On the representations of a number as a sum of squares.

By

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Introduction.

If \( r_s(n) \) denotes the number of solutions of the equation

\[ \sum_{i=1}^{s} x_i^2 = n \]

in integers \( x_1, x_2, \ldots, x_s \) and \( 1 \)

\[
\Theta_s(t) = \sum_{n=1}^{\infty} e^{i\pi nt} \quad (S > 0),
\]

then

\[
\Theta_s(t)^{-s} = \sum_{n=1}^{\infty} r_s(n) e^{i\pi nt} \quad (S > 0).
\]

The object of this paper is to use (2) for the evaluation of \( r_s(n) \) in the cases \( s = 5, 6, 7, 8 \) in a more elementary way than has been done before\( ^1 \). Thus I hope to make the subject accessible even to those

\( ^1 \) Readers familiar with elliptic functions will perhaps prefer the notation \( \Phi_s(0) \), but the simpler notation \( \Phi_s(t) \) is sufficient for the present purpose.


Dickson, Studies in the Theory of Numbers (1920), ch. XIII.
who know nothing of the theories of modular functions, theta functions, and Gaussian sums.

The main result of Part 1 is this:

THEOREM 1. Let

\[ \xi_m = e^{2\pi i m}, \]

\[ A_k = \sum_{h} \left( \frac{1}{2k} \sum_{\substack{\ell \\ \text{odd}}} \frac{1}{h^{1/2}} \right) \xi_{2k h}, \]

where \( h \) runs through all positive integers \( \leq 2k \) and prime to \( k \), and

\[ S(n) = \sum_{k \geq 1} A_k. \]

Then, for any positive integer \( n \),

\[ r_s(n) = cn^{1/2 - 1/s} S(n) \quad (s = 5, 6, 7, 8), \]

where \( c \) depends only on \( s \).

In Part 2 I obtain expressions for \( S(n) \) in the cases \( s = 8 \) and \( s = 5 \) which, when substituted in (6), lead to the following two theorems:

THEOREM 2. Let \( a_s(x) \) denote the sum \(^1\) of the cubes of the positive divisors of \( x \). Then, for any positive integer \( n \),

\[ r_s(n) = 16 a_s(n) - 32 a_s \left( \frac{1}{2} \right) + 256 a_s \left( \frac{1}{4} \right). \]

THEOREM 3. Let

\[ R(l) = C_l \pi^{-2} l \sum_{m=1}^{\infty} \left( \left\lfloor \frac{l}{m} \right\rfloor \right) m^{-2}, \]

where \( \left\lfloor \frac{l}{m} \right\rfloor \) is Jacobi's residue symbol \(^1\) if \( (m, 2l) = 1 \), \( \left\lfloor \frac{l}{m} \right\rfloor = 0 \) otherwise, \( C_l = 80 \) if \( l \equiv 0 \) (mod 4) or \( l = 1 \) (mod 8), \( C_l = 160 \) if \( l = 2 \) or 3 (mod 4), and \( C_l = 112 \) if \( l = 5 \) (mod 8). Then, for any positive integer \( n \),

\[ \sum_{p, \gamma \mid 2l} \psi(p, \gamma) = \psi(0, 0), \]

where \( \psi(p, \gamma) \) is the number of primitive representations of \( l \) as a sum of 5 squares, i.e., the number of solutions of the equation

\[ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = l \]

in integers \( x_1, x_2, x_3, x_4, x_5 \) with greatest common divisor 1.

None of these results are new, and for the general ideas underlying my proof of Theorem 1 I am greatly indebted to the papers quoted, especially the first, but I hope the publication of Part 1 is justified by the simplifications obtained in it. The method used in Part 2 is my own.

**Part 1.**

1.1. Notation.

1.11. \( x \) and \( y \) are real numbers, and \( \pi \) is a number whose imaginary part is positive.

1.12. \( r \) is a rational number.

1.13. In \( \sum_{p, \gamma \mid 2l} \psi(p, \gamma) \), \( r \) runs through all rational numbers satisfying the condition stated.

These sums are said to exist only if they are absolutely convergent. It follows that, if \( \sum_{p} f(p) \) exists, then

\[ \sum_{p, \gamma \mid 2l} \psi(p, \gamma) = \sum_{p} f(-\frac{1}{r}), \]

and if \( \sum_{p} f(p) \) exists, then

\[ \sum_{p} f(-r) = \sum_{p} f(r) = \sum_{p} \sum_{\gamma \mid 2l} \psi(p, \gamma) f(r + 2m). \]

1.14. \( \log z \) is the principal value of the logarithm of \( z \), so that

\[-\pi < \log z \leq \pi \quad (z \neq 0), \]

\( z^\pi \) means \( \exp(\pi \log z) \).

\( \psi \) is from [2].
On this definition, the equation \( (x, y)^n = x^n y^n \) is not always true, but it is true if \( \Re x > 0, \Re y > 0, \) and \( x \neq y \).

1.15. \( \lim_{y \to \infty} f(y) = l \) means \( \lim_{y \to \infty} f(y + i y) = l \) for every \( x \).

1.151. It easily follows that, if \( \lim f(y) = l, a > 0, \) and \( b \) is any number, then \( \lim_{y \to \infty} f(a z + b) = l. \)

1.16. \( f(t) \) is an abbreviation for \( \{f(t)\}^n. \)

1.17. \( \bar{z} = \Re z - i \Im z \) (i.e., \( \bar{z} \) is the conjugate complex number to \( z \)).

1.2. Proof of Theorem 1.

1.201. Let \( f(t) = \sum_{m=0}^{\infty} (-1)^m e^{i m t} \)

and

\[
\theta_2(x) = \sum_{m=0}^{\infty} e^{i n + \frac{1}{2} t}.
\]

Then, by (1), (11), and (12),

1.202. We have

\[
\theta_2(x + 1) = \theta_2(x), \quad \phi_2(x + 1) = \phi_2(x), \quad \phi_2(x + 1) = e^{-i \pi} \phi_2(x).
\]

On the representation of a number as a sum of squares.

\[
= \int_{-\infty}^{\infty} e^{-x^2} \left(1 + \sum_{n=1}^{\infty} e^{2 \pi i n x}ight) d x = a_0 + 2 \sum_{n=1}^{\infty} a_n.
\]

where

\[
a_n = \int_{-\infty}^{\infty} e^{-i \pi n y} e^{2 \pi i m n} d y
\]

as is shown by the substitution \( z = \Re z + i \Im z \) and a subsequent application of Cauchy's theorem. Hence

\[
a_n = e^{-i \pi n \gamma} \sum_{n=0}^{\infty} e^{-i \pi n \gamma} d w = c_0 \eta \frac{1}{2} e^{-i \pi n \gamma},
\]

where \( c_0 = \int_{-\infty}^{\infty} e^{-i \pi n} d x, \) and we obtain

\[
\sum_{m=0}^{\infty} e^{-i \pi m \eta} = a_0 + 2 \sum_{n=1}^{\infty} a_n = c_0 \eta \frac{1}{2} \sum_{m=0}^{\infty} e^{-i \pi m \eta}.
\]

Since this holds, in particular, for \( \eta = 1, \) we have \( c_0 = 1, \) which, together with (16), proves (15). Incidentally, we have proved the well-known formula

\[
\int_{-\infty}^{\infty} e^{-x^2} d x = 1.
\]

1.203. We have

\[
\beta_2(x) = (-i \pi) \frac{1}{2} \phi_2\left(\frac{1}{x}\right).
\]

Proof. By (12) and (1),

\[
\theta_2(x) = \sum_{m=0}^{\infty} e^{i \pi m + i \pi m}\sum_{n=0}^{\infty} \frac{1}{x^n} e^{i \pi n}.
\]
\[ \sum_{n=0}^{\infty} e^{i n \pi y} = \sum_{n=0}^{\infty} e^{i n \pi y x} = \frac{1}{4} - \sum_{n=0}^{\infty} e^{i n \pi y x} = \frac{1}{4} - \delta_y \left( \frac{1}{4} \right) - \delta_y \left( \frac{1}{4} \right) \]

Hence, by (14), (1), and (11).

\[ \delta_y \left( \frac{1}{4} \right) = \left( \frac{1}{4} - \frac{1}{2} \right) \delta_y \left( \frac{1}{4} \right) = \left( \frac{1}{4} \right) \delta_y \left( \frac{1}{4} \right) \]

\[ \left( \frac{1}{4} - \frac{1}{2} \right) \delta_y \left( \frac{1}{4} \right) = \left( \frac{1}{4} \right) \delta_y \left( \frac{1}{4} \right) \]

\[ \left( \frac{1}{4} \right) \delta_y \left( \frac{1}{4} \right) \]

1. **204.** Let us call a function \( \varphi(x) \) the comparison function of dimension \( -\alpha \) or, more briefly, the c.l. \( -\alpha \), of \( f(x) \), if the following conditions hold:

1. \( \alpha > 0 \).
2. \( f(x) \) is regular for \( 3^\alpha x > 0 \).
3. There is a number \( L \) and a function \( l(r) \), defined for every \( r \) (cf. 1.12), such that

   a. \( \varphi(x) = L + \sum_{r} l(r) (r - i \xi)^{-\alpha} \) for every \( r \) (which implies the existence of the last sum as defined in 1.13),

   b. \( \lim_{x \to \infty} f(x) = L \), and

   c. \( \lim_{x \to \infty} \left( \frac{-1}{i} \right)^{-\alpha} f \left( \frac{1}{4} \right) = l(r) \) for every \( r \).

1. **205.** It is obvious that any function \( f(x) \) cannot have more than one c.l. \( -\alpha \) (for a given \( \alpha \)).

1. **206.** Let \( \varphi(x) \) be the c.l. \( -\alpha \) of \( f(x) \), and let \( a \) be a constant.

Then

1. \( a \varphi(x) \) is the c.l. \( -\alpha \) of \( a f(x) \),
2. \( \varphi(x + 1) \) is the c.l. \( -\alpha \) of \( f(x + 1) \),

and

3. \( (x - i \xi)^{-\alpha} \) is the c.l. \( -\alpha \) of \( (x - i \xi)^{-\alpha} f \left( \frac{1}{4} \right) \).

It may be left to the reader to prove (i) and (ii).

**Proof of (iii).** We are given that there is a number \( L \) and a function \( l(r) \) such that

\[ L = \lim_{r \to \infty} f(x), \]

\[ l(r) = \lim_{r \to \infty} \left( \frac{-1}{i} \right)^{-\alpha} f \left( \frac{1}{4} \right), \]

and

\[ \varphi(x) = L + \sum_{r} l(r) (r - i \xi)^{-\alpha} \]

Putting

\[ f_i(x) = \left( \frac{-1}{i} \right)^{-\alpha} f \left( \frac{1}{4} \right), \]

we have to prove that there is a number \( L_i \) and a function \( l_i(r) \) such that

\[ L_i = \lim_{r \to \infty} f_i(x), \]

\[ l_i(r) = \lim_{r \to \infty} \left( \frac{-1}{i} \right)^{-\alpha} f \left( \frac{1}{4} \right), \]

and

\[ (x - i \xi)^{-\alpha} \]
On the representations of a number as a sum of squares.

By (19) and (33),

$$ l \left( \frac{1}{r} \right) = \lim_{r \to \infty} g(r) \frac{1}{r}.$$

and hence, by 1·151,

$$ l \left( \frac{1}{r} \right) = \lim_{r \to \infty} g(r^2 \tau - r).$$

By (30), (34), and (32),

$$ l \left( \frac{1}{r} \right) = \lim_{r \to \infty} \left( \frac{1}{r} \right)^n g(r^2 \tau - r) = \lim_{r \to \infty} \left( \frac{1}{r} \right)^n f_1 \left( r^2 \tau - r \right).$$

which, together with (31), proves (23).

1·207. Let $f(t)$ be such that

$$ f(t) = f(t + \tau),$$

for every $t$, and let $\tau(t)$ be the c. f. $-a$ of $f(t)$. Then

$$ \tau(t) = \tau(t),$$

Proof. By 1·204,

$$ \tau(t) = L + \sum_{r} f(t) (r \tau - r)^n,$$

where

$$ L = \lim_{r \to \infty} f(t),$$

and

$$ f(t) = \lim_{r \to \infty} \left( \frac{1}{r} \right)^n f \left( r \tau - r \right).$$

Now, by (37) and (10),

$$ \tau(t) = L + \sum_{r} f(t) (r \tau - r)^n.$$
Using (35) with \( \varepsilon = -r + i/y \) \((y > 0)\), we find that \( f(r + i/y) \) and \( f(-r + i/y) \) are conjugate complex numbers. Hence, by (42) and (43),

\[
\phi(-r) = \overline{\phi(r)}.
\]

Similarly, using (35) with \( \varepsilon = iy \), we deduce from (41) that

\[
\lambda(-r) = \overline{\lambda(r)}
\]

(which, of course, means that \( \lambda \) is real). Also \( f-r-i\bar{\varepsilon} \) and \( -ir + i\bar{\varepsilon} \) are conjugate complex numbers and, by 1·11, certainly not \( \leq 0 \). Hence, by 1·14, \( (f-r-i\bar{\varepsilon})^- \) and \( (-ir + i\bar{\varepsilon})^- \) are conjugate complex numbers. From this and (46), (37), (45), and (44) we obtain (36).

1·208. For any integers \( h, k \), such that \( k > 0 \) and \((h, k) = 1\), let

\[
\lambda \left( \frac{h}{k} \right) = \frac{1}{2k} \sum_{\xi \equiv h \pmod{k}} \xi^{4k},
\]

where \( \lambda_p \) is defined by (3). Then \( \lambda \left( \frac{h}{k} \right) \) is defined for every \( r > 0 \) and \((h, k) = 1\).

\[
\lambda \left( \frac{h}{k} \right) \leq k^2 \bigg( \frac{1}{2k} \sum_{\xi \equiv h \pmod{k}} \xi^{2k} \bigg)^2.
\]

Proof. By (46), (3), and 1·17,

\[
2k \lambda \left( \frac{h}{k} \right) = \sum_{\xi \equiv h \pmod{2k}} \xi^{2k} + \sum_{\xi \equiv h \pmod{2k}} \xi^{2k} = \sum_{\xi \equiv h \pmod{2k}} \xi^{2k},
\]

for any integer \( m \), and

\[
2k \lambda \left( \frac{h}{k} \right) = \sum_{\xi \equiv h \pmod{2k}} \xi^{2k}.
\]

Hence

\[
4k^2 \lambda \left( \frac{h}{k} \right)^2 = \sum_{\xi \equiv h \pmod{2k}} \xi^{2k} \sum_{\xi \equiv h \pmod{2k}} \xi^{2k} = \sum_{\xi \equiv h \pmod{2k}} \xi^{4k}.
\]

Observing that \( \sum_{\xi \equiv h \pmod{2k}} \xi^{4k} \) is equal to \( 2k \) or 0 according as \( q \) is or is not a multiple of \( k \), we deduce from the last formula that

\[
4k^2 \lambda \left( \frac{h}{k} \right)^2 = 2k \left( \sum_{\xi \equiv h \pmod{2k}} \xi^{4k} \right) = 2k \left( -1 \right)^h + 1 = 4k,
\]

which implies (47).

On the representations of a number as a sum of squares.

1.210. We have

\[
\lambda \left( \frac{h}{k} \right) = \lim_{t \to \infty} \left( \phi \left( \frac{h}{k} - \frac{t}{k} \right) \phi \left( \frac{1}{k} \right) \right).
\]

Proof. Let

\[
r = \frac{h}{k}, \quad k > 0, \quad (h, k) = 1.
\]

Then, by (1) and (3),

\[
(-i \varepsilon)^{-\frac{1}{2}} \psi (r - \frac{1}{r}) = (-i \varepsilon)^{-\frac{1}{2}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \xi^{2m} e^{-\pi n^2 r}
\]

\[
= \frac{1}{2k} \sum_{\xi \equiv h \pmod{2k}} \xi^{4k},
\]

where

\[
u = -i \varepsilon \frac{1}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} e^{-\pi n^2 r},
\]

and \( \psi (\xi) \) is the number of those integers \( m \) for which \( m = q \pmod{2k} \) and \( m^2 \leq \nu \), so that

\[
\psi (\xi) - \frac{1}{\nu} \xi \leq 1.
\]

To evaluate the integral

\[
\int_{0}^{
\infty} \frac{1}{\nu} (-i \varepsilon)^{-\frac{1}{2}} e^{-\pi n^2 r} \psi (\xi) d \nu = \int_{0}^{
\infty} \frac{1}{\nu} e^{-\pi \xi \nu} \xi d \nu = \frac{1}{2},
\]

we put \( \varepsilon = -i \varepsilon \), and replace the new path of integration (a half-line in the half-plane \( \Re z > 0 \)) by the positive real axis, which does not alter the value of the integral, as can be shown in a well-known way by means of Cauchy's theorem. Thus we obtain

\[
\int_{0}^{
\infty} \frac{1}{\nu} (-i \varepsilon)^{-\frac{1}{2}} e^{-\pi n^2 r} \psi (\xi) d \nu = \int_{0}^{
\infty} \frac{1}{\nu} e^{-\pi \xi \nu} \xi d \nu = \frac{1}{2}.
\]
(Readers not familiar with the \( \Gamma \)-function may deduce the last equation from the formula at the end of 1 \( \cdot \) 202.) By (51), (53), and (52),
\[
|u_\rho - \frac{1}{2 \pi}| = \int_0^\infty \pi \left( -\frac{\rho}{\pi} \right)^{-\frac{1}{2}} e^{-\rho \pi/2 \left( \frac{\pi}{2} \right)^2} d\rho = \int_0^\infty \pi \left| -\frac{1}{2} \rho \right| \exp \left( -\pi \rho \left| 1 - \frac{1}{2} \right| \right) d\rho = |\frac{1}{2} (\frac{1}{2} - 1)^{-1}.
\]
Hence, by 1 \( \cdot \) 15,
\[
\lim_{\rho \to \infty} u_\rho = \frac{1}{2 \pi}.
\]
From this and (50), (46), and (49) we obtain (48).

1 \( \cdot \) 211. Henceforth let \( \rho \geq 5 \). Then it easily follows from (47) that
\[
\sum_{\rho} \rho (\rho - i \rho)^{-\frac{1}{2}}
\]
exists for any \( s \). Also, by (1),
\[
\lim_{\rho \to \infty} |\rho| = 1.
\]
Put
\[
\varphi_\rho (\rho) = 1 + \sum_{\rho} \rho (\rho - i \rho)^{-\frac{1}{2}}.
\]
Then, by 1 \( \cdot \) 204, (54), and (48), \( \varphi_\rho (\rho) \) is the c.f. \( -\frac{1}{2} \) of \( \rho \rho' (\rho) \).

Put
\[
\varphi_\rho (\rho) = \varphi_\rho (\rho + 1), \quad \varphi_\rho (\rho) = \left( -i \rho \right)^{-\frac{1}{2}} \varphi_\rho (\rho^\frac{1}{2}) = 1.
\]
Then, by 1 \( \cdot \) 206, (13), and (17) \( \varphi_\rho (\rho) \) and \( \varphi_\rho (\rho) \) are the c.f. \( -\frac{1}{2} \) of \( \rho \rho' (\rho) \) and \( \rho \rho' (\rho) \) respectively.

1 \( \cdot \) 212. Putting
\[
\varphi_\rho (\rho) = \varphi_\rho (\rho) \rho \rho' (\rho) \quad (\rho = 0, 2, 3),
\]
we have, by (13), (17), and (56),
\[
\varphi_\rho (\rho) = \varphi_\rho (\rho + 1)
\]
and
\[
\varphi_\rho (\rho) = \varphi_\rho (\rho + 1).
\]
Also, by 1 \( \cdot \) 206, 1 \( \cdot \) 211, and (13), \( \varphi_\rho (\rho + 1) = \rho \rho' (\rho) \) is the c.f. \( -\frac{1}{2} \) of \( \rho \rho' (\rho) \).

Hence, by 1 \( \cdot \) 205,
\[
\varphi_\rho (\rho + 1) = \varphi_\rho (\rho).
\]
Similarly, by 1 \( \cdot \) 206, 1 \( \cdot \) 211, and (14), \( -i \rho \rho' (\rho) \) is the c.f. \( -\frac{1}{2} \) of \( \rho \rho' (\rho) \), and hence, by 1 \( \cdot \) 205,
\[
\left( -i \rho \rho' (\rho) \right)^{-\frac{1}{2}} \varphi_\rho (\rho) = \varphi_\rho (\rho).
\]
By (57), (60), and (13),
\[
\varphi_\rho (\rho + 1) = \varphi_\rho (\rho).
\]
By (57), (61), and (14),
\[
\varphi_\rho (\rho + 1) = \varphi_\rho (\rho).
\]
Also, by 1 \( \cdot \) 206 and 1 \( \cdot \) 211, \( \varphi_\rho (\rho + 1) = \rho \rho' (\rho) \) is the c.f. \( -\frac{1}{2} \) of \( \rho \rho' (\rho + 1) \), and \( \rho \rho' (\rho + 1) = \rho \rho' (\rho) \) is the c.f. \( -\frac{1}{2} \) of \( \rho \rho' (\rho) \). Hence, by (13) and 1.205,
\[
\varphi_\rho (\rho + 1) = \rho \rho' (\rho) = \rho \rho' (\rho + 1).
\]
By (57), (64), and (13),
\[
\varphi_\rho (\rho + 1) = \varphi_\rho (\rho).
\]
Finally, on substituting \( -\frac{1}{2} \) for \( \rho \) in (59), we obtain
\[
\varphi_\rho (\rho) = \varphi_\rho (\rho).
\]
Put
\[
\varphi_\rho (\rho) = \varphi_\rho (\rho) \varphi_\rho (\rho) \rho \rho' (\rho) \quad (\rho = 0, 2, 3),
\]
we have, by (13), (17), and (56),
\[
\varphi_\rho (\rho) = \varphi_\rho (\rho + 1)
\]
\[ F_q(\tau + 1) = F_q(\tau) \quad (q = 1, 2, 3), \]
and by (59), (66), and (63),
\[ F_q\left(-\frac{1}{\tau}\right) = F_q(\tau) \quad (q = 1, 2, 3). \]

1·213. The functions \( F_1(\tau) \), \( F_2(\tau) \), and \( F_3(\tau) \) are regular for \( \Im \tau > \frac{1}{2} \).

*Proof.* It is easily seen that any comparison function in the sense of 1·204 is regular throughout the half-plane \( \Im \tau > 0 \). Hence, by (67), (57), and 1·211, it is sufficient to prove that
\[ \phi_q(\tau) \neq 0 \quad (q = 0, 2, 3; \ \Im \tau > \frac{1}{2}). \]

Suppose, then,
\[ \Im \tau > \frac{1}{2}. \]

Then, by (1) and (11),
\[ |\phi_q(\tau) - 1| \leq 2 \sum_{n=1}^{\infty} e^{\Im \tau n} < 2 \sum_{n=1}^{\infty} e^{-\frac{1}{2} \Im \tau n}, \]
\[ < 2 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = 1 \quad (q = 0, 3), \]
and hence
\[ \phi_q(\tau) \neq 0 \quad (q = 0, 3). \]

Also, by (12),
\[ |e^{-\frac{1}{\Im \tau} \phi_q(\tau)} - 2| = 2 \sum_{n=1}^{\infty} e^{\Im \tau n} \left| e^{(\Re \tau + \Im \tau) n} \right| \]
\[ < 2 \sum_{n=1}^{\infty} e^{-\frac{1}{2} \Im \tau (\Re \tau + \Im \tau)} < 1. \]

so that \( \phi_q(\tau) \neq 0 \), and (70) is proved.

1·214. We have
\[ F_q(-\tau) = F_q(\tau) \quad (q = 1, 2, 3). \]

*Proof.* By (1), (11), and (12),
\[ \phi_q(-\tau) = \phi_q(\tau) \quad (q = 0, 2, 3). \]

Hence, by 1·207 and 1·211.

On the representations of a number as a sum of squares.

\[ \gamma_q(-\tau) = \gamma_q(\tau) \quad (q = 0, 2, 3). \]

From this and (72) and (57) we obtain
\[ \gamma_q(-\tau) = \gamma_q(\tau) \quad (q = 0, 2, 3), \]

which, together with (67), proves (71).

1·215. Let
\[ G_q(\tau) = F_q \left(\frac{1}{2 \pi i} \log \tau\right) \quad (q = 1, 2, 3). \]

Then \( G_q(\tau) \) is regular for \( 0 < |\tau| < e^{-\pi}. \)

This follows from (68) and 1·213.

1·216. Let \( \tau > 0 \), let \( \sum_{r} |l(r)| |ir - i\tau|^{-\nu} \) exist for \( \tau = i \), and let \( (U) \)
be an abbreviation for
\[ \text{uniformly for } -\frac{1}{2} < x < \frac{1}{2}. \]

Then
\[ \lim_{\nu \to 0} \sum_{r} |l(r)| |ir - i(x + iy)|^{-\nu} = 0 \quad (U). \]

*Proof.* Put
\[ \sum_{r} |l(r)| |ir - 1|^{-\nu} = c_{\nu}, \]

which is permissible by 1·13. Then
\[ \lim_{\nu \to 0} \sum_{r \neq 0} |l(r)| |ir - 1|^{-\nu} = c_{\nu}, \]

and hence, by (75),
\[ \lim_{\nu \to 0} \sum_{r \neq 0} |l(r)| |ir + 1|^{-\nu} = 0. \]

Now let \( \epsilon \) be any positive number. Then, by (76), there is an \( a \) such that
\[ \sum_{|r| > a} |l(r)| |ir + 1|^{-\nu} < 2^{-a-1} \epsilon. \]

Let \( y \geq 1 \) and \( -\frac{1}{2} < x < \frac{1}{2} \). Then
This follows from $1 \cdot 204$ and $1 \cdot 216$.

1 \cdot 218. Hancelorth let $s \leq 8$, so that $s$ is now restricted to the values 5, 6, 7, and 8. Then the three functions $G_s(z)$ defined by (73) are regular also at the origin.

\textbf{Proof.} It is sufficient to prove that

$$\lim_{z \to 0} \{ z \ G_s(z) \} = 0.$$\hfill (U).

This is equivalent to

$$\lim_{z \to 0} \{ e^{-2\sqrt{y}} F_s(x+i y) \} = 0 \quad (U).$$\hfill (79)

Hence, by (73), it is sufficient to prove that

$$\lim_{y \to \infty} \{ e^{-2\sqrt{y}} F_s(x+i y) \} = 0 \quad (U).$$\hfill (80)

Now, by (11) and (1),

$$\lim_{y \to \infty} \{ x+i y \} = \lim_{y \to \infty} \{ x+i y \} = 1 \quad (U).$$

and hence, by $1 \cdot 211$, $1 \cdot 217$, and (57),

$$\lim_{y \to \infty} \{ e^{-\frac{1}{4} z^2} \varphi(z+i y) \} = 2 \quad (U).$$\hfill (82)

and $\lim \varphi(t) = 0$, so that, by $1 \cdot 211$ and $1 \cdot 217$.

$$\lim_{y \to \infty} \{ e^{-\frac{1}{4} z^2} \varphi(z+i y) \} = 0 \quad (U).$$\hfill (83)

By (57), (82), and (83),

$$\lim_{y \to \infty} \{ e^{-\frac{1}{4} z^2} \varphi(z+i y) \} = 0 \quad (U).$$

which means that

$$\lim_{y \to \infty} \{ e^{-\frac{1}{4} z^2} \varphi(z+i y) \} = 0 \quad (U).$$

Since $s \leq 8$, it follows that
\[
\lim_{\gamma \to 0} e^{-2\pi \gamma} g_\delta(x + i\gamma) = 0 \quad (U).
\]

From (67), (81), and (84) we obtain (80).

1.219. Let the set \( A \) consist of the origin and those points \( z \) for which \( |z| < 1 \) and \( \log z \) is real. Then it is easily seen that \( A \) is closed and contained in the circle \( |z| < e^{-2\pi} \), that it contains the circle \( |z| < e^{-2\pi} \), and that its boundary consists of those points \( z \) for which \( |z| < 1 \) and \( \log z \) is real. Now \( z \) be any point on the boundary of \( A \). Then \( G_\delta(z) \) is real \((\delta = 1, 2, 3)\).

Proof. By the last part of 1.219, \( |z| < 1 \) and \( \log z = 2\pi \).

Hence the number

\[ \tau = \frac{1}{2\pi i} \log z \]

satisfies 1.11 and 85.

Also, by (73),

\[ G_\delta(z) = F_\delta(t). \]

Now, by (85), \( \frac{1}{\tau} = -\bar{\tau} \), and hence, by (69) and (71), \( F_\delta(t) = F_\delta\left(-\frac{1}{\tau}\right) \)

\[ = F_\delta(-\bar{\tau}) = F_\delta(t), \]

which implies that \( F_\delta(t) \) is real. Hence, by (86), \( G_\delta(z) \) is real.

1.221. Let \( D_1 \) and \( D_2 \) be domains, let \( E \) be a closed bounded set contained in \( D_1 \) and containing \( D_2 \), and let \( f(z) \) be regular in \( D_1 \) and real on the boundary of \( E \). Then \( f(t) \) is a constant.

Proof. The imaginary part of a regular function, considered in a closed bounded set, assumes its maximum and its minimum on the boundary of the set. Since \( \Im f(z) = 0 \) on the boundary of \( E \), it follows that \( \Im f(z) = 0 \) throughout \( E \). Hence \( f(t) \) is real throughout the domain \( D_1 \), and this implies the result stated.

1.222. \( G_\delta(z), G_\delta^2(z), \) and \( G_\delta^3(z) \) are constants.

Proof. We apply 1.221, taking for \( E \) the set \( A \) of 1.219 and for \( D_1 \) and \( D_2 \) the circles \( |z| < e^{-2\pi} \) respectively. Then, by 1.219, 1.220, \( G_\delta(z) \) is regular in \( D_1 \) and real on the boundary of \( E \). Hence, by 1.221, \( G_\delta(z) \) is a constant.

1.223. \( G_\delta(z) = 1. \)

Proof. It follows from (67) that \( g_\delta(z) \) is a root of the cubic

\[ u^3 - F_\delta(t) u^2 + F_\delta(t) u - F_\delta(t) = 0. \]

By (73) and 1.222, this cubic has constant coefficients. Hence \( g_\delta(t) \) is a constant, and it follows from (81) that this constant is 1.

1.224. \( q_\delta(t) = \tau_2(t). \)

This follows from (57) and 1.223.

1.225. By (55) and (10),

\[ \tau_2(t) = 1 + \sum_{q \neq 0} \sum_{q \neq 0} \lambda(t, 2) \frac{1}{2\pi} \sum_{q \neq 0} \lambda(t, q)\]

Now it follows from (48) and (1) that \( \lambda(t, 2) = \lambda(t) \) for any integer \( q \).

Hence, putting

\[ F(t) = \sum_{q \neq 0} \lambda(t, q)\]

we have, by (87),

\[ \tau_2(t) = 1 + \sum_{q \neq 0} \lambda(t, q)\]

It easily follows from (88) that \( F(t) \) has period 2 and that \( \lim_{y \to \infty} F(x + i\gamma) = 0 \) uniformly in \( x \). Hence

\[ F(t) = \sum_{q \neq 0} b_\delta e^{i\tau_\delta}, \]

where

\[ b_\delta = \frac{1}{2\pi} \int F(t) e^{-i\tau_\delta} d\tau, \]

\( \tau_\delta \) being any number in the upper half-plane. Taking, in particular, \( \tau_\delta = 0 \), we obtain from (91) and (88)

\[ b_\delta = \frac{1}{2\pi} \int F(t) e^{-i\tau_\delta} d\tau = \frac{1}{2\pi} \int F(t) e^{-i\tau_\delta} d\tau. \]
\[ \lambda(r) = \lambda(1) = 0. \]

Theorem 1 is thus established.

**Part 2.**

2.1. **Evaluation of** \( \lambda^4(r) \).

2.11. We have

\[ \lambda(0) = 1, \lambda(1) = 0. \]

On the representations of a number as a sum of squares.

\[ \lambda(r + 2) = \lambda(r), \]

and

\[ \lambda^2 \left( \frac{1}{r} \right) = (\lfloor \frac{r}{4} \rfloor) \lambda^2(r) \quad (r \neq 0). \]

(96) and (97) follow immediately from (46) and (3).

**Proof of (98).** By (48) and 1 \cdot 151,

\[ \lambda^2(r) = \lim_{n \rightarrow \infty} \left\{ (-r)^{r-2} \frac{1}{\Gamma(\frac{r}{2})} \sum_{n=0}^{\infty} \lambda^2(n) \right\}. \]

Hence, by (14) and (48),

\[ \left( \lfloor \frac{r}{4} \rfloor \right) \lambda^2(r) = \lim_{n \rightarrow \infty} \left( \frac{1}{\Gamma(\frac{r}{2})} \sum_{n=0}^{\infty} \lambda^2(n) \right) \]

where 1 \cdot 226. By 1 \cdot 13 and (3),

\[ \sum_{n=0}^{\infty} \lambda^2(n) = \sum_{n=0, \text{odd}}^{\infty} \lambda^2(n) = \sum_{n=0, \text{even}}^{\infty} \lambda^2(n) = S(n), \]

which, together with (95), proves (6).

Theorem 1 is thus established.

To be in \( \beta \) if and only if there is an \( r \) such that \( |h(r)| = 2k(r) = n \).

Suppose \( \beta \) does not contain all rational numbers. Then \( \beta \) is not empty, and so \( \beta \) has a least member \( n_k \), say. There is an \( n_k \), not in \( \beta \), such that

\[ |h(r_k)| = 2k(r_k) = n_k. \]

Now 0, 1, and \(-1\) are in \( \beta \), so that \( |r_0| \) is neither 0 nor 1. Put
\[ r_i = \begin{cases} r_i - 2 & \text{if } r_i > 1, \\ r_i + 2 & \text{if } r_i < -1, \\ -1/r_i & \text{if } 0 < |r_i| < 1. \end{cases} \]

and

\[ n_i = \left| h(r_i) \right| = |r_i| \left| h(r_i) \right|. \]

Then \( r_i \) is not in \( a \), and hence \( n_i \) is in \( b \). On the other hand, \( n_i \) is less than \( n_i \), the least member of \( b \). This is a contradiction.

2·13. Let two functions \( f_m(r) \) \((m = 1, 2)\) be defined for every \( r \) and have the following properties:

\begin{align*}
(99) & \quad f_1(0) = f_2(0), \quad f_1(1) = f_2(1), \\
(100) & \quad f_m(r + 2) = f_m(r), \\
& \quad f_m\left(-\frac{1}{r}\right) = -r^2 f_m(r) \quad (r \neq 0).
\end{align*}

Then

\[ f_1(r) = f_2(r) \]

for every \( r \).

This follows from 2·12 on taking for \( a \) the aggregate of those numbers \( r \) for which (102) holds.

2·14. We have

\[ \lambda^4(r) = \begin{cases} 0 & \text{if } (k, h(r) \neq 1), \\
\left( k, r, r \right) \left( 2 + h(r)k(r) \right) & \text{if } (k, h(r) = 1). \end{cases} \]

This follows from 2·13 on taking for \( f_1(r) \) and \( f_2(r) \) the two sides of (103), and applying 2·11.

2·21. Henceforth \( h, k, l, m, n, q, r, \) and \( v \) denote positive integers, and \( f, x, y \) denote integers.

\( c_v(x) \) denotes the sum of the \( x \)-th powers of the primitive \( v \)-th roots of unity (Ramanujan's sum).

It follows that

\[ \sum_{x=1}^{\infty} c_v(x) \]

is the sum of the \( x \)-th powers of all \( v \)-th roots of unity, so that

\[ \sum_{x=1}^{\infty} c_v(x) = \begin{cases} v & \text{if } (v, v) = 1, \\
0 & \text{otherwise}. \end{cases} \]

If \( v \) is odd, and \( p_1, p_2, \ldots, p_m \) are the primitive \( v \)-th roots of unity, it is easily seen that \(-p_1, -p_2, \ldots, -p_m\) are the primitive \( 2v \)-th roots of unity.

Hence

\[ c_{2v}(x) = (-1)^v c_v(x) \]

and

\[ (2v) \cdot \sum_{x=1}^{\infty} c_v(x) = (2v) \cdot \sum_{x=1}^{\infty} c_v(x). \]

2·22. Let \((h, k) = 1\). Then, by (103),

\[ \lambda^4(\frac{h}{k}) = \begin{cases} 0 & \text{if } (h, k) \neq 1, \\
\left( k, k \right) \left( 2v \right) & \text{if } (h, k) = 1. \end{cases} \]

Also, by (4) and (46),

\[ A_k = \sum_{x=1}^{\infty} \lambda^4(\frac{h}{k}) \zeta_{2v}^{k-1}. \]

It follows from (106), (107), and (3) that

\[ \lambda^4(\frac{h}{k}) = \begin{cases} 0 & \text{if } (h, k) \neq 1, \\
\left( k, k \right) \left( 2v \right) & \text{if } (h, k) = 1. \end{cases} \]

Then, by (5) and (108),

\[ S(n) = S_2 + 16 S_4. \]

Also, by (109) and (105),

\[ S_2 - S_4 = \sum_{x=1}^{\infty} \phi^{-1} c_v(x) = \sum_{x=1}^{\infty} (2v) \cdot c_v(x) = \frac{(-1)^v S_2}{16}, \]

and hence, by (105),

\[ S(n) = 16 S_2 - 15 S_2 = ( -1 )^v S_2. \]

2·23. It remains to evaluate \( S_2 \) and \( S_4 \).

Let

\[ a = \sum_{x=1}^{\infty} \phi^{-1}. \]

Then

\[ \sum_{x=1}^{\infty} \phi^{-1} = a = \sum_{x=1}^{\infty} (2v) + \frac{15}{16}. \]

By (109), (112), and (104),

\[ \alpha S_2 = \sum_{x=1}^{\infty} (2v) \cdot c_v(x) = \sum_{x=1}^{\infty} \phi^{-1} \sum_{x=1}^{\infty} c_v(x) = \sum_{x=1}^{\infty} \phi^{-1} \cdot c_v(x). \]
Similarly, by (109), (113), and (104),

\[ \frac{15}{16} a \sum_{\nu \equiv 0} q^{-\nu} = \sum_{\nu \equiv 0} q^{-\nu} - \sum_{\nu \equiv 0} q^{-\nu} = -n^{-1} a_q(n) - n^{-1} a_q \left( \frac{1}{2} n \right). \]

Now, if \( n \) is odd, then \( a_q \left( \frac{1}{2} n \right) = 0 \). Hence, by (111), (114), and (115),

\[ \frac{15}{16} a n^3 S(n) = \left[ a_q(n) \right] \left( 2 \mid n \right) \]

If \( n \) is even, and \( u_1, u_2, \ldots, u_q \) are all those positive divisors of \( \frac{1}{2} n \) which do not divide \( \frac{1}{4} \), then \( 2 u_1, 2 u_2, \ldots, 2 u_q \) are all those positive divisors of \( n \) which do not divide \( \frac{1}{2} n \). Hence

\[ a_q(n) - a_q \left( \frac{1}{2} n \right) = \sum_{\nu = 1}^{\frac{1}{4}} \left( 2 u_1 \right)^{\nu} - \sum_{\nu = 1}^{\frac{1}{4}} u_1^{\nu} \]

\[ = 8 a_q \left( \frac{1}{2} n \right) - 8 a_q \left( \frac{1}{4} n \right) \]

and hence, by (111),

\[ \frac{15}{16} a n^3 S(n) = a_q(n) - 2 a_q \left( \frac{1}{2} n \right) + 16 a_q \left( \frac{1}{4} n \right). \]

From this and (6) we obtain

\[ r_1(n) = a_q(n) - 2 a_q \left( \frac{1}{2} n \right) + 16 a_q \left( \frac{1}{4} n \right). \]

where \( a_q \) is a constant. Substituting 1 for \( n \) in this formula, we obtain

\[ a_q = 16, \]

which, together with (118), proves Theorem 2.

2.3. Evaluation of \( S(n) \) for \( s = 5 \).

2.301. If \( k \) is odd, then, by (4), (46), (103), and (3),

\[ A_k = \sum_{[m,n]=1} \lambda^x \left[ \frac{2 m}{k} \right] \frac{1}{2 k} \sum_{n=1}^{\frac{2 m}{k}} a_q(n) \]

\[ = k^{-3} \sum_{[m,n]=1} \lambda^x a_q(n) = k^{-3} \sum_{n=1}^{\lambda^x} a_q \left( q^x - n \right), \]

in the notation introduced in 2.21.

Let

\[ d_4(x) = \frac{1}{m} \sum_{\nu = 1}^{\frac{1}{2}} c_4 \left( q^{\nu} - x \right) \]

and

\[ \nu(m, \ell) = \sum_{\nu = 1}^{\frac{1}{2}} \frac{1}{\nu \equiv (\ell \mod m)} \]

(which means that \( \nu(m, \ell) \) is the number of solutions of the congruence \( x^2 = \ell \mod m \)). Then, by (119) and (120),

\[ A_4 = k^{-3} d_4(n) \]

Similarly

\[ A_4 = -k^{-3} d_4(n) \]

Now \( c_4 \left( q^{\nu} - x \right) \), considered as a function of \( q \), has period \( h \). Hence it follows from (120) that, if \( k \mid m \), then

\[ d_4(x) = \frac{1}{m} \sum_{\nu = 1}^{\frac{1}{m}} c_4 \left( q^{\nu} - x \right), \]

and hence, by (104) and (121),

\[ \sum_{n=1}^{\lambda^x} d_4(x) = \frac{1}{m} \sum_{\nu = 1}^{\frac{1}{m}} c_4 \left( q^{\nu} - x \right) \]

\[ = \sum_{n=1}^{\lambda^x} 1 = \nu(m, x). \]

2.302. Let \( k \) be odd. Then

\[ d_{2k}(x) = 0. \]

Proof. It has been observed that \( c_4 \left( q^{\nu} - x \right) \), considered as a function of \( q \), has period \( k \). From this, (120), (105), and the identity

\[ \sum_{\nu = 1}^{\frac{1}{k}} f(q) = \sum_{\nu = 1}^{\frac{1}{k}} \left[ f(q) + f(q + k) \right] \]

we obtain

\[ 2 k d_{2k}(x) = \sum_{\nu = 1}^{\frac{1}{k}} c_{2k} \left( q^{\nu} - x \right) = \sum_{\nu = 1}^{\frac{1}{k}} (-1)^{\nu-x} c_4 \left( q^{\nu} - x \right) \]

\[ = (-1)^{x} \sum_{\nu = 1}^{\frac{1}{k}} c_4 \left( q^{\nu} - x \right) \left[ (-1)^{\nu} + (-1)^{\nu+k} \right], \]
and 
\[ (-1)^n + (-1)^{a+n} = 0 \]
since \( k \) is odd.

(126) \( |d_4(n)| \leq 2n^{-\frac{1}{2}} \).

**Proof.** It follows from (4), (46), and (47) that

(127) \( |A_k| \leq 2k^{-\frac{3}{2}} \).

From this and (122) we obtain (126) immediately if \( n \) is odd. If \( 4 \mid u \), it follows from (123) that

\[ d_u(n) = -\left(\frac{1}{2}\right)\frac{1}{\eta}A_{\frac{1}{2}} \eta. \]

which, together with (127), again proves (126). Finally, if \( u = 2 \pmod{4} \), it follows from (2.302) that \( d_u(n) = 0 \). Thus (126) holds in all cases.

(128) \( S_4 = \sum_{n=1}^{\infty} u^{-2}d_u(n) \), \( S_u = \sum_{n=1}^{\infty} u^{-3}d_u(n) \).

These sums are absolutely convergent by (126), and it follows from (2.302) that

(129) \( S_4 - S_1 = \sum_{n=1}^{\infty} u^{-3}d_u(n) = \sum_{n=1}^{\infty} (2k)^{-2}d_{2k}(n) \).

By (5), (122), (123), (128), and (129),

(130) \( S(u) = S_1 - 4(S_4 - S_1) = 5S_4 - 4S_1 \).

Let

(131) \( a_u = \sum_{n=1}^{\infty} \psi^{-2} \).

Then

(132) \( \sum_{n=1}^{\infty} \psi^{-2} = a_u - \sum_{n=1}^{\infty} (2u)^{-2} = \frac{3}{4} a_u \).

By (128), (131), and (124),

(133) \( a_u S_4 = \sum_{n=1}^{\infty} (u \psi)^{-3}d_u(n) = \sum_{n=1}^{\infty} m^{-4} \sum_{n=1}^{\infty} d_u(n) \)

\[ = \sum_{n=1}^{\infty} m^{-4} \psi(m, n). \]

Similarly, by (128), (122), and (124),

(134) \( \frac{3}{4} a_u S_4 = \sum_{m=1}^{\infty} \psi(m, n) \).

(2.305). A function \( f(a) \) is said to be **multiplicative** if \( f(uv) = f(u)f(v) \)

whenever \( (u, v) = 1 \). This notion will be used several times in the remainder of this paper.

Use will also be made of the following elementary lemmas:

(i) If \( f_1(u) \) and \( f_2(u) \) are **multiplicative**, and

\[ f_2(u) = \sum_{\psi(u) \neq 0} f_1(\psi) \psi_2(\psi), \]

then \( f_2(u) \) is **multiplicative**.

(ii) If \( (u, v) = 1 \), and \( f(x) \) has period \( x, \psi \), then

\[ \sum_{\psi(x) \neq 0} f(q) = \sum_{\psi(x) \neq 0} \sum_{\psi(q) \neq 0} f(x, \psi). \]

(iii) If \( (u, v) = 1 \), and \( f(x) \) has period \( x, \psi \), then

\[ \sum_{\psi(x) \neq 0} f(q) = \sum_{\psi(q) \neq 0} f(q, \psi). \]

(iv) If \( f(x) \) has period \( m, \psi \), then

\[ \sum_{\psi(x) \neq 0} f(q) = k \sum_{\psi(q) \neq 0} f(q). \]

(2.306). Let \( (u, v) = 1 \). Then

(135) \( \psi(uv, \psi) = \psi(u, \psi) \psi(v, \psi). \)

In other words: \( \psi(u, \psi) \) is a **multiplicative function** of \( u \).

**Proof.** Define the auxiliary function \( g(x, t, m) \) as 1 if \( x^2 \equiv t \pmod{m} \)

and 0 otherwise. Then, by (121),

(136) \( \psi(m, \psi) = \sum_{t=0}^{m} g(t, m, \psi). \)

Hence, by lemma (ii) of (2.305),

(137) \( \psi(u, \psi) = \sum_{t=0}^{u} \sum_{t=0}^{v} g(u, \psi, t, u, \psi). \)

Now it follows from the definition of \( g(x, t, m) \) that
(138) \[ g(u + v, y, t, u + v) = g(v, y, t, u) g(u, t, v), \]
and from lemma (iii) of 2·305 and (136) that

(139) \[ \sum_{q \in \mathcal{Q}} g(q, y, t, u) = \sum_{q \in \mathcal{Q}} g(q, t, u) = v(u, t) \]

and similarly

(140) \[ \sum_{q \in \mathcal{Q}} g(u, x, t, v) = v(x, t). \]

From (137) — (140) we obtain (135).

2·307. We have

(141) \[ v(u^2 m, u^2 t) = u v(m, t). \]

Proof. By (136),

(142) \[ v(u^2 m, u^2 t) = \sum_{q \in \mathcal{Q}} g(q, u^2 t, u^2 m). \]

Now \( g(q, u^2 t, u^2 m) = 0 \) unless \( q \) is a multiple of \( u \). Hence

(143) \[ \sum_{q \in \mathcal{Q}} g(q, u^2 t, u^2 m) = \sum_{q \in \mathcal{Q}} g(u^2 t, u^2 m). \]

and it follows from the definition of \( g(x, t, m) \) that

(144) \[ g(u^2 t, u^2 m) = g(t, m). \]

By (142), (143), (144), and lemma (iv) of 2·305,

(145) \[ v(u^2 m, u^2 t) = \sum_{q \in \mathcal{Q}} g(v, t, m) = u \sum_{q \in \mathcal{Q}} g(v, t, m), \]

which, together with (136), proves (141).

2·308. An integer is said to be square-free (kubikerfrei) if it is not divisible by any square other than 1. Let us define the auxiliary function \( x(m) \) as 1 or 0 according as \( m \) is or is not square-free. This function is obviously multiplicative. Hence, if we put

(146) \[ v(m, t) = x((m, t)) v(m, t), \]

the inner pair of brackets in \( x((m, t)) \) belonging to the symbol for the greatest common divisor, it follows from 2·306 that \( v(m, t) \) is a multiplicative function of \( m \). Also

In fact, the sum on the right, in spite of its three variables of summation, has only one possible non-vanishing term, namely that in which \( q \) is the greatest integer whose square divides \( m \) and \( n \), and it follows from (145) and (141) that this term is equal to \( v(m, n) \).

2·309. By (133) and (146),

(147) \[ a_s S_s = \sum_{m=1}^{2} \sum_{n=1}^{2} q^{-2} u^{-2} v(u, v) \]

\[ = \sum_{q \in \mathcal{Q}} q^{-2} u^{-2} v(u, v) = \sum_{q \in \mathcal{Q}} q^{-2} T_1(v), \]

where

(148) \[ T_1(v) = \sum_{q \in \mathcal{Q}} u^{-2} v(u, v). \]

Similarly, by (133) and (146),

(149) \[ \frac{3}{4} a_s S_s = \sum_{q \in \mathcal{Q}} q^{-2} T_2(v), \]

where

(150) \[ T_2(v) = \sum_{q \in \mathcal{Q}} u^{-2} v(u, v). \]

By (149),

(151) \[ \frac{3}{4} a_s S_s = S_1 - S_1, \]

where

(152) \[ S_1 = \sum_{q \in \mathcal{Q}} q^{-2} T_1(v), \quad S_1 = \sum_{q \in \mathcal{Q}} q^{-2} T_2(v). \]

Substituting 2\( m \) for \( q \) and \( \frac{1}{4} l \) for \( v \) in the last sum, we obtain

(153) \[ S_1 = \sum_{m=1}^{2} \sum_{l=1}^{2} q^{-2} T_1\left(\frac{1}{4} l\right) = \frac{1}{8} \sum_{m=1}^{2} m^{-3} T_1\left(\frac{1}{4} l\right), \]

where \( T_1(0) = 0 \) if \( w \) is not an integer.
By (130) and (151),
\[ 6a_S S(n) = -24 a_S T_4 + 40 S_2 - 40 S_7. \]
Hence, putting
\[ T_6(n) = -24 T_1(n) + 40 T_2(n) - 5 T_3 \left( \frac{1}{4} \right), \]
we have, by (147), (152), and (153),
\[ a_S S(n) = \sum_{q \mid n} q^{-1} T_6(n). \]

\[ 2 \cdot 310. \] Let \( p \) be a prime. Then
\[ \chi'(p^n, t) = \begin{cases} 1 & (p \nmid t, p > 2), \\ \frac{t}{p} & (p \mid t), \\ 0 & (p \mid t, m > 1). \end{cases} \]
and
\[ \chi'(p^n, t) = \begin{cases} 1 & (p \nmid t, p > 2), \\ \frac{t}{p} & (p \mid t), \end{cases} \]

**Proof.** If \( p \nmid t \) and \( p > 2 \), it is known that \( \chi(p^n, t) \) (as defined in 2.301) is 2 or 0 according as \( t \) is or is not a quadratic residue mod \( p \), and we have \((p^n, t) = 1\), so that \( \chi((p^n, t)) = 1\). From this and (145) we obtain (156).

If \( p \mid t \), we have, by (121),
\[ \chi(p, t) = \sum_{q \mid t} \chi(q) = 1, \]
and \( \chi((p^n, t)) = \chi(p) = 1 \). From these formulae and (145) we obtain (157).

If \( p \nmid t \) and \( m > 1 \), we consider the cases \( p \nmid t \) and \( p^2 \nmid t \) separately. In the former, \((p^n, t)\) is divisible by \( p^3 \) and therefore not square-free, so that \( \chi((p^n, t)) = 0 \). In the latter, by (121),
\[ \chi(p^n, t) = \sum_{q \mid t} \chi(q) = 0, \]
since the condition \( q^2 \equiv t (mod p^n) \) now implies that \( p \mid q^2 \) and \( p^2 \nmid q^2 \), which is impossible, so that the sum is empty. Thus it follows from (145) that (158) holds in either case.

\[ 2 \cdot 311. \] We have
\[ \chi'(1, t) = \chi'(2, t) = 1, \]

\[ \chi'(4, t) = \begin{cases} 2 & (t \equiv 1 (mod 4)), \\ 0 & (otherwise), \end{cases} \]

and
\[ \chi'(2^n, t) = \begin{cases} 4 & (t \equiv 1 (mod 8), m = 3), \\ 0 & (t \equiv 1 (mod 8), m = 3). \end{cases} \]

(159) and (160) follow easily from (145) and (121). If \( t \) is odd, (161) can be established by an argument similar to the proof of (156). If \( t \) is even, (161) is implied in (158).

\[ 2 \cdot 312. \] Let \( p \) be an odd prime. Then
\[ \chi'(p^n, t) = \sum_{q \mid t} \chi(q) \]

This follows easily from 2.310.

\[ 2 \cdot 313. \] We have
\[ \chi(u, l) = \sum_{|\omega|} \omega^{-1} \chi(\omega) \]

**Proof.** It follows from 2.308 and lemma (i) of 2.305 that both sides of (163) are multiplicative functions of \( u \), and the equation is obviously true for \( u = 1 \). Hence it is sufficient to prove that (163) holds if \( u \) is a power of an odd prime, and this follows from (162).

\[ 2 \cdot 314. \] Let
\[ a_u = \sum_{\omega \mid u} \omega^{-1} \chi(\omega). \]
Then, by (150) and (163),
\[ T_2(l) = \sum_{\omega \mid l} \omega^{-2} \chi(\omega) \]

(164)

\[ = \sum_{\omega \mid l} \sum_{\omega' \mid \omega} \omega^{-2} \chi(\omega) = a_u \sum_{\omega \mid l} \left( \frac{l}{\omega} \right) q^{-2}, \]
since \( \left( \frac{l}{\omega} \right) = 0 \) if \( q \) is even.

Since \( \chi'(u, l) \) is a multiplicative function of \( u \), it follows from (148) and (150) that
On the representations of a number as a sum of squares.

\[ \frac{3}{l^2} T_6(l) = a_4 R(l), \]
where \( a_4 \) is a constant. Hence, by (6) and (155),

\[ r_3(n) = c n^3 S(n) = (6 a_4)^{-1} c \sum_{q \mid m} \frac{3}{l^2} T_6(l) \]

where \( a_3 \) is a constant. In particular

\[ r_3(1) = a_3 R(1). \]

Now \( r_3(1) = 10 \), and it follows from (7) that

\[ R(1) = 80 \sum_{m=1}^{\infty} m^{-3} = 10. \]

Hence \( a_4 = 1 \), which, together with (172), proves Theorem 3.

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