+1)=g_2(\tau).$$

Finally, on substituting $-\frac{1}{\tau}$ for τ in (59), we obtain

(66) $$g_2\left(-\frac{1}{\tau}\right)=g_0(\tau).$$

Put

(67) $$\begin{cases}F_1(\tau)=g_0(\tau)+g_2(\tau)+g_3(\tau),\\ F_2(\tau)=g_0(\tau)\,g_2(\tau)+g_0(\tau)\,g_3(\tau)+g_2(\tau)\,g_3(\tau),\\ F_3(\tau)=g_0(\tau)\,g_2(\tau)\,g_3(\tau).\end{cases}$$

Then, by (62), (65), and (58),

(68) $$F_q(\tau+1) = F_q(\tau) \qquad (q=1,2,3),$$

and, by (59), (66), and (63),

(69) $$F_q\left(-\frac{1}{\tau}\right) = F_q(\tau) \qquad (q=1,2,3).$$

1·213. *The functions $F_1(\tau)$, $F_2(\tau)$, and $F_3(\tau)$ are regular for $\Im\tau > \frac{1}{2}$.*

Proof. It is easily seen that any comparison function in the sense of 1·204 is regular throughout the half-plane $\Im\tau > 0$. Hence, by (67), (57), and 1·211, it is sufficient to prove that

(70) $$\vartheta_q(\tau) \neq 0 \qquad \left(q=0,2,3;\ \Im\tau > \frac{1}{2}\right).$$

Suppose, then,

$$\Im\tau > \frac{1}{2}.$$

Then, by (1) and (11),

$$|\vartheta_q(\tau) - 1| \leqq 2\sum_{m=1}^{\infty} |e^{\pi i m^2\tau}| < 2\sum_{m=1}^{\infty} e^{-\frac{1}{2}\pi m^2}$$

$$< 2\sum_{m=1}^{\infty}\left(\frac{1}{3}\right)^m = 1 \qquad (q=0,3),$$

and hence

$$\vartheta_q(\tau) \neq 0 \qquad (q=0,3).$$

Also, by (12),

$$\left|e^{-\frac{1}{4}\pi i\tau}\vartheta_2(\tau) - 2\right| = 2\left|\sum_{m=1}^{\infty} e^{\pi i(m^2+m)\tau}\right|$$

$$< 2\sum_{m=1}^{\infty} e^{-\frac{1}{2}\pi(m^2+m)} < 1,$$

so that $\vartheta_2(\tau) \neq 0$, and (70) is proved.

1·214. *We have*

(71) $$F_q(-\bar\tau) = \overline{F_q(\tau)} \qquad (q=1,2,3).$$

Proof. By (1), (11), and (12),

(72) $$\vartheta_q(-\bar\tau) = \overline{\vartheta_q(\tau)} \qquad (q=0,2,3).$$

Hence, by 1·207 and 1·211,

$$\varphi_q(-\bar\tau) = \overline{\varphi_q(\tau)} \qquad (q=0,2,3).$$

From this and (72) and (57) we obtoin

$$g_q(-\bar\tau) = \overline{g_q(\tau)} \qquad (q=0,2,3),$$

which, together with (67), proves (71).

1·215. *Let*

(73) $$G_q(z) = F_q\left(\frac{1}{2\pi i}\log z\right) \qquad (q=1,2,3).$$

Then $G_q(z)$ is regular for $0 < |z| < e^{-\pi}$.

This follows from (68) and 1·213.

1·216. *Let $\alpha > 0$, let $\sum_r l(r)(ir - i\tau)^{-\alpha}$ exist for $\tau = i$, and let (U) be an abbreviation for*

$$\text{„uniformly for } -\frac{1}{2} < x \leqq \frac{1}{2}\text{"}.$$

Then

(74) $$\lim_{y\to\infty}\sum_r l(r)\{ir - i(x+iy)\}^{-\alpha} = 0 \qquad (U).$$

Proof. Put

(75) $$\sum_r |l(r)|\,|ir+1|^{-\alpha} = c_1,$$

which is permissible by 1·13. Then

$$\lim_{a\to\infty}\sum_{|r|\leqq a} |l(r)|\,|ir+1|^{-\alpha} = c_1,$$

and hence, by (75),

(76) $$\lim_{a\to\infty}\sum_{|r|>a} |l(r)|\,|ir+1|^{-\alpha} = 0.$$

Now let ε be any positive number. Then, by (76), there is an a such that

(77) $$\sum_{|r|>a} |l(r)|\,|ir+1|^{-\alpha} < 2^{-\alpha-1}\varepsilon.$$

Let $y \geqq 1$ and $-\frac{1}{2} < x \leqq \frac{1}{2}$. Then

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$$|ir-ix+y| \geqq |ir+1| - \frac{1}{2} \geqq \frac{1}{2}|ir+1|,$$

and hence

$$|ir-i(x+iy)|^{-\alpha} \leqq 2^{\alpha}|ir+1|^{-\alpha},$$

so that, by (77),

(78) $$\left|\sum_{|r|>a} l(r)\{ir-i(x+iy)\}^{-\alpha}\right| < \frac{1}{2}\varepsilon.$$

Also, if $|r| \leqq a$ and $y>0$, then

$$|ir-ix+y|^{-\alpha} \leqq y^{-\alpha} \leqq y^{-\alpha}|ir+1|^{-\alpha}(a+1)^{\alpha},$$

so that, by (75),

$$\left|\sum_{|r|\leqq a} l(r)\{ir-i(x+iy)\}^{-\alpha}\right| \leqq y^{-\alpha}(a+1)^{\alpha}\sum_{|r|\leqq a}|l(r)|\;|ir+1|^{-\alpha}$$
$$\leqq y^{-\alpha}(a+1)^{\alpha}c_1.$$

Hence there is a $y_0 \geqq 1$ such that

$$\left|\sum_{|r|\leqq a} l(r)\{ir-i(x+iy)\}^{-\alpha}\right| \leqq \frac{1}{2}\varepsilon \qquad (y \geqq y_0).$$

From this and (78) it follows that

$$\left|\sum_{r} l(r)\{ir-i(x+iy)\}^{-\alpha}\right| < \varepsilon \qquad (y \geqq y_0).$$

We have thus established the following result:

To every positive ε there is a y_0 such that, for every x satisfying $-\frac{1}{2} < x \leqq \frac{1}{2}$ and every $y \geqq y_0$, we have

$$\left|\sum_{r} l(r)\{ir-i(x+iy)\}^{-\alpha}\right| < \varepsilon.$$

Formula (74) is, of course, only a shorter enunciation of this result.

1·217. *Let*

$$\lim_{\Im\tau\to\infty} f(\tau) = L,$$

and let $\varphi(\tau)$ *be the c. f. —* α *of* $f(\tau)$. *Then*

$$\lim_{y\to\infty} \varphi(x+iy) = L \qquad (U).$$

This follows from 1·204 and 1·216.

1·218. *Henceforth let* $s \leqq 8$, *so that* s *is now restricted to the values 5, 6, 7, and 8. Then the three functions* $G_q(z)$ *defined by (73) are regular also at the origin.*

Proof. It is sufficient to prove that

$$\lim_{z\to 0}\{z\,G_q(z)\} = 0.$$

This is equivalent to

(79) $$\lim_{y\to\infty}\{e^{2\pi i(x+iy)}\,G_q(e^{2\pi i(x+iy)})\} = 0 \qquad (U).$$

Hence, by (73), it is sufficient to prove that

(80) $$\lim_{y\to\infty}\{e^{-2\pi y}F_q(x+iy)\} = 0 \qquad (U).$$

Now, by (11) and (1),

$$\lim_{y\to\infty}\vartheta_0(x+iy) = \lim_{y\to\infty}\vartheta_3(x+iy) = 1 \qquad (U),$$

and hence, by 1·211, 1·217, and (57),

(81) $$\lim_{y\to\infty} g_0(x+iy) = \lim_{y\to\infty} g_3(x+iy) = 1 \qquad (U).$$

Also, by (12),

(82) $$\lim_{y\to\infty}\left\{e^{-\frac{1}{4}\pi i(x+iy)}\,\vartheta_2(x+iy)\right\} = 2 \qquad (U)$$

and $\lim\limits_{\Im\tau\to\infty}\vartheta_2(\tau) = 0$, so that, by 1·211 and 1·217,

(83) $$\lim_{y\to\infty}\varphi_2(x+iy) = 0 \qquad (U).$$

By (57), (82), and (83),

$$\lim_{y\to\infty}\left\{e^{\frac{1}{4}s\pi i(x+iy)}\,g_2(x+iy)\right\} = 0 \qquad (U),$$

which means that

$$\lim_{y\to\infty}\left\{e^{-\frac{1}{4}s\pi y}\,g_2(x+iy)\right\} = 0 \qquad (U).$$

Since $s \leqq 8$, it follows that

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$$\lim_{y\to\infty} \{ e^{-2\pi y} g_2(x+iy) \} = 0 \qquad (U). \tag{84}$$

From (67), (81), and (84) we obtain (80).

1·219. Let the set A consist of the origin and those points z for which $|z| < 1$ and $|\log z| \geqq 2\pi$. Then it is easily seen that A is closed and contained in the circle $|z| < e^{-\pi}$, that it contains the circle $|z| < e^{-2\pi}$, and that its boundary consists of those points z for which $|z| < 1$ and $|\log z| = 2\pi$.

1·220. *Let z be any point on the boundary of A. Then $G_q(z)$ is real* $(q = 1, 2, 3)$.

Proof. By the last part of 1·219, $|z| < 1$ and $|\log z| = 2\pi$. Hence the number

$$\tau = \frac{1}{2\pi i} \log z$$

satisfies 1·11 and

$$|\tau| = 1. \tag{85}$$

Also, by (73),

$$G_q(z) = F_q(\tau). \tag{86}$$

Now, by (85), $-\dfrac{1}{\tau} = -\bar{\tau}$, and hence, by (69) and (71), $F_q(\tau) = F_q\left(-\dfrac{1}{\tau}\right)$ $= F_q(-\bar{\tau}) = \overline{F_q(\tau)}$, which implies that $F_q(\tau)$ is real, Hence, by (86), $G_q(z)$ is real.

1·221. *Let D_1 and D_2 be domains, let E be a closed bounded set contained in D_1 and containing D_2, and let $f(z)$ be regular in D_1 and real on the boundary of E. Then $f(z)$ is a constant.*

Proof. The imaginary part of a regular function, considered in a closed bounded set, assumes its maximum and its minimum on the boundary of the set. Since $\Im f(z) = 0$ on the boundary of E, it follows that $\Im f(z) = 0$ throughout E. Hence $f(z)$ is real throughout the domain D_2, and this implies the result stated.

1·222. *$G_1(z)$, $G_2(z)$, and $G_3(z)$ are constants.*

Proof. We apply 1·221, taking for E the set A of 1·219 and for D_1 and D_2 the circles $|z| < e^{-\pi}$ and $|z| < e^{-2\pi}$ respectively. Then, by 1·215, 1·218, and 1·220, $G_q(z)$ is regular in D_1 and real on the boundary of E. Hence, by 1·221, $G_q(z)$ is a constant.

1·223. $g_3(\tau) = 1$.

Proof. It follows from (67) that $g_3(\tau)$ is a root of the cubic

$$u^3 - F_1(\tau)u^2 + F_2(\tau)u - F_3(\tau) = 0.$$

By (73) and 1·222, this cubic has constant coefficients. Hence $g_3(\tau)$ is a constant, and it follows from (81) that this constant is 1.

1·224. $\vartheta_3^s(\tau) = \varphi_3(\tau)$.

This follows from (57) nnd 1·223.

1·225. By (55) and (10),

$$\varphi_3(\tau) = 1 + \sum_{0<r\leqq 2} \sum_{q=-\infty}^{\infty} \lambda^s(r+2q)(ir+2iq-i\tau)^{-\frac{1}{2}s}. \tag{87}$$

Now it follows from (48) and (1) that $\lambda(r+2q) = \lambda(r)$ for any integer q. Hence, putting

$$F(\tau) = \sum_{q=-\infty}^{\infty} (2iq - i\tau)^{-\frac{1}{2}s}, \tag{88}$$

we have, by (87),

$$\varphi_3(\tau) = 1 + \sum_{0<r\leqq 2} \lambda^s(r) F(\tau - r). \tag{89}$$

It easily follows from (88) that $F(\tau)$ has period 2 and that $\lim\limits_{y\to\infty} F(x+iy) = 0$ uniformly in x. Hence

$$F(\tau) = \sum_{n=1}^{\infty} b_n e^{\pi i n\tau}, \tag{90}$$

where

$$b_n = \frac{1}{2} \int_{\tau_0}^{\tau_0+2} F(\tau) e^{-\pi i n\tau}\, d\tau, \tag{91}$$

τ_0 being any number in the upper half-plane. Taking, in particular, $\tau_0 = i/n$, we obtain from (91) and (88)

$$b_n = \frac{1}{2} \int_{i/n}^{i/n+2} \sum_{q=-\infty}^{\infty} (2iq - i\tau)^{-\frac{1}{2}s} e^{-\pi i n\tau}\, d\tau \tag{92}$$

$$= \frac{1}{2} \sum_{q=-\infty}^{\infty} \int_{i/n}^{i/n+2} (2iq - i\tau)^{-\frac{1}{2}s} e^{-\pi i n(\tau - 2q)}\, d\tau$$

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$$=\frac{1}{2}\sum_{q=-\infty}^{\infty}\int_{i/n-2q}^{i/n-2q+2}(-iz)^{-\frac{1}{2}s}e^{-\pi inz}\,dz$$

$$=\frac{1}{2}\int_{i/n-\infty}^{i/n+\infty}(-iz)^{-\frac{1}{2}s}e^{-\pi inz}\,dz=c\,n^{\frac{1}{2}s-1},$$

where

(93) $$c=\frac{1}{2}\int_{i-\infty}^{i+\infty}(-iw)^{-\frac{1}{2}s}e^{-\pi iw}\,dw.$$

By (89), (90), and (92),

(94) $$\varphi_3(\tau)=1+c\sum_{0<r\leqq 2}\lambda^s(r)\sum_{n=1}^{\infty}n^{\frac{1}{2}s-1}e^{\pi in(\tau-r)}$$

$$=1+c\sum_{n=1}^{\infty}n^{\frac{1}{2}s-1}e^{\pi in\tau}\sum_{0<r\leqq 2}\lambda^s(r)\,e^{-\pi inr}.$$

(The inversion of the order of the summations is justified by (47), since $s\geqq 5$). From (2), (94), and $1\cdot 224$ we obtain, on equating the coefficients of $e^{\pi in\tau}$,

(95) $$r_s(n)=c\,n^{\frac{1}{2}s-1}\sum_{0<r\leqq 2}\lambda^s(r)\,e^{-\pi inr}\qquad(n=1,2,\ldots).$$

$1\cdot 226$. By $1\cdot 13$ and (3),

$$\sum_{0<r\leqq 2}\lambda^s(r)\,e^{-\pi inr}=\sum_{k=1}^{\infty}\sum_{h}\lambda^s\left(\frac{h}{k}\right)\xi_{2k}^{-nh},$$

where h runs through the same values as in (4). From this and (46), (4), and (5) it follows that

$$\sum_{0<r\leqq 2}\lambda^s(r)\,e^{-\pi inr}=S(n),$$

which, together with (95), proves (6).

Theorem 1 is thus established.

Part 2.

$2\cdot 1$. *Evaluation of* $\lambda^4(r)$.

$2\cdot 11$. *We have*

(96) $$\lambda(0)=1,\ \lambda(1)=0,$$

(97) $$\lambda(r+2)=\lambda(r),$$

and

(98) $$\lambda^2\left(-\frac{1}{r}\right)=(i\,r)^{-1}\lambda^2(r)\qquad(r\neq 0).$$

(96) and (97) follow immediately from (46) and (3).

Proof of (98). By (48) and $1\cdot 151$,

$$\lambda^2(r)=\lim_{\Im\tau\to\infty}\left\{(-i\,r^{-2}\tau-ir^{-1})^{-1}\vartheta_3^2\left(r-\frac{1}{r^{-2}\tau+r^{-1}}\right)\right\}.$$

Hence, by (14) and (48),

$$(i\,r)^{-1}\lambda^2(r)=\lim_{\Im\tau\to\infty}\left\{\frac{r}{\tau+r}\vartheta_3^2\left(\frac{r\tau}{\tau+r}\right)\right\}$$

$$=\lim_{\Im\tau\to\infty}\left\{\frac{r}{\tau+r}\left(\frac{-i\,r\,\tau}{\tau+r}\right)^{-1}\vartheta_3^2\left(-\frac{\tau+r}{r\,\tau}\right)\right\}$$

$$=\lim_{\Im\tau\to\infty}\left\{(-i\,\tau)^{-1}\vartheta_3^2\left(-\frac{1}{r}-\frac{1}{\tau}\right)\right\}=\lambda^2\left(-\frac{1}{r}\right),$$

q. e. d.

$2\cdot 12$. *Let* α *be an aggregate of rational numbers, containing the numbers* 0 *and* 1, *and such that, to every* r *which it contains, it also contains the numbers* $r+2$ *and* $r-2$ *and, if* $r\neq 0$, *the number* $-\dfrac{1}{r}$. *Then* α *contains all rational numbers.*

Proof. Let $h(r)$ and $k(r)$ be the numerator and the denominator of r when expressed as a fraction in its lowest terms, the denominator being taken positive, so that

$$r=h(r)/k(r),\ (h(r),\ k(r))=1,\ k(r)>0.$$

Define an aggregate β of positive integers as follows:

The number n is to be in β if and only if there is an r, *not* in α, such that $|h(r)|+2k(r)=n$.

Suppose α does not contain all rational numbers. Then β is not empty, and so β has a least member n_0, say. There is an r_0, not in α, such that

$$|h(r_0)|+2k(r_0)=n_0.$$

Now 0, 1, and -1 are in α, so that $|r_0|$ is neither 0 nor 1. Put

$$r_1=\begin{cases} r_0-2 \text{ if } r_0>1, \\ r_0+2 \text{ if } r_0<-1, \\ -1/r_0 \text{ if } 0<|r_0|<1. \end{cases}$$

and

$$n_1=|h(r_1)|+2k(r_1).$$

Then r_1 is not in α, and hence n_1 is in β. On the other hand, n_1 is less than n_0, the least member of β. This is a contradiction.

$2 \cdot 13$. *Let two functions $f_m(r)\ (m=1,2)$ be defined for every r and have the following properties:*

(99) $$f_1(0)=f_2(0),\ f_1(1)=f_2(1),$$

(100) $$f_m(r+2)=f_m(r),$$

and

(101) $$f_m\left(-\frac{1}{r}\right)=-r^{-2}f_m(r) \quad (r\neq 0).$$

Then

(102) $$f_1(r)=f_2(r)$$

for every r.

This follows from $2 \cdot 12$ on taking for α the aggregate of those numbers r for which (102) holds.

$2 \cdot 14$. *We have*

(103) $$\lambda^4(r)=\begin{cases} 0 & (2\nmid h(r)\,k(r)) \\ k^{-2}(r)(-1)^{k(r)-1} & (2\mid h(r)\,k(r)). \end{cases}$$

This follows from $2 \cdot 13$ on taking for $f_1(r)$ and $f_2(r)$ the two sides of (103), and applying $2 \cdot 11$.

$2 \cdot 2$. *Evaluation of $S(n)$ for $s=8$.*

$2 \cdot 21$. Henceforth h, k, l, m, n, q, u, and v denote positive integers, and t, x, and y denote integers.

$c_u(x)$ denotes the sum of the x-th powers of the primitive u-th roots of unity (Ramanujan's sum).

It follows that

$$\sum_{u\mid v} c_u(x)$$

is the sum of the x-th powers of all v-th roots of unity, so that

(104) $$\sum_{u\mid v} c_u(x)=\begin{cases} v & (v\mid x) \\ 0 & (v\nmid x). \end{cases}$$

If u is odd, and $\rho_1, \rho_2, \ldots, \rho_m$ are the primitive u-th roots of unity, it is easily seen that $-\rho_1, -\rho_2, \ldots, -\rho_m$ are the primitive $2u$-th roots of unity.

Hence

(105) $$c_{2u}(x)=(-1)^x c_u(x) \quad (2\nmid u).$$

$2 \cdot 22$. Let $(h,k)=1$. Then, by (103),

(106) $$\lambda^8\left(\frac{h}{k}\right)=\begin{cases} 0 & (2\nmid hk) \\ k^{-4} & (2\mid hk). \end{cases}$$

Also, by (4) and (46),

(107) $$A_k=\sum_{\substack{h\leqq 2k \\ (h,k)=1}} \lambda^8\left(\frac{h}{k}\right)(\xi_{2k}^{-h})^n.$$

It follows from (106), (107), and (3) that

(108) $$A_k=\begin{cases} k^{-4}c_k(n) & (2\nmid k) \\ k^{-4}c_{2k}(n) & (2\mid k). \end{cases}$$

Let

(109) $$S_1=\sum_{u=1}^{\infty} u^{-4}c_u(n),\quad S_2=\sum_{u\,\mathrm{odd}} u^{-4}c_u(n),\quad S_3=\sum_{4\mid u} u^{-4}c_u(n).$$

Then, by (5) and (108),

(110) $$S(n)=S_2+16\,S_3.$$

Also, by (109) and (105),

$$S_1-S_2-S_3=\sum_{v\equiv 2 \pmod 4} v^{-4}c_v(n)=\sum_{u\,\mathrm{odd}}(2u)^{-4}c_{2u}(n)=\frac{(-1)^n}{16}S_2,$$

and hence, by (110),

(111) $$S(n)=16\,S_1-15\,S_2-(-1)^n S_2.$$

$2 \cdot 23$. It remains to evaluate S_1 and S_2.

Let

(112) $$a=\sum_{v=1}^{\infty} v^{-4}.$$

Then

(113) $$\sum_{v\,\mathrm{odd}} v^{-4}=a-\sum_{u=1}^{\infty}(2u)^{-4}=\frac{15}{16}a.$$

By (109), (112), and (104),

(114) $$a\,S_1=\sum_{u,v}(uv)^{-4}c_u(n)=\sum_{q=1}^{\infty} q^{-4}\sum_{u\mid q} c_u(n)$$

$$=\sum_{q\mid n} q^{-3}=n^{-3}\sigma_3(n).$$

Similarly, by (109), (113), and (104),

$$(115)\quad \frac{15}{16} a S_2 = \sum_{\substack{q \text{ odd} \\ q|n}} q^{-3} = \sum_{q|n} q^{-3} - \sum_{\substack{q \text{ even} \\ q|n}} q^{-3} = n^{-3}\sigma_3(n) - n^{-3}\sigma_3\left(\frac{1}{2}n\right).$$

Now, if n is odd, then $\sigma_3\left(\frac{1}{2}n\right) = 0$. Hence, by (111), (114), and (115),

$$(116)\quad \frac{15}{16} a n^3 S(n) = \begin{cases} \sigma_3(n) & (2 \nmid n) \\ -\sigma_3(n) + 16\,\sigma_3\left(\frac{1}{2}n\right) & (2 \mid n). \end{cases}$$

If n is even, and $u_1, u_2, \ldots, u_q$ are all those positive divisors of $\frac{1}{2}n$ which do not divide $\frac{1}{4}n$, then $2u_1, 2u_2, \ldots, 2u_q$ are all those positive divisors of n which do not divide $\frac{1}{2}n$. Hence

$$\sigma_3(n) - \sigma_3\left(\frac{1}{2}n\right) = \sum_{m=1}^{q} (2u_m)^3 = 8\sum_{m=1}^{q} u_m^3$$

$$= 8\,\sigma_3\left(\frac{1}{2}n\right) - 8\,\sigma_3\left(\frac{1}{4}n\right) \qquad (2 \mid n),$$

and hence, by (116),

$$(117)\quad \frac{15}{16} a n^3 S(n) = \sigma_3(n) - 2\,\sigma_3\left(\frac{1}{2}n\right) + 16\,\sigma_3\left(\frac{1}{4}n\right).$$

From this and (6) we obtain

$$(118)\quad r_8(n) = a_1\left\{\sigma_3(n) - 2\,\sigma_3\left(\frac{1}{2}n\right) + 16\,\sigma_3\left(\frac{1}{4}n\right)\right\},$$

where a_1 is a constant. Substituting 1 for n in this formula, we obtain $a_1 = 16$, which, together with (118), proves Theorem 2.

2·3. *Evaluation of $S(n)$ for $s = 5$.*

2·301. If k is odd, then, by (4), (46), (103), and (3),

$$(119)\quad A_k = \sum_{\substack{m \leq k \\ (m,k)=1}} \lambda^4\left(\frac{2m}{k}\right)\frac{1}{2k}\sum_{q=1}^{2k} \xi_{2k}^{2m(q^2-n)}$$

$$= k^{-3}\sum_{\substack{m \leq k \\ (m,k)=1}} \sum_{q=1}^{k} \xi_k^{m(q^2-n)} = k^{-3}\sum_{q=1}^{k} c_k(q^2-n),$$

in the notation introduced in 2·21.

Let

$$(120)\quad d_k(x) = \frac{1}{k}\sum_{q=1}^{k} c_k(q^2 - x)$$

and

$$(121)\quad \nu(m,t) = \sum_{\substack{q \leq m \\ q^2 \equiv t \pmod m}} 1$$

(which means that $\nu(m,t)$ is the number of solutions of the congruence $x^2 \equiv t \pmod m$). Then, by (119) and (120),

$$(122)\quad A_k = k^{-2} d_k(n) \qquad (2 \nmid k).$$

Similarly

$$(123)\quad A_k = -k^{-2} d_{2k}(n) \qquad (2 \mid k).$$

Now $c_k(q^2 - x)$, considered as a function of q, has period k. Hence it follows from (120) that, if $k \mid m$, then

$$d_k(x) = \frac{1}{m}\sum_{q=1}^{m} c_k(q^2 - x),$$

and hence, by (104) and (121),

$$(124)\quad \sum_{k \mid m} d_k(x) = \sum_{q=1}^{m} \frac{1}{m}\sum_{k \mid m} c_k(q^2 - x)$$

$$= \sum_{\substack{q \leq m \\ m \mid q^2 - x}} 1 = \nu(m, x).$$

2·302. *Let k be odd. Then*

$$(125)\quad d_{2k}(x) = 0.$$

Proof. It has been observed that $c_k(q^2 - x)$, considered as a function of q, has period k. From this, (120), (105), and the identity

$$\sum_{q=1}^{2k} f(q) = \sum_{q=1}^{k}\{f(q) + f(q+k)\}$$

we obtain

$$2k\,d_{2k}(x) = \sum_{q=1}^{2k} c_{2k}(q^2 - x) = \sum_{q=1}^{2k} (-1)^{q^2 - x} c_k(q^2 - x)$$

$$= (-1)^x \sum_{q=1}^{k} c_k(q^2 - x)\left\{(-1)^{q^2} + (-1)^{(q+k)^2}\right\},$$

and
$$(-1)^{q^2}+(-1)^{(q+k)^2}=0$$
since k is odd.

2·303. *We have*

(126) $$|d_u(n)| \leqq 2u^{\frac{1}{2}}.$$

Proof. It follows from (4), (46), and (47) that

(127) $$|A_k| \leqq 2k^{-\frac{3}{2}}.$$

From this and (122) we obtain (126) immediately if u is odd. If $4 \mid u$ it follows from (123) that
$$d_u(n) = -\left(\frac{1}{2}u\right)^2 A_{\frac{1}{2}u},$$
which, together with (127), again proves (126). Finally, if $u \equiv 2 \pmod 4$, it follows from 2·302 that $d_u(n)=0$. Thus (126) holds in all cases,

2·304. Let

(128) $$S_4=\sum_{u=1}^{\infty} u^{-2} d_u(n), \quad S_5=\sum_{u\,\mathrm{odd}} u^{-2} d_u(n).$$

These sums are absolutely convergent by (126), and it follows from 2·302 that

(129) $$S_4-S_5=\sum_{4\mid u} u^{-2} d_u(n)=\sum_{2\mid k}(2k)^{-2} d_{2k}(n).$$

By (5), (122), (123), (128), and (129),

(130) $$S(n)=S_5-4(S_4-S_5)=5S_5-4S_4.$$

Let

(131) $$a_2=\sum_{v=1}^{\infty} v^{-2}.$$

Then

(132) $$\sum_{v\,\mathrm{odd}} v^{-2}=a_2-\sum_{u=1}^{\infty}(2u)^{-2}=\frac{3}{4}a_2.$$

By (128), (131), and (124),

(133) $$a_2 S_4=\sum_{u,v}(uv)^{-2} d_u(n)=\sum_{m=1}^{\infty} m^{-2}\sum_{u\mid m} d_u(n)$$
$$=\sum_{m=1}^{\infty} m^{-2}\nu(m,n).$$

Similarly, by (128), (132), and (124),

(134) $$\frac{3}{4}a_2 S_5=\sum_{m\,\mathrm{odd}} m^{-2}\nu(m,n).$$

2·305. A function $f(u)$ is said to be *multiplicative* if $f(uv)=f(u)f(v)$ whenever $(u,v)=1$. This notion will be used several times in the remainder of this paper.

Use will also be made of the following elementary lemmas:

(i) *If $f_1(u)$ and $f_2(u)$ are multiplicative, and*
$$f_3(u)=\sum_{\substack{q,v\\ qv=u}} f_1(q) f_2(v),$$
then $f_3(u)$ is multiplicative.

(ii) *If $(u,v)=1$, and $f(x)$ has period uv, then*
$$\sum_{q=1}^{uv} f(q)=\sum_{x=1}^{v}\sum_{y=1}^{u} f(ux+vy).$$

(iii) *If $(u,v)=1$, and $f(x)$ has period u, then*
$$\sum_{q=1}^{u} f(q)=\sum_{q=1}^{u} f(qv).$$

(iv) *If $f(x)$ has period m, then*
$$\sum_{q=1}^{km} f(q)=k\sum_{q=1}^{m} f(q).$$

2·306. *Let $(u,v)=1$. Then*

(135) $$\nu(uv,t)=\nu(u,t)\,\nu(v,t).$$

In other words: $\nu(u,t)$ is a multiplicative function of u.

Proof. Define the auxiliary function $g(x,t,m)$ as 1 if $x^2 \equiv t \pmod m$, and 0 otherwise. Then, by (121),

(136) $$\nu(m,t)=\sum_{q=1}^{m} g(q,t,m).$$

Hence, by lemma (ii) of 2·305,

(137) $$\nu(uv,t)=\sum_{x=1}^{v}\sum_{y=1}^{u} g(ux+vy,t,uv).$$

Now it follows from the definition of $g(x,t,m)$ that

(138) $$g(ux+vy,t,uv)=g(vy,t,u)\,g(ux,t,v),$$

and from lemma (iii) of $2\cdot305$ and (136) that

(139) $$\sum_{y=1}^{u} g(vy,t,u)=\sum_{q=1}^{u} g(q,t,u)=\nu(u,t)$$

and similarly

(140) $$\sum_{x=1}^{v} g(ux,t,v)=\nu(v,t).$$

From (137)—(140) we obtain (135).

$2\cdot307$. *We have*

(141) $$\nu(u^2m,u^2t)=u\,\nu(m,t).$$

Proof. By (136),

(142) $$\nu(u^2m,u^2t)=\sum_{q=1}^{u^2m} g(q,u^2t,u^2m).$$

Now $g(q,u^2t,u^2m)=0$ unless q is a multiple of u. Hence

(143) $$\sum_{q=1}^{u^2m} g(q,u^2t,u^2m)=\sum_{v=1}^{um} g(uv,u^2t,u^2m),$$

and it follows from the definition of $g(x,t,m)$ that

(144) $$g(uv,u^2t,u^2m)=g(v,t,m).$$

By (142), (143), (144), and lemma (iv) of $2\cdot305$,

$$\nu(u^2m,u^2t)=\sum_{v=1}^{um} g(v,t,m)=u\sum_{v=1}^{m} g(v,t,m),$$

which, together with (136), proves (141).

$2\cdot308$. An integer is said to be square-free (quadratfrei) if it is not divisible by any square other than 1. Let us define the auxiliary function $\varkappa(m)$ as 1 or 0 according as m is or is not square-free. This function is obviously multiplicative. Hence, if we put

(145) $$\nu'(m,t)=\varkappa((m,t))\,\nu(m,t),$$

the inner pair of brackets in $\varkappa((m,t))$ belonging to the symbol for the greatest common divisor, it follows from $2\cdot306$ that $\nu'(m,t)$ is a multiplicative function of m. Also

(146) $$\nu(m,n)=\sum_{\substack{q,u,v\\ q^2u=m\\ q^2v=n}} q\,\nu'(u,v).$$

In fact, the sum on the right, in spite of its three variables of summation, has only one possibly non-vanishing term, namely that in which q is the greatest integer whose square divides m and n, and it follows from (145) and (141) that this term is equal to $\nu(m,n)$.

$2\cdot309$. By (133) and (146),

(147) $$a_2S_4=\sum_{m=1}^{\infty}\sum_{\substack{q,u,v\\ q^2u=m\\ q^2v=n}} q^{-3}u^{-2}\nu'(u,v)$$

$$=\sum_{\substack{q,u,v\\ q^2v=n}} q^{-3}u^{-2}\nu'(u,v)=\sum_{\substack{q,v\\ q^2v=n}} q^{-3}T_1(v),$$

where

(148) $$T_1(v)=\sum_{u=1}^{\infty}u^{-2}\nu'(u,v).$$

Similarly, by (134) and (146),

(149) $$\frac{3}{4}a_2S_5=\sum_{\substack{q,v\\ q\text{ odd}\\ q^2v=n}} q^{-3}T_2(v),$$

where

(150) $$T_2(v)=\sum_{u\text{ odd}} u^{-2}\nu'(u,v).$$

By (149),

(151) $$\frac{3}{4}a_2S_5=S_6-S_7,$$

where

(152) $$S_6=\sum_{\substack{q,v\\ q^2v=n}} q^{-3}T_2(v),\qquad S_7=\sum_{\substack{q,v\\ q\text{ even}\\ q^2v=n}} q^{-3}T_2(v).$$

Substituting $2m$ for q and $\frac{1}{4}l$ for v in the last sum, we obtain

(153) $$S_7=\sum_{\substack{m,l\\ 4\mid l\\ m^2l=n}}(2m)^{-3}T_2\left(\frac{1}{4}l\right)=\frac{1}{8}\sum_{\substack{m,l\\ m^2l=n}} m^{-3}T_2\left(\frac{1}{4}l\right),$$

where $T_2(w)=0$ if w is not an integer.

By (130) and (151),

$$6\,a_2\,S(n) = -24\,a_2\,S_4 + 40\,S_6 - 40\,S_7\,.$$

Hence, putting

(154) $$T_3(l) = -24\,T_1(l) + 40\,T_2(l) - 5\,T_2\left(\frac{1}{4}\,l\right),$$

we have, by (147), (152), and (153),

(155) $$6a_2\,S(n) = \sum_{\substack{q,\,l \\ q^2 l = n}} q^{-3}\,T_3(l)\,.$$

2·310. *Let p be a prime. Then*

(156) $$\nu'(p^m, t) = 1 + \left(\frac{t}{p}\right) \qquad (p \nmid t,\ p > 2),$$

(157) $$\nu'(p, t) = 1 \qquad (p \mid t),$$

and

(158) $$\nu'(p^m, t) = 0 \qquad (p \mid t,\ m > 1).$$

Proof. If $p \nmid t$ and $p > 2$, it is known that $\nu\ (p^m, t)$ (as defined in 2·301) is 2 or 0 according as t is or is not a quadratic residue mod p, and we have $(p^m, t) = 1$, so that $\varkappa\,((p^m, t)) = 1$. From this and (145) we obtain (156).

If $p \mid t$, we have, by (121),

$$\nu(p, t) = \sum_{\substack{q \leq p \\ q^2 \equiv 0 \pmod p}} 1 = 1\,,$$

and $\varkappa\,((p, t)) = \varkappa\,(p) = 1$. From these formulae and (145) we obtain (157).

If $p \mid t$ and $m > 1$, we consider the cases $p^2 \mid t$ and $p^2 \nmid t$ separately. In the former, (p^m, t) is divisible by p^2 and therefore not square-free, so that $\varkappa\,((p^m, t)) = 0$. In the latter, by (121),

$$\nu\,(p^m, t) = \sum_{\substack{q \leq p^m \\ q^2 \equiv t \pmod{p^m}}} 1 = 0\,,$$

since the condition $q^2 \equiv t \pmod{p^m}$ now implies that $p \mid q^2$ and $p^2 \nmid q^2$, which is impossible, so that the sum is empty. Thus it follows from (145) that (158) holds in either case.

2·311. *We have*

(159) $$\nu'(1, t) = \nu'(2, t) = 1,$$

(160) $$\nu'(4, t) = \begin{cases} 2 & (t \equiv 1 \pmod 4) \\ 0 & (\text{otherwise}) \end{cases},$$

and

(161) $$\nu'(2^m, t) = \begin{cases} 4 & (t \equiv 1 \pmod 8,\ m \geqq 3) \\ 0 & (t \not\equiv 1 \pmod 8,\ m \geqq 3). \end{cases}$$

(159) and (160) follow easily from (145) and (121). If t is odd, (161) can be established by an argument similar to the proof of (156). If t is even, (161) is implied in (158).

2·312. *Let p be an odd prime. Then*

(162) $$\nu'(p^m, t) = \left(\frac{t}{p^m}\right) + \left(\frac{t}{p^{m-1}}\right).$$

This follows easily from 2·310.

2·313. *We have*

(163) $$\nu'(u, t) = \sum_{\substack{q,\,v \\ qv = u}} \left(\frac{t}{q}\right) \varkappa\,(v) \qquad (2 \nmid u).$$

Proof. It follows from 2·308 and lemma (i) of 2·305 that both sides of (163) are multiplicative functions of u, and the equation is obviously true for $u = 1$. Hence it is sufficient to prove that (163) holds if u is a power of an odd prime, and this follows from (162).

2·314. Let

(164) $$a_3 = \sum_{v \text{ odd}} v^{-2}\,\varkappa\,(v).$$

Then, by (150) and (163),

(165) $$T_2(l) = \sum_{u \text{ odd}} \sum_{\substack{q,\,v \\ qv = u}} (q\,v)^{-2} \left(\frac{l}{q}\right) \varkappa\,(v)$$

$$= \sum_{q \text{ odd}} \sum_{v \text{ odd}} q^{-2} \left(\frac{l}{q}\right) v^{-2}\,\varkappa\,(v) = a_3 \sum_{q=1}^{\infty} \left(\frac{l}{q}\right) q^{-2},$$

since $\left(\frac{l}{q}\right) = 0$ if q is even.

Since $\nu'(u, l)$ is a multiplicative function of u, it follows from (148) and (150) that

(166) $$T_1(l) = \sum_{u \text{ odd}} \sum_{x=0}^{\infty} (2^x u)^{-2} \nu'(2^x, l)\, \nu'(u, l)$$

$$= d_l\, T_2(l),$$

where

(167) $$d_l = \sum_{x=0}^{\infty} 2^{-2x} \nu'(2^x, l).$$

Since $T_2(w)$ has been defined as 0 if w is not an integer, it follows from (165) that

(168) $$T_2\left(\frac{1}{4} l\right) = \begin{cases} T_2(l) & (4 \mid l) \\ 0 & (4 \nmid l). \end{cases}$$

By (154), (166), and (168),

(169) $$T_3(l) = e_l\, T_2(l),$$

where

(170) $$e_l = \begin{cases} -24\, d_l + 35 & (4 \mid l) \\ -24\, d_l + 40 & (4 \nmid l). \end{cases}$$

2·315. By (167) and 2·311,

$$24\, d_l = \begin{cases} 30 & (l \not\equiv 1 \pmod 4) \\ 35 & (l \equiv 1 \pmod 8) \\ 33 & (l \equiv 5 \pmod 8). \end{cases}$$

Hence, by (170),

$$e_l = \begin{cases} 5 & (4 \mid l) \\ 10 & (l \equiv 2 \text{ or } 3 \pmod 4) \\ 5 & (l \equiv 1 \pmod 8) \\ 7 & (l \equiv 5 \pmod 8). \end{cases}$$

From this and the definition of C_l (in the enunciation of Theorem 3) it follows that

(171) $$C_l = 16\, e_l.$$

By (169), (171), (165), and (7),

$$l^{\frac{3}{2}}\, T_3(l) = a_4\, R(l),$$

where a_4 is a constant. Hence, by (6) and (155),

(172) $$r_5(n) = c\, n^{\frac{3}{2}}\, S(n) = (6\, a_2)^{-1} c \sum_{\substack{q, l \\ q^2 l = n}} l^{\frac{3}{2}}\, T_3(l)$$

$$= a_5 \sum_{\substack{q, l \\ q^2 l = n}} R(l) = a_5 \sum_{q^2 \mid n} R\left(\frac{n}{q^2}\right),$$

where a_5 is a constant. In particular

$$r_5(1) = a_5\, R(1).$$

Now $r_5(1) = 10$, and it follows from (7) that

$$R(1) = 80\, \pi^{-2} \sum_{m \text{ odd}} m^{-2} = 10.$$

Hence $a_5 = 1$, which, together with (172), proves Theorem 3.

University College, London.

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