Theorem 4 gives that among the Lüroth type expansions the Lüroth expansion is the slowest in convergence. Also, among the well known expansions, for almost all x, the Lüroth expansion requires the largest number of terms to provide the same accuracy.

The case $h = 1$ was recently investigated in detail in [2].

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On the simultaneous diophantine approximation of values of certain algebraic functions*

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INTRODUCTION

In a recent paper [3] the present author obtained a result, which is sketched immediately below, about the simultaneous diophantine approximation of the values of $(N+s_1)^{k_1k^{-1}}, \ldots, (N+s_n)^{k_1k^{-1}}$, where $0 = s_1 < s_2 < \ldots < s_n$ were $n \geqslant 2$ integers, $k \geqslant 2$ was an integer, $1 \leqslant k_1 \leqslant k$ was an integer satisfying $(k_1, k) = 1$, N was a sufficiently large positive integer, and the kth roots above were the positive real kth roots.

Let ε denote a positive real number, $(p_1, \ldots, p_n)$ denote any nonzero vector of nonnegative integers, C denote a real number, and q denote a positive integer. Then three functions $\psi = \psi(s_1, \ldots, s_n, k, k_1, \varepsilon, N)$, $\varphi = \varphi(s_1, \ldots, s_n, k, k_1, \varepsilon, N)$ and $\Lambda = \Lambda(s_1, \ldots, s_n, k, k_1, N)$ were given explicitly(1). It was shown that if $\varepsilon < (2n-4)^{-1}$, $N \geqslant \psi$, $q \geqslant \varphi$, and $0 \leqslant C \leqslant 1$, then

$$\max_{1\leqslant j\leqslant n} \{|C(N+s_j)^{k_1k^{-1}} - p_j q^{-1}|\} \geqslant \frac{1}{n}(2q)^{-\left(1+\frac{1+\varepsilon}{\Lambda}\right)}. \tag{1}$$

Further, as $N \to +\infty$ (and all of the other parameters were held constant) Λ increased to $n-1$.

In this paper we shall prove results allowing us to make statements analogous to (1) about a larger class of algebraic functions. In these statements the auxiliary functions corresponding to φ and ψ above are not given explicitly; however, it is shown that they are effectively computable.

Let Q denote the rational field, C the complex field, $Q(i)$ the Gaussian field, Z the integers, and $Z[i]$ the Gaussian integers. In what follows N will always denote a Gaussian integer.

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(1) The present notation differs slightly from that used in [3].

DEFINITION. For each $N \in Z[i]$ a set of $k \geqslant 2$ functions $w_1(z), \ldots, w_j(z), \ldots, w_k(z)$ will be called N *admissible* if the $w_j(z)$ are the distinct branches of the algebraic function defined by $p(w) = z$ for some

$$p(w) = \sum_{l=0}^{k} a_l w^l$$

with each $a_l \in Z[i]$, $a_k = 1$, $a_0 = 0$, and every $|a_l| \leqslant (\frac{2}{3}k^{-1})^{k^3}|N|^{1-lk^{-1}}$.

Let $0 = a_1, \ldots, a_r, \ldots, a_n$ denote $n \geqslant 2$ distinct Gaussian integers. By q we denote a nonzero Gaussian integer and by $\{p_{r,l}|\ 1 \leqslant r \leqslant n, 1 \leqslant l \leqslant k-1\}$ a collection of Gaussian integers containing at least one nonzero number. Let the $C_j, 1 \leqslant j \leqslant k$, denote any k complex numbers satisfying for some real a independent of N,

$$0 \leqslant |C_j| \leqslant |N|^a, \quad \text{and} \quad \sum_{j=1}^{k} C_j = 0.$$

Let ε denote a positive real number.

THEOREM I. *There exist effectively computable functions $\psi_1 = \psi_1(a_1, \ldots, a_n, a, k, \varepsilon)$ and $\varphi_1 = \varphi_1(n, a, k, \varepsilon)$ such that if N is any Gaussian integer with $|N| \geqslant \psi_1$ and $w_1(z), \ldots, w_k(z)$ are any set of N admissible functions then*

$$(2) \qquad \max_{\substack{1 \leqslant r \leqslant n \\ 1 \leqslant l \leqslant k-1}} \left\{\left|\sum_{j=1}^{k} C_j w_j^l(N+a_r) - p_{r,l}q^{-1}\right|\right\} \geqslant |q|^{-\left(1+\frac{1+\varepsilon}{n-1}\right)}$$

for all q with $|q| \geqslant |N|^{\varphi_1}$.

One may apply a transference theorem to Theorem I, after choosing the constants C_j appropriately, to obtain the result given next concerning linear forms in the $w_j^l(N+a_r)$.

Let the $B_{r,l}$, $1 \leqslant r \leqslant n$ and $1 \leqslant l \leqslant k-1$, and A denote $n(k-1)+1$ Gaussian integers. Let ε denote a positive real number.

THEOREM II. *There exist effectively computable functions $\psi_2 = \psi_2(a_1, \ldots, a_n, k, \varepsilon)$ and $\varphi_2 = \varphi_2(n, k, \varepsilon)$ such that if N is any Gaussian integer with $|N| \geqslant \psi_2$ and $w_1(z), \ldots, w_k(z)$ are any set of N admissible functions then*

$$(3) \qquad \max_{1 \leqslant j \leqslant k} \left\{\left|\sum_{r=1}^{n} \sum_{l=1}^{k-1} B_{r,l} w_j^l(N+a_r) + A\right|\right\} \geqslant (\max_{r,l}\{|B_{r,l}|\})^{-(n-1+\varepsilon)},$$

if $\max_{r,l}\{|B_{r,l}|\} \geqslant |N|^{\varphi_2}$.

Theorems I and II are "best possible" in the sense that if we choose any negative ε above then the respective statements are false for any, effectively computable or not effectively computable, functions $\psi_1, \psi_2, \varphi_1$, and φ_2. We shall see shortly a proof that Theorem II is best "possible".

It follows from the proof which we shall later give for Theorem II that if Theorem I holds for some negative ε then Theorem II holds for some (possibly different) negative ε, so Theorem I is best possible also.

We shall see later that each $w_j(z)$ above may be obtained by analytic continuation of any branch about $z = \infty$ an appropriate number of times. This leads us to:

COROLLARY I. *Let $w_j(z)$ be any function satisfying $p(w_j(z)) = z$ for some $p(w) \in Q[i, w]$ of degree $k \geqslant 2$ and let $\delta_1, \ldots, \delta_n$ be any $n \geqslant 2$ distinct elements of $Q(i)$. Given $\varepsilon > 0$ and $h > \varphi_2(k, n, \varepsilon)$ it is impossible to find a collection of $B_{r,l}(z) \in C[z]$, $1 \leqslant r \leqslant n$ and $1 \leqslant l \leqslant k-1$, with $\max_{r,l}\{\deg B_{r,l}(z)\} \geqslant h$ and any $A(z) \in C(z)$ such that*

$$(4) \qquad \sum_{r=1}^{n} \sum_{l=1}^{k-1} B_{r,l}(z) w_j^l(z+\delta_r) + A(z)$$

vanishes at $z = \infty$ to an order larger than

$$(5) \qquad (n-1+\varepsilon)\left(\max_{r,l}\{\deg B_{r,l}(z)\}\right).$$

Proof. If $p(w)$ belonged to $Z[i, w]$; $p(w)$ were monic; the $\delta_1, \ldots, \delta_n$ were Gaussian integers; the $B_{r,l}(z)$ and $A(z)$ each belonged to $Z[i, z]$; and Corollary I were false; we would have by Theorem II a contradiction for z equal to a sufficiently large positive integer, since each analytic continuation of (4) about $z = \infty$ would vanish at $z = \infty$ to the same order as (4).

Suppose in the paragraph above we now allow the $B_{r,l}(z)$ and $A(z)$ to be in $C[z]$ and obtain a counterexample to Corollary I. We shall next show that then we may obtain a counterexample with the $B_{r,l}(z)$ and $A(z)$ in $Q[i, z]$; hence, one may obtain a counterexample with these polynomials in $Z[i, z]$.

Suppose that $p(w) = w^k + a_{k-1}w^{k-1} + \ldots + a_1 w$. We shall see later in this paper that the expansions of the $w_j(z)$ about $z = \infty$ are each in descending powers of $z^{k^{-1}}$ and begin with the power $z^{k^{-1}}$. Thus one solution is given by $w_1(z) = z^{k^{-1}} - k^{-1}a_{k-1}z^0 + \ldots + b_l z^{-lk^{-1}} + \ldots$ Substituting this series in $p(w) = z$ we see that for each $l \geqslant 1$, b_l equals a linear combination over $Q(i)$ of $b_{l-1}, \ldots, b_{-1}$ and 1. Thus each $b_l \in Q(i)$. Note that in our supposed counterexample $\deg A(z) \leqslant \max_{r,l}\{\deg B_{r,l}(z)\} = d$. We see that because of the existence of the counterexample a certain system of homogeneous linear equations in $(n(k-1)+1)(d+1)$ variables with coefficients in $Q(i)$ has rank less than $(n(k-1)+1)(d+1)$. Then there exist $e_1, \ldots, e_{d_1}$ (for $d_1 \geqslant 1$), a maximal linearly independent set (over $Q(i)$) of solution vectors of the system of homogeneous equations, with each e_j, $1 \leqslant j \leqslant d_1$, having entries in $Q(i)$. Each solution vector with complex entries is a complex linear combination of $e_1, \ldots, e_{d_1}$. Therefore, corresponding to one of the vectors e_j we have an expression of type (4) in $w_1(z)$ with $B_{r,l}(z)$ and $A(z)$ belonging to $Q[i,z]$ and with $\max_{r,l}\{\deg B_{r,l}\} = d \geqslant h$, which vanishes to an order larger than or equal to (5). This is a contradiction.

Now suppose that we are in the general case. There exist positive integers d_2 and d_3 such that for a monic $p_1(w)$ (of degree k) belonging to $Z[i,w]$ each $p_1(d_2 w_j(z)) = d_3 z$ $(1 \leqslant j \leqslant k)$ and each $d_3\delta_r$ $(1 \leqslant r \leqslant n)$ belongs to $Z[i]$. The cases proven apply to show that

$$\sum_{r=1}^{n}\sum_{l=1}^{k-1} B_{r,l}(d_2 z)\big(d_2 w_1(z+\delta_r)\big)^l + A(d_2 z)$$

can not vanish at $d_2 z = \infty$ to an order greater than (5) in $d_2 z$ if $\max_{r,l}\{\deg B_{r,l}(z)\} \geqslant h$. Corollary I follows.

Corollary II. (i) *Inequality* (3) *will hold for all sufficiently large* $|N|$ *and* $\max_{r,l}\{\log_{|N|}(|B_{r,l}|)\}$ *if and only if* $\varepsilon > 0$.

(ii) *Line* (4) *will always vanish to an order less than or equal to* $(n-1+\varepsilon)\max_{r,l}\{\deg B_{r,l}(z)\}$ *at* $z=\infty$ *for all sufficiently large* $\max_{r,l}\{\deg B_{r,l}(z)\}$ *if and only if* $\varepsilon > 0$.

Proof. In each case the "if" part has been shown. Also, it will suffice to show the "only if" case for statement (ii). Consider for any positive integer h_1 the problem of constructing a collection of $B_{r,l}(z)$, each belonging to $Q[i,z]$ and having degree less than or equal to h_1, such that

$$\sum_{r=1}^{n}\sum_{l=1}^{k-1} B_{r,l}(z)\Big[k w_1^l(z+\delta_r) - \sum_{t=1}^{k} w_t^l(z+\delta_r)\Big]$$

vanishes at $z=\infty$ to at least the order $(n-1)(h_1+1)-k^{-1} > (n-1)h_1$. We see that this leads to $n(k-1)(h_1+1)-1$ simultaneous linear equations in $n(k-1)(h_1+1)$ unknowns with coefficients in $Q(i)$. Thus we may construct our $B_{r,l}(z)$. Set

$$A(z) = -\sum_{r=1}^{n}\sum_{l=1}^{k-1} B_{r,l}(z)\sum_{t=1}^{k} w_t^l(z+\delta_r).$$

Now we would be through if we knew that going through this procedure for $h_1 = 1, 2, \ldots$ the sequence of values $\max_{r,l}\{\deg B_{r,l}(z)\}$ obtained is unbounded. On the other hand if the sequence is bounded then for each positive integer t

$$\big((n-1)(h_1+1)-k^{-1}-t\big)\big(t+\max_{r,l}\{\deg B_{r,l}(z)\}\big)^{-1} \to \infty \quad \text{with} \quad h_1.$$

Hence for each t if h_1 is sufficiently large the

$$\sum_{r=1}^{n}\sum_{l=1}^{k-1} z^t B_{r,l}(z) w_1^l(z+\delta_r) + z^t A(z)$$

each vanish at $z=\infty$ to an order larger than $(n-1)\max_{r,l}\{z^t B_{r,l}(z)\}$. This proves Corollary II.

In the course of proving Theorem I we shall construct forms of the type appearing on the left hand side of (3), which come very near to violating (3). It follows from the proof of Theorem I that we in fact have:

Corollary of proof of Theorem I. *For each* $\varepsilon_1 > 0$ *there exists an effectively computable constant* $\beta_0(\varepsilon_1)$ *such that, for every* $N \in Z[i]$ *with* $|N| > \beta_0(\varepsilon_1)$, *inequality* (3) *is violated infinitely often if* ε *is less than* $-\varepsilon_1$.

Now to consider a slightly different sort of problem. Let

$$p(w,z) \stackrel{\text{def}}{=} w^k + \sum_{l=0}^{k-1} a_l(z) w^l$$

where each $a_l(z) \in Z[i,z]$, $k \geqslant 2$, and $\deg a_0(z) = d > 0$. Suppose, also, that the k roots $w_1(z), \ldots, w_k(z)$ of $p(w(z), z) = 0$ have expansions about $z=\infty$ in series involving decreasing fractional powers of z where *no two* $w_j(z)$ *have identical initial* (i.e. *dominant*) *terms*. Next consider the equation $p(w,z) = u$. For each $u \in C$ the set of dominant terms of the expansions about $z=\infty$ of the roots of $p(w,z) = u$ is the same, as will be shown in Lemma XV. Let us enumerate these dominant terms. For each $u \in C$ let $w_j(u,z)$, $1 \leqslant j \leqslant k$, denote that root of $p(w,z) = u$ which has the jth dominant term. (In fact each $w_j(u,z)$ is an analytic function of both u and z on appropriate subregions of $C \times C$.) Set

$$\gamma = \max_{0 \leqslant l \leqslant k-1}\big\{(\deg a_l(z))(\deg a_0(z))^{-1} k(k-l)^{-1}\big\} \geqslant 1.$$

Let $\alpha_1, \ldots, \alpha_n$ denote $n \geqslant 2$ distinct elements of $Z[i]$. Let the $p_{j,r}$ $(1 \leqslant j \leqslant k, 1 \leqslant r \leqslant n)$ and q denote elements of $Z[i]$ with $q \neq 0$. Let $B_{r,l}$ $(1 \leqslant j \leqslant k, 1 \leqslant r \leqslant n)$ and $B_{0,0}$ denote elements of $Z[i]$. Let a denote a real number and ε a positive real number. Let $C_1, \ldots, C_k$ denote k complex numbers satisfying

$$\sum_{j=1}^{k} C_j = 0 \quad \text{and} \quad \max_{1 \leqslant j \leqslant k}\{|C_j|\} \leqslant |N|^a$$

where a is independent of N.

THEOREM III. (a) *There exist two effectively computable functions* $\psi_3 = \psi_3(p(w, z), \alpha_1, \ldots, \alpha_n, \alpha, \varepsilon)$ *and* $\varphi_3 = \varphi_3(\gamma, d, k, n, \alpha, \varepsilon)$ *such that if N is any Gaussian integer with* $|N| \geqslant \psi_3$ *then*

$$\max_{r,l}\Big\{\Big|\sum_{j=1}^{k} C_j w_j^l(\alpha_r, N) - p_{r,l}q^{-1}\Big|\Big\} \geqslant |q|^{-\left(1+\frac{\gamma(1+\varepsilon)}{n-1}\right)}, \tag{6}$$

if $|q| > |N|^{\varphi_3}$.

(b) *If* $\gamma = 1$ *there exist effectively computable functions* $\psi_3' = \psi_3'(p(w, z), \alpha_1, \ldots, \alpha_n, \varepsilon)$ *and* $\varphi_3' = \varphi_3'(d, k, n, \varepsilon)$ *such that for all Gaussian integers N with* $|N| \geqslant \psi_3'$

$$\max_{1\leqslant j\leqslant k}\Big\{\Big|\sum_{r=1}^{n}\sum_{l=1}^{k} B_{r,l}w_j^l(\alpha_r, N) + B_{0,0}\Big|\Big\} \geqslant \big(\max_{r,l}\{|B_{r,l}|\}\big)^{-(n-1+\varepsilon)}, \tag{7}$$

if $\max\limits_{r,l}\{|B_{r,l}|\} > |N|^{\varphi_3'}$.

Inequality (7) is obtained from inequality (6) by the same proceedure as gives Theorem II from Theorem I. If $\gamma \neq 1$ is sufficiently close to 1 a statement of type (7) may be obtained but with a power of $\max\limits_{r,l}\{|B_{r,l}|\}$ less than $-(n-1+\varepsilon)$ appearing on the right hand side. We shall now show that if the $a_l(z)$, for $1 \leqslant l \leqslant k-1$, belong to $Z[i, z]$ and, for all sufficiently large $|z|$, $|a_l(z)| \leqslant (\frac{2}{3}k^{-1})^{k^3}\cdot|a_0(z)|^{1-lk^{-1}}$ then the k different solutions $w_j(z)$ of $w^k + \sum_{l=1}^{k-1} a_l(z)w^l = 0$ each have different dominant terms in their expansions about $z = \infty$. This will show that Theorem III (b) is at least as strong as any statement of its type which might be proven directly from Theorem II. (To see that Theorem III (b) is stronger consider the example $p(w) = (w+z)(w+2z)\ldots(w+(k-1)z)$.)

First we show that if $\gamma = 1$ the expansion of each root about $z = \infty$ begins with the same power of z. Let $d = \deg a_0(z) \geqslant 1$. One sees that the dominant terms of the expansion of each $w_j(z)$ about $z = \infty$ must involve z to a power less than or equal to dk^{-1}, by examination of the equation $p(w, z) = 0$. Since $\deg\big(\prod_{j=1}^{k} w_j(z)\big) = d$ it follows that each dominant term must be of the form $\gamma_j z^{dk^{-1}}$ for some $\gamma_j \neq 0$. Let $p_1(w) = w^k + \sum_{l=0}^{k-1} b_l w^l$ where b_l is the coefficient of $z^{d(1-lk^{-1})}$ in $a_l(z)$. Note $b_0 \neq 0$. Then for each $1 \leqslant l \leqslant k-1$, $|b_l| \leqslant (\frac{2}{3}k^{-1})^{k^3}$. Also $p_1(\gamma_j) = 0$ for each $1 \leqslant j \leqslant k$. In Lemma XI we show that under these conditions each $p_1'(\gamma_j) \neq 0$; therefore, the dominant terms of the expansions of the $w_j(z)$ about $z = \infty$ are distinct.

If $\gamma = (k, \deg a_0(z)) = 1$ the analogue of Corollary I of Theorem II follows easily with the $w_j^l(\delta_r, z)$ replacing the $w_j^l(z+\delta_r)$ except that we must require that the $B_{r,l}(z)$ and $A(z)$ belong to $Q(i, z)$. We need this latter condition since the exact form of the coefficients of the expansions of the $w_j(\delta_r, z)$ about $z = \infty$ is not too clear in general and we might not be able to pair the vanishing of

$$\sum_{r=1}^{n}\sum_{l=1}^{k-1} B_{r,l}(z)w_j(\delta_r, z) + A(z)$$

to a given order at infinity with a system of homogeneous linear equations with coefficients in $Q(i)$ having a nontrivial solution. Also, this difficulty makes a result analogous to Corollary II of Theorem II appear to be doubtful. However one may prove the following result.

Suppose that $p(w, z)\in Z[w, z]$, $p(w, z) = w^k + \sum_{l=0}^{k-1} a_l(z)w^l$ for some prime integer $k > 2$, $d = \deg a_0(z) \geqslant 1$, $(k, d) = 1$, and

$$\max_{0\leqslant l\leqslant k-1}\{\deg a_l(z)(\deg a_0(z))^{-1}k(k-l)^{-1}\} = 1.$$

Let $\delta_1, \ldots, \delta_r, \ldots, \delta_n$ denote $n \geqslant 1$ distinct elements of Q. Let the $B_{r,l}(z)$ $(1 \leqslant r \leqslant n, 1 \leqslant l \leqslant k-1)$ and $A(z)$ denote elements of $Q[z]$. Let $\{j(r)\}$ denote the collection of all functions from $\{1, \ldots, n\}$ to $\{1, \ldots, k\}$. Let the $w_j(u, z)$ $(1 \leqslant j \leqslant k)$, denote the k distinct solutions of $p(w, z) = u$. Let ε denote a positive real number.

THEOREM IV. *There exists an effectively computable function* $\varphi_4 = \varphi_4(d, k, n, \varepsilon)$ *such that*

$$\deg\Big(\prod_{\{j(r)\}}\Big(\sum_{r=1}^{n}\sum_{l=1}^{k-1} B_{r,l}(z)w_{j(r)}^l(\delta_r, z) + A(z)\Big)\Big) \geqslant \Big(k^n - k^{n-1} - \sum_{j=1}^{n-1} k^j - \varepsilon\Big)\big(\max_{r,l}\{\deg B_{r,l}(z)\}\big) \tag{8}$$

if $\max\{\deg B_{r,l}(z), 1 \leqslant r \leqslant n, 1 \leqslant l \leqslant k-1; \deg A(z)\} \geqslant \varphi_4$.

We note that the product in (8) must belong to $Q[i, z, B_{1,1}, \ldots, B_{n,k-1}, A]$. An application of the above theorem is to equations of the form

$$\prod_{\{j(r)\}}\Big(\sum_{r=1}^{n}\sum_{l=1}^{k-1} X_{r,l}(z)w_{j(r)}^l(\delta_r, z) + X(z)\Big) = Y(z) \tag{9}$$

where the $X_{r,l}(z)$ and $X(z)$ are unknown elements of $Q[z]$ and $Y(z)$ is a polynomial in the $X_{r,l}(z)$, $X(z)$, and z. In many cases there will exist an effectively computable bound on $\max\{\deg X_{r,l}(z), 1 \leqslant r \leqslant n, 1 \leqslant l \leqslant k-1, \deg X(z)\}$, because of Theorem IV.

One could also prove a result analogous to Theorem IV for the case where $\gamma > 1$ but γ is "sufficiently close" to 1.

In a future paper the present author will extend the methods of this paper to treat other functions including hypergeometric and generalized hypergeometric functions. Further, for any non-polynomial function which is algebraic over $Q(i, z)$, we shall obtain lower bounds upon the simultaneous diophantine approximation of a set of numbers consisting not only of values at $z = N+a_1, \ldots, N+a_n$ of powers of the algebraic function but also (regretably) *values at* $z = N+a_1, \ldots, N+a_n$ *of certain generally nonalgebraic repeated integrals of the algebraic function*. (Note that if $w_j(z)$ is a solution of $p(w) = z$ then $\frac{dz}{dw_j(z)} = p'(w_j(z))$, so each repeated integral of $w_j(z)$ with respect to z is an algebraic function over $Q(i, z)$ if the constants of integration are chosen appropriately.) In Sections I and II of this paper the approach is more general than is needed for the proofs of Theorems I–IV. In particular Theorem V in Section I, below, will be helpful in this future paper.

Section I

Suppose that, where $c_1, \ldots, c_t$ are $t \geqslant 1$ indeterminants and $y_1(z), \ldots \ldots, y_t(z)$ are t functions, $y = \sum_{j=1}^{t} c_j y_j(z)$ satisfies a linear homogeneous differential equation with coefficients in $Q(i, z)$. One may put this equation in the form

$$q_0(zD)y(z) = \sum_{l \geqslant 1} D^l q_l(zD)y(z) \tag{10}$$

where $q_0(zD) \not\equiv 0$, and each $q_l(zD)$ $(0 \leqslant l)$ belongs to $Q[i, zD]$. (If necessary multiply through by a power of z and or by a power of D and then use $zD = Dz-1$ repeatedly.) The equation $q_0(t) = 0$ we call the indicial equation of (10) corresponding to an expansion about $z = \infty$. The roots of $q_0(t)$ are the roots of the indicial equation about $z = \infty$ of the original equation for $y(z)$ plus $-1, -2, \ldots, -\mu$ where μ is the number of times we differentiated the original equation. (To see this substitute z^m for y in the undifferentiated equation and then differentiate μ times.) Since $D(zD) = (zD+1)D$ we may easily write down the differential equation of type (10) which is satisfied by $D^\lambda y(z)$ for $\lambda = 0, 1, \ldots$ By substituting, for some $\lambda = 0, 1, \ldots, D^\lambda y(z)$ for $y(z)$ we may obtain an equation of type (10) with the property that $q_0(t) \neq 0$ if t is a nonnegative integer. In what follows we shall assume that in (10) $q_0(t) \neq 0$ *if t is a nonnegative integer*.

Choose K to be a positive integer larger than $\max_{l \geqslant 0}\{l + \deg q_l(zD)\}$ = the order of (10). Given $a_1, \ldots, a_n$, $n \geqslant 2$, distinct Gaussian integers, set $\varphi(z) = \prod_{r=1}^{n} (z+a_r)$. Where here only $[x]$ denotes the greatest integer function, set for each positive integer M, $m = [n^{-1}(M+n-1)]$ and $h = mn-M$, so $M = mn-h$ with $0 \leqslant h \leqslant n-1$. Let $E^t y(z)$ denote any t-fold primitive of $y(z)$ if t is a positive integer and denote $D^{-t}y(z)$ if t is a nonpositive integer.

For each positive integer M and each $-K \leqslant \delta \leqslant \min(K, M-m)$ form the function

$$R_M^\delta(y) = \int_c \big(\varphi(z-t)\big)^{-m}(z-t)^h \big(E^{m-1+\delta}y(t)\big)\,dt$$

where c is a closed path in the t plane which winds once about $\{z+a_r | r = 1, \ldots, n\}$ in the positive direction and $|z|$ is sufficiently large that $y(z)$ is analytic in a simply connected region containing the path c. We shall also write

$$R_M^\delta = \big((\varphi(z))^{-m}z^h\big) * \big(E^{m-1+\delta}y(z)\big).$$

It follows that $R_M^\delta(y)$ is well defined regardless of the choice of $E^{m-1+\delta}y(t)$, since $\big(\varphi(z-t)\big)^{-m}(z-t)^h$ vanishes to the order $M \geqslant m+\delta$.

One may verify that if f and g are each analytic on c then

$$f*(Dg) = (Df)*g \tag{i}$$

and

$$f*(zg) = z(f*g)-(zf)*g. \tag{ii}$$

From this it follows that

$$f*(zDg) = z(Df*g)-(Dzf*g). \tag{iii}$$

Also if $\varphi(z)$, h, m, and δ are as above then,

$$\begin{aligned}\big((\varphi(z))^{-m}z^h\big)*\big(E^{m-1+\delta}(zD)g\big) \\ = \big((\varphi(z))^{-m}z^h\big)*\big(zE^{m-2+\delta}g-(m-1+\delta)E^{m-1+\delta}g\big).\end{aligned} \tag{iv}$$

Using (i)–(iv) we verify that

$$\begin{aligned}
&\big((\varphi(z))^{-m}z^h\big)*\big(E^{m-1+\delta}(zD)y(z)\big)\\
&= \big((\varphi(z))^{-m}z^h\big)*\big((zD)E^{m-1+\delta}y(z)\big)-\\
&\quad -\big((\varphi(z))^{-m}z^h\big)*\big((m-1+\delta)E^{m-1+\delta}y(z)\big)\\
&= -\big(D(\varphi(z))^{-m}z^{h+1}\big)*\big(E^{m-1+\delta}y(z)\big)+\\
&\quad +z\Big[\big(D(\varphi(z))^{-m}z^h\big)*E^{m-1+\delta}y(z)\Big]-\\
&\quad -(m-1+\delta)\Big[\big((\varphi(z))^{-m}z^h\big)*E^{m-1+\delta}y(z)\Big]\\
&= \big(M-(m+\delta)\big)\Big[\big((\varphi(z))^{-m}z^h\big)*\big(E^{m-1+\delta}y(z)\big)+\\
&\quad +\big((\varphi(z))^{-(m+1)}\big(mz^{h+1}\varphi'(z)-mnz^h\varphi(z)\big)\big)*E^{m-1+\delta}\big(y(z)\big)+\\
&\quad +z\big[(\varphi(z))^{-(m+1)}\big(hz^{h-1}\varphi(z)-mz^h\varphi'(z)\big)*E^{m-1+\delta}y(z)\Big].
\end{aligned} \tag{11}$$

Now

$$\deg[mz^{h+1}\varphi'(z)-mnz^h\varphi(z)] \quad \text{and} \quad \deg[hz^{h-1}\varphi(z)-mz^h\varphi'(z)]$$

are each at most $n+h-1$. Therefore we may rewrite the final expression in (11), if $\delta \geqslant 0$, as a linear combination over $Q[i,z]$ of terms of the form

$$\big((\varphi(z))^{-m'}z^{h'}\big)*\big(E^{m'-1-\delta'}y(z)\big)$$

where $0 \leqslant h' \leqslant n-1$, $m'+\delta' = m+\delta$ and m' equals m or $m+1$. The coefficient of the term where $h' = h$, $m' = m$, and (necessarily) $\delta' = \delta$ is $M-(m+\delta)$. If we now write for each $\delta \geqslant 0$

$$(12) \qquad \big((\varphi(z))^{-m}z^h\big)*\big(E^{m-1+\delta}[q_0(zD)-\sum_{l\geqslant 1}D^l q_l(zD)]y(z)\big) = 0$$

we see that this leads to a linear relation among the different $R_{M'}^{\delta'}(y)$ where $M \leqslant M' \leqslant n(m+K) \leqslant M+n(K+1)$ and $\delta' \geqslant \delta-K$. Also the coefficient of the term where $M' = M$ and $\delta' = \delta$ is $q_0\big(M-(m+\delta)\big)$. Since $M-(m+\delta) \geqslant 0$, $q_0\big(M-(m+\delta)\big) \neq 0$.

LEMMA I. *Each of the* $R_M^\delta(y), \ldots, R_{M+n(K+1)-1}^\delta(y)$ *may be written as a linear combination over* $Q[i,z]$ *of the* $R_{M+1}^\delta(y), \ldots, R_{M+n(K+1)}^\delta(y)$ *for each* $M \geqslant 1$.

Proof. We must show how to write each $R_M^\delta(y)$ as a linear combination over $Q[i,z]$ of the stated objects. If $\delta \geqslant 0$ is as large as is allowed we apply the relation obtained from (12) and obtain terms of the stated sort plus terms where $M' = M$ and $-K \leqslant \delta' < \delta$. Thus we need only consider the case when δ is not the largest value allowable. We note that then

$$\big((\varphi(z))^{-m}z^h\big)*\big(E^{m-1+\delta}y(z)\big) = \big(D(\varphi(z))^{-m}z^h\big)*\big(E^{m-1+(\delta+1)}y(z)\big)$$

which leads to an expression of $R_M^\delta(y)$ as a linear combination over $Q(i)$ of terms of the form $R_{M'}^{\delta'}$ where $M+1 \leqslant M' \leqslant M+2n$ and δ' equals δ or $\delta+1$. Since $K+1 \geqslant 2$ we are through.

LEMMA II. *For each* $M \geqslant 1$ *the module* P_M *over* $Q[i,z]$ *spanned by the* $(2\pi i)^{-1}R_M^\delta(y), \ldots, (2\pi i)^{-1}R_{M+n(K+1)-1}^\delta(y)$ *contains the elements* $D^j y(z+a_r)$, *for* $0 \leqslant j \leqslant K$ *and* $1 \leqslant r \leqslant n$, *and the elements*

$$(2\pi i)^{-1}\int_c \big(E^{l-1}y(t)\big)(z+a_{r_1}-t)^{-1}\ldots(z+a_{r_l}-t)^{-1}dt$$

where $r_1, \ldots, r_l$ *are any* $1 \leqslant l \leqslant n$ *distinct elements of* $\{1, 2, \ldots, n\}$.

Proof. By Lemma I we need only show this for $M = 1$. Each

$$(2\pi i)^{-1}\int_c \big(D^j y(t)\big)(z+a_r-t)^{-1}dt, \quad 0 \leqslant j \leqslant K,$$

may be written as a linear combination over $Q[i,z]$ of the

$$(2\pi i)^{-1}\Big[\big((\varphi(z))^{-1}z^h\big)*\big(E^{-j}y(z)\big)\Big]$$

where $0 \leqslant h \leqslant n-1$, so this gives the result for the $D^j y(z+a_r)$. Each

$$(2\pi i)^{-1}\int_c \big(E^{l-1}y(t)\big)(z+a_{r_1}-t)^{-1}\ldots(z+a_{r_l}-t)^{-1}dt$$

may be written as a linear combination over $Q[i,z]$ of the

$$(2\pi i)^{-1}\Big[\big((\varphi(z))^{-1}z^h\big)*\big(E^{l-1}y(z)\big)\Big] \quad \text{where} \quad 0 \leqslant h \leqslant n-1.$$

We see that then $0 \leqslant l-1 \leqslant M-1 = M-m$, so we are through.

(Obviously one may write down other elements which are in each P_M, but we shall need to know that the $D^j y(z+a_r)$ are included for the proof of the next lemma; also, the fact that certain simple linear combinations of integrals of the $y(z+a_r)$ are included indicates that in general each $(2\pi i)^{-1}R_M^\delta$ may not be expressed in terms of the derivatives of the $y(z+a_r)$ alone.)

In equation (10) let us use $zD = Dz-1$ repeatedly to obtain an equation of the form

$$\sum_{K\geqslant j\geqslant 0} s_j(z)D^j y(z) = 0$$

where each $s_j(z) \in Z[i,z]$. Let $s_{j_1}(z)$ be the coefficient of the highest derivative of $y(z)$ actually appearing above. Set $s(z) = \prod_{r=1}^{n} s_{j_1}(z+a_r)$. Note that if $\theta \geqslant K+1$ then each $D^\theta y(z+a_r)$ belongs to $\big(s(z)\big)^{\theta-K}P_1 \subseteq \big(s(z)\big)^{\theta-K}P_M$ for $M = 1, 2, \ldots$

LEMMA III. *Let* K *be chosen sufficiently large that* $-K$ *is less than the smallest integral zero of* $q_0(t)$. *Then for each* $M \geqslant 1$ *there exists a nonnegative integer* $J(M)$ *such that* $\big(s(z)\big)^{J(M)}P_M \subseteq P_1$.

Proof. All that we need to do is to show that every $(2\pi i)^{-1}R_M^\delta(y)$ may be written as a linear combination of $Q[i,z]$ of elements belonging to P_1 and different $D^\theta y(z+a_r)$, for $\theta \geqslant 0$.

From (10) we see that for each nonnegative integer λ

$$q_0(zD-\lambda)E^\lambda y(z) = \sum_{l\geqslant 1} q_l(zD-\lambda+l)E^{\lambda-l}y(z)+r_\lambda(z)$$

where $r_\lambda(z)$ is a polynomial of degree at most $\lambda-1$. If $\lambda \geqslant K$ then, under the hypotheses of the lemma, we are able to write an equation of the form

$$(13) \qquad E^\lambda y(z) = \sum_{\substack{l\geqslant 1\\ K\geqslant l\geqslant j\geqslant 0}} \gamma_{j,l}z^j E^{\lambda-l}y(z)+r_\lambda(z)$$

where the $\gamma_{j,l} = \gamma_{j,l}(\lambda)$ are elements of $Q(i)$.

Now we suppose that we are given

$$(2\pi i)^{-1}\Big[\big((\varphi(z))^{-m}z^h\big)*E^\lambda y(z)\Big]$$

where $\lambda \geqslant K$, $mn-h > \lambda$, and $0 \leqslant h < +\infty$. We may use line (13) to write

$$(2\pi i)^{-1}\Big[\big((\varphi(z))^{-m}z^h\big)*\big(E^\lambda y(z)\big)\Big] = \sum_{\substack{l\geqslant 1\\ K\geqslant l\geqslant j\geqslant 0}} \gamma_{j,l}(2\pi i)^{-1}\Big[\big((\varphi(z))^{-m}z^h\big)*\big(z^j E^{\lambda-l}y(z)\big)\Big].$$

Using property (ii) which was established for $*$ products we see that for a collection of $t_{j,l}(z)$ in $Q[i,z]$ we have

$$(2\pi i)^{-1}\Big[\big((\varphi(z))^{-m}z^h\big)*\big(E^\lambda y(z)\big)\Big] = \sum_{\substack{l\geqslant 1\\ K\geqslant l\geqslant j\geqslant 0}} t_{j,l}(z)(2\pi i)^{-1}\Big[\big((\varphi(z))^{-m}z^{h+j}\big)*\big(E^{\lambda-l}y(z)\big)\Big].$$

We note that $mn-(h+j) > \lambda - l$, so one may apply the same procedure, repeatedly, to all terms where the power of E is larger than or equal to K. One finally arrives at a linear combination over $Q[i,z]$ of terms of the form

$$(2\pi i)^{-1}\Big[\big((\varphi(z))^{-m}z^{h+l}\big)*(E^\theta y)\Big]$$

where $0 \leqslant \theta < K$.

We next use property (i) of $*$ products $K-1-\theta$ times to obtain in each case terms of the form

$$(2\pi i)^{-1}\Big[\Big(D^{K-1-\theta}\big((\varphi(z))^{-m}z^{h+l}\big)\Big)*(E^{K-1}y)\Big],$$

and may write the collected terms in the form

$$(14)\qquad (2\pi i)^{-1}\int_c \big(\varphi(z-t)\big)^{-\tau}r(z,z-t)\big(E^{K-1}y(t)\big)\,dt$$

for some positive integer τ and some $r(X_1,X_2)\in Q[i,X_1,X_2]$. The order of vanishing of $\big(\varphi(X_2)\big)^{-\tau}r(X_1,X_2)$ at $X_2=\infty$ (where X_1 remains indeterminant) we denote by Φ. Without loss of generality we may assume that $\Phi \geqslant 1$, i.e. subtract away terms which give rise to integrals which are zero.

Now restricting our choice of h to $0 \leqslant h \leqslant n-1$ we see that the quantity in (14) is $(2\pi i)^{-1}R_M^\delta(y)$. Notice that $(2\pi i)^{-1}R_M^\delta(y)$ is defined independently of the definition of $E^{K-1}y(t)$. If $1 \leqslant \Phi \leqslant K-1$, however, then replacing $E^{K-1}y(t)$ by $E^{K-1}y(t)+t^{\Phi-1}$ would change (14) by the addition of a nonzero polynomial in z. Thus $\Phi \geqslant K$. We may write the

partial fraction decomposition of $\big(\varphi(X_2)\big)^{-\tau}r(X_1,X_2)$ considered as a function of X_2 and recombine certain of the terms to obtain the identity

$$\big(\varphi(X_2)\big)^{-\tau}r(X_1,X_2) = \big(\varphi(X_2)\big)^{-(K-1)}\mu(X_1,X_2) + \sum_{j\geqslant K,r}\mu_{r,j}(X_1)(X_2+a_r)^{-j}$$

for some $\mu(X_1,X_2)$ in $Q[i,X_1,X_2]$ and collection of $\mu_{r,j}(X_1)$ in $Q[i,X_1]$, where $\big(\varphi(X_2)\big)^{-(K-1)}\mu(X_1,X_2)$ vanishes at $X_2=\infty$ to an order larger than or equal to K.

From the above it follows that $(2\pi i)^{-1}R_M^\delta(y)$ equals a linear combination over $Q[i,z]$ of derivatives of the $y(z+a_r)$ plus

$$(2\pi i)^{-1}\int_c \big(\varphi(z-t)\big)^{-(K-1)}\mu(z,z-t)\big(E^{K-1}y(t)\big)\,dt.$$

We may write this last term as a linear combination over $Q[i,z]$ of terms of the form

$$(2\pi i)^{-1}\int_c \big(\varphi(z-t)\big)^{-m}(z-t)^h\big(E^{K-1}y(t)\big)\,dt$$

where $1 \leqslant m \leqslant K-1$, $0 \leqslant h \leqslant n-1$, and $M = mn-h \geqslant \Phi \geqslant K$. Setting $K-1 = m-1+\delta$ we see $\delta \geqslant 1$, $\delta = K-m \leqslant M-m$, and $\delta = K-m \leqslant K-1 < K$. Since the number of derivatives of the $y(z+a_r)$ which can appear is finite we see that there exists $J(M)$ such that $\big(s(z)\big)^{J(M)}P_M \subseteq P_1$. This proves Lemma III.

Lemma IV. *There exists a positive integer p, depending on $y(z)$ and K and independent of M such that each module P_M has a module basis consisting of exactly p linearly independent elements.*

Proof. By Lemmas I and III the different modules P_M generate the same vector space V over $Q(i,z)$. Since $y(z) \not\equiv 0$, $[V: Q(i,z)] = p > 0$. If K is a field then any finitely generated module over $K[x]$ has a linearly independent module basis. Since here any module basis for P_M is a vector space basis for $V = Q(i,z)P_M$ we see that this basis must have exactly p elements. This proves Lemma IV.

Suppose next that $v_1,\ldots,v_p$ are p linearly independent elements of P_1 and that for a sequence of nonzero $T_M(z)\in Q[i,z]$ each

$$T_M(z)P_M \subseteq Q[i,z]v_1+\ldots+Q[i,z]v_j+\ldots+Q[i,z]v_p.$$

Lemma V. *For each $M \geqslant 1$ the matrix of coefficients of the $T_M(z)\times$ $\times(2\pi i)^{-1}R_{M+j}^\delta(y)$, $0 \leqslant j \leqslant (K+1)n-1$, when written in terms of the $v_1,\ldots,v_p$ has rank exactly p for all complex numbers z other than the zeros of $T_M(z)$.*

Proof. We note that the rank is at most p because of the dimensions of the matrix. Let $h_1(M),\ldots,h_p(M)$ be a linearly independent module basis for P_M. Let $\Delta_M(z)$ be the matrix of coefficients when the $T_M(z)\times$

$\times h_1(M), \ldots, T_M(z)h_p(M)$ are written in terms of $v_1, \ldots, v_p$. Since each $v_j \in P_1 \subseteq P_M$ we see that there exists $\Delta'_M(z)$, a p by p matrix of elements in $Q[i, z]$, such that $\Delta_M(z)\Delta'_M(z)$ gives the coefficients when the $T_M(z)h_1(M), \ldots, T_M(z)h_p(M)$ are written in terms of the basis $h_1(M), \ldots, h_p(M)$. Thus $\det(\Delta_M(z)\cdot\Delta'_M(z)) = (T_M(z))^p$, so $\Delta_M(z)$ has rank p if z is not a zero of $T_M(z)$.

Since each $T_M(z)h_k(M)$, $1 \leqslant k \leqslant p$, may be written as a linear combination of the $T_M(z)(2\pi i)^{-1}R^{\delta}_{M+j}(y)$, $0 \leqslant j \leqslant (K+1)n-1$, over $Q[i, z]$ we see that $\Delta_M(z) = \Delta''_M(z)\,\Delta'''_M(z)$ where $\Delta''_M(z)$ and $\Delta'''_M(z)$ are each matrices with entries in $Q[i, z]$ and $\Delta'''(z)$ is the matrix of coefficients of the $T_M(z)(2\pi i)^{-1}R^{\delta}_{M+j}(y)$, $0 \leqslant j \leqslant (K+1)n-1$, when written in terms of the basis $v_1, \ldots, v_p$. Thus the rank of Δ'''_M is at least p if z is not a zero of $T_M(z)$. This proves Lemma V.

Below for each $R^{\delta}_M(y)$ we denote by $E^{\theta}y(z)$, $0 \leqslant \theta \leqslant m+\delta$, the function $D^{m+\delta-\theta}E^{m+\delta}y(z)$. Note that the definition of E^{θ} depends upon M and δ and that $E^{\theta_1}E^{\theta_2} \neq E^{\theta_1+\theta_2}$ in general. In this paper β with a subscript will always denote an effectively computable constant.

LEMMA VI. *The functions*

$$(2\pi i)^{-1}(m+2K)!\Big(\prod_{k>l}(a_k-a_l)^2\Big)^{m+K+1}R^{\delta}_{M+j}(y)$$

for $0 \leqslant \delta \leqslant K$, $0 \leqslant j \leqslant (K+1)n-1$ *are each equal to a linear combination over* $Z[i]$ *of the functions* $(\theta-1)!E^{\theta}y(z+a_r)$, $1 \leqslant \theta \leqslant m+2K+1$, *and the functions* $D^{\varphi}y(z+a_r)$, $0 \leqslant \varphi \leqslant K$, *with coefficients which are less in absolute value than* β_1^m *for an effectively computable* β_1 *independent of* m.

Proof. It will suffice to show that each

$$(2\pi i)^{-1}\Big(\prod_{k\geqslant l}(a_k-a_l)^2\Big)^m(m+K-1)!\,R^{\delta}_M(y)$$

can be expressed as a linear combination over $Z[i]$ of the $(\theta-1)!\times \times E^{\theta}y(z+a_r)$ and the $D^{\varphi}y(z+a_r)$, but for $1 \leqslant \theta \leqslant m+K$, with coefficients which are less in absolute value than β_1^m. Then the result stated in the lemma will hold for a possibly larger effectively computable constant β_1.

Writing each $(z+a_k-t)^{-m}$ as

$$\big((z+a_l-t)(a_k-a_l)^{-1}+1\big)^{-m}(a_k-a_l)^{-m};$$

if $l \neq k$, and $E^{m-1+\delta}y(t)$ as

$$\sum_{p=0}^{\infty}\big(E^{m-1+\delta-p}y(z+a_r)\big)(t-z-a_l)^p(p!)^{-1},$$

we proceed to evaluate the residue of the integrand of $R^{\delta}_M(y)$ at $t = z+a_l$. After multiplication through by the quantity $\Big(\prod_{k>l}(a_k-a_l)^2\Big)^m(m+K-1)!$

we obtain a sum of terms of the form a Gaussian integer times

$$(m+K-1)!(p!)^{-1}E^{m-1+\delta-p}y(z+a_l) \quad \text{where} \quad 0 \leqslant p \leqslant m-1.$$

If $m-1+\delta-p \geqslant 1$ set $\theta = m-1+\delta-p \geqslant 1$ and note that we have a Gaussian integer times

$$(m+K-1)\ldots(m+\delta-1)\big[(m+\delta-2)!\big((\theta-1)!p!\big)^{-1}\big](\theta-1)!E^{\theta}y(z+a_l)$$

with $1 \leqslant \theta < m+\delta \leqslant m+K$. If $m-1+\delta-p < 0$ set $\varphi = -(m-1+\delta-p)$. Note here $1 \leqslant m-p \leqslant K$. Then our term is of the form a Gaussian integer times $(m+K-1)\ldots(p+1)$ times $D^{\varphi}y(z+a_r)$ where $1 \leqslant \varphi \leqslant K$.

The Gaussian integers mentioned above are in each case $\Big(\prod_{k>l}(a_k-a_l)^2\Big)^m$ times the residue of $(\varphi(z-t))^{-m}(z-t)^h(t-z-a_l)^p$ at $t = z+a_l$. This equals

$$\Big(\prod_{k>l}(a_k-a_l)^2\Big)^m(2\pi i)^{-1}\int_{c'}(\varphi(-t))^{-m}(-t)^h(t-a_l)^p\,dt$$

where c' is any closed path which winds once about $t = a_l$ in the positive direction and not at all about the other poles of the integrand. The absolute value of the above expression may be bounded from above by β_2^m for an effectively computable β_2 independent of m. This suffices to prove the Lemma since $(m+K-1)\ldots(m+\delta) \leqslant (m+K)^{2K}$, $(m+\delta-2)!\big((\theta-1)!p!\big)^{-1} \leqslant 2^{m+K-2}$, and $(m+K-1)\ldots(p+1) \leqslant (m+K)^K$ if $1 \leqslant m-p \leqslant K$. This proves Lemma VI.

We next state the first of two conditions which will be formulated in this section. They repeat, somewhat, the assumptions which we have already made on each $y(z)$ but demand, additionally, uniformity with respect to all $y(z)$ in a given class of functions, along with other assumptions.

CONDITION A. Suppose that we are given a class π of "functions" $y(z) = \sum_{j=1}^{t} c_j y_j(z)$ where the c_j are indeterminants, the $y_j(z)$ are functions, and t is effectively bounded from above (¹). Suppose that each $y(z) = y \in \pi$ satisfies a linear homogeneous differential equation of the form

$$(15) \qquad q_0(zD)y(z) = \sum_{l\geqslant 1} D^l q_l(zD)y(z)$$

where $q_0(zD) \not\equiv 0$ and each $q_l(zD)$ $(0 \leqslant l)$ belong to $Q[i, zD]$. Suppose that the orders of the differential equations of type (15) are uniformly bounded from above by some effectively computable $K \geqslant 1$ for all $y \in \pi$, and that in each equation $q_0(s) \neq 0$ if s is a nonnegative integer or if s is a negative integer less than or equal to $-K$. Let $a_1, \ldots, a_n$ denote $n \geqslant 1$ distinct Gaussian integers. Suppose that where $0 < \eta < 1$ is effecti-

(¹) Except in the expression $2\pi i$, the symbol π will never refer in this paper to the real number.

vely computable and depends only on π, there exist three effectively computable functions $K_1 = K_1(y, a_1, \ldots, a_n)$, $K_2(y)$, and $K_3(y)$, with $K_1(y, a_1, \ldots, a_n) \geqslant 1+2(\max_r\{|a_r|\})$ being real valued, $K_2(y)$ being complex valued, and $K_3(y)$ taking values on $[0, +\infty]$, such that, for all z and t satisfying $|z| \geqslant K_1(y, a_1, \ldots, a_n)$, $|z-K_2(y)| \leqslant K_3(y)$, and $|z-t| \leqslant \eta|z|$, $y(t)$ is analytic and satisfies $|y(t)| \leqslant \beta_3|t|^{\beta_4}$ for effectively computable constants β_3 and β_4 depending only on π, where $|\sum_{j=1}^{t} c_j f_j(z)| \stackrel{\text{def}}{=} \max_j\{|f_j(z)|\}$.

Lemma VII. *Under Condition A there exist effectively computable numbers β_5 and β_6 dependent only on π and n such that for all $y \in \pi$ and z satisfying $|z| \geqslant K_1$ and $|z-K_2| < K_3$,*

$$\left|(2\pi i)^{-1}(m+2K)!\Big(\prod_{k>l}(a_k-a_l)^2\Big)^{m+K+1} R^{\delta}_{M+j}(y)\right| \leqslant \beta_5^m |z|^{-(n-1)m+\beta_6},$$

for all $0 \leqslant j \leqslant n(K+1)-1$ and $0 \leqslant \delta \leqslant K$.

Proof. We shall obtain, for each $M \geqslant 1$ and $-K \leqslant \delta \leqslant \min\{M-m, K\}$, the upper bound $\big((m+2K)!\big)^{-1}\beta_7^m|z|^{-(n-1)m+\beta_8}$ for $|(2\pi i)^{-1}R^{\delta}_M(y)|$, where β_7 and β_8 are effectively computable constants depending only on π and n. Then slightly larger values of β_7 and β_8 will suffice to bound each $|(2\pi i)^{-1}R^{\delta}_{M+j}(y)|$ for $0 \leqslant j \leqslant n(K+1)-1$. This will prove Lemma VII.

Suppose that $|z| \geqslant K_1 > 2$ and $c = c(t)$ is a path going around $\{t|\ |z-t| = \eta(|z|-1)\}$ once in the positive direction. Then if t lies on c, $(1+\eta)|z| > |t| > (1-\eta)|z|$ and each $|z-t+a_r| \geqslant \eta(|z|-1)-\frac{1}{2}\eta(K_1-1) \geqslant \frac{1}{2}\eta(|z|-1) \geqslant \frac{1}{4}\eta|z|$.

If $m-1+\delta \geqslant 1$ set

$$E^{m-1+\delta}y(t) = \int_z^t \big((m-2+\delta)!\big)^{-1}(s-z)^{m-2+\delta}y(s)\,ds$$

where the path of integration is the line segment between z and t. Then for each δ,

$$\begin{aligned}&|E^{m-1+\delta}y(t)|\\ &\quad\leqslant \big((m+2K)!\big)^{-1}\Big[\big((m+2K)!\big)\big((m-2+\delta)!\,(2K-2+\delta)!\big)^{-1}\Big]\beta_9^m|z|^{m+\beta_{10}}\end{aligned}$$

for all $t \in c$, where β_9 and β_{10} are effectively computable and independent of m, δ, and z. (Recall K is a constant.)

In the finite number of cases in which $m-1+\delta \leqslant 0$ we may estimate $|E^{m-1+\delta}y(t)|$ for all $t \in c$ by an integral with respect to s about a path which goes around $\{s|\ |s-t| = \frac{1}{2}\eta\}$ once in the positive direction and obtain again the above bound, for possibly a larger set of constants β_9 and β_{10}.

Hence

$$\begin{aligned}&|(2\pi i)^{-1}R^{\delta}_M(y)|\\ &\quad\leqslant \big((m+2K)!\big)^{-1}2^{m+2K}(2K-2+\delta)!\,\beta_9^m|z|^{m+\beta_{10}}(4\eta^{-1}|z|^{-1})^{mn}|2z|^{h+1}\\ &\quad\leqslant \big((m+2K)!\big)^{-1}\beta_7^m|z|^{-(n-1)m+\beta_8}.\end{aligned}$$

This proves Lemma VII.

Definitions. By the *height* of a polynomial in a finite number of variables we mean the maximum of the absolute values of its coefficients.

If $r_1(z)$ and $r_2(z)$ belong to $Q[i, z]$ by $r_1(z) \ll r_2(z)$ we mean that the absolute value of the coefficient of each power of z in $r_1(z)$ is less than or equal to the coefficient of the corresponding power of z in $r_2(z)$. (Hence $r_2(z)$ has real, nonnegative coefficients.)

Suppose that Condition A holds. Let $a_1, \ldots, a_n$ denote n distinct Gaussian integers. For each positive integer $1 \leqslant M \leqslant n(K+1)-1$ write $M = mn-h$ where m and h are integers with $1 \leqslant m \leqslant K+1$ and $0 \leqslant h \leqslant n-1$. For each integer δ with $-K \leqslant \delta \leqslant \min(K, 1-m)$ form the functions

$$(16)\qquad (2\pi i)^{-1}\int_c (t-z-a_1)^{-m}\ldots(t-z-a_n)^{-m}(z-t)^h\big(E^{m-1-\delta}y(t)\big)\,dt$$

where $|z| > K_1$, $|z-K_2| \leqslant K_3$, and c winds once about $\{z|\ |z-t| = \eta(|z|-1)\}$ once in the positive direction. Let $P_1(y)$ denote the module over $Z[i, z]$ generated by the functions in (16). Denote by $p = p(y)$ the dimension of $Q(i, z)P_1(y)$ over $Q(i, z)$. Let $\beta_{11}, \ldots, \beta_{16}$ denote effectively computable constants depending on π and n.

Condition B. Suppose that for each $y \in \pi$ there exists a collection of $p_1 = p_1(y) \geqslant p(y)$ linearly independent functions over $Q(i, z)$, $U_{1,y}(z), \ldots, U_{p_1,y}(z)$, which is such that each $U_{j,y}(z)$, $1 \leqslant j \leqslant p_1$, belongs to $Q(i, z)P_1(y)$. Suppose that for each $y \in \pi$ there exists $T_{1,y}(z) \in Z[i, z]$ such that (a) $T_{1,y}(z)$ has no zeros z satisfying both $|z| \geqslant K_1$ and $|z-K_2| \leqslant K_3$ and (b) each of the functions $T_{1,y}(z)U_{1,y}(z), \ldots, T_{1,y}(z)U_{p,y}(z)$ belongs to $P_1(y)$. Suppose also, that for every $y \in \pi$ there exist (i) some sequence of repeated integrals of $y, E^1y, \ldots, E^{\theta}y, \ldots$; (ii) some $T_y(z) \in Z[i, z]$ with degree less than β_{11}, height less than $K_1^{\beta_{12}}$, and no zeros z satisfying both $|z| \geqslant K_1$ and $|z-K_2| \leqslant K_3$; (iii) some positive integral valued function $S_y(m) < \beta_{13}^m$; and (iv) some $\gamma \geqslant 1$; such that for each $1 \leqslant r \leqslant n$, $0 \leqslant \theta \leqslant m+K+1$, and $0 \leqslant \varphi \leqslant K$, every $S_y(m)T_y(z)(\theta-1)!\,E_y^{\theta}y(z+a_r)$ and every $S_y(m)T_y(z)Dy(z+a_r)$ equals a linear combination over $Z[i, z]$ of $U_{1,y}(z), \ldots, U_{p_1,y}(z)$ with coefficients

$$\ll \beta_{14}^m K_1^{\beta_{15}}(z^{m+\beta_{16}}-K_1^{\gamma(m+\beta_{16})})(z-K_1^{\gamma})^{-1}.$$

LEMMA VIII. *Under Conditions* A *and* B, *for each* $y(z)\in\pi$ *if* $|z|\geqslant K_1$ *and* $|z-K_2|\leqslant K_3$ *every*

$$L_M^{\delta,j} \overset{\text{def}}{=} \big(S_y(m+2K+1)\big)T_y(z)(2\pi i)^{-1}\Big(\prod_{k>l}(a_k-a_l)^2\Big)^{m+K+1}(m+K+1)!\,R^{\delta}_{M+j}(y)$$

where $0\leqslant\delta\leqslant K$ *and* $0\leqslant j\leqslant n(K+1)-1$, *may be written as a linear combination over* $Z[i,z]$ *of* $U_{1,y}(z),\ldots,U_{p,y}(z)$ *with coefficients*

$$\ll \beta_{17}^m K_1^{\beta_{15}}\big(z^{m+\beta_{16}}-K_1^{\gamma(m+\beta_{16})}\big)(z-K_1^{\gamma})^{-1},$$

where β_{17} *is effectively computable and dependent only on* π *and* n. *Further each* $|L_M^{\delta,j}(y)|\leqslant\beta_{18}^m K_1^{\beta_{12}}|z|^{-(n-1)m+\beta_{19}}$ *for effectively computable constants* β_{18} *and* β_{19} *dependent only on* π *and* n.

Proof. The $U_{1,y}(z),\ldots,U_{p,y}(z)$ are a basis for the vector space over $Q(i,z)$ generated by $P_1(y)$. Since each $L_M^{\delta,j}(y)$ belongs to $Q(i,z)P_1(y)$ the coefficients of $U_{p+1,y}(z),\ldots,U_{p_1,y}(z)$ obtained when one writes $L_M^{\delta,j}(y)$ in terms of $U_{1,y}(z),\ldots,U_{p_1,y}(z)$ must be zero. The various estimates follow almost immediately from Lemma VI and Condition B, and from Lemma VII and Condition B, respectively. This proves Lemma VIII.

DEFINITION. Let the s_j, $1\leqslant j\leqslant p(y)$, denote a collection of Gaussian integers which are not all zero.

THEOREM V. *Suppose that Conditions* A *and* B *each hold. Then for every* $r,\varepsilon>0$ *there exists an effectively computable number* $\psi_5=\psi_5(\pi,a_1,\ldots,a_n,r,\varepsilon)$ *such that if* N *is a Gaussian integer and* $y(z)$ *is any element of* π *with* $|N|\geqslant\max\{\psi_5,K_1(y,a_1,\ldots,a_n)\}$ *and* $|N-K_2(y)|\leqslant K_3(y)$ *we have, for all t-tuples of complex numbers satisfying* $\max_j\{|c_j|\}\leqslant|N|^r$,

$$\max_{1\leqslant j\leqslant p(y)}\{|U_j(N)-s_jq^{-1}|\}\geqslant|q|^{-\left(1+\frac{\gamma(1+\varepsilon)}{n-1}\right)} \tag{17}$$

for all nonzero Gaussian integers q *with* $|q|\geqslant|N|^{\beta_{20}}$, *for an effectively computable* β_{20} *dependent only on* π,n,r, *and* ε.

Proof. First we present with only minor changes in notation the Lemma from [1]. Below $\|\text{matrix}\|$ denotes the maximum of the absolute values of its entries.

Suppose that for some positive integer t we have a sequence Δ_m of t by t nonsingular matrices over the integers and a t by 1 matrix $V\neq0$ with real entries. Let $f(m)$ be a monotone increasing function from the nonnegative reals onto the interval $[1,+\infty)$. Let $0<\varepsilon<+\infty$, $0<r_1<+\infty$, and $\left(1+\frac{\varepsilon}{r_1}\right)^2\left(1-\frac{\varepsilon}{r_1}\right)^{-1}<1+\varepsilon$. Suppose that for all nonnegative integers $m\geqslant m_1\geqslant1$

(i) $\|\Delta_m\|\leqslant(f(m))^{1+\frac{\varepsilon}{r_1}}$,

(ii) $\|\Delta_m V\|\leqslant(f(m))^{-\Delta\left(1-\frac{\varepsilon}{r_1}\right)}$ for some $\Delta>0$; and

(iii) $f(m)\leqslant(f(m-1))^{1+\frac{\varepsilon}{r_1}}$.

Let q denote a nonnegative integer, S denote a general t by 1 matrix of integers with not all entries zero.

LEMMA. *If* $q\geqslant\frac12(f(m_1))^{\Delta\left(1-\frac{\varepsilon}{r_1}\right)}$ *then*

$$\|V-Sq^{-1}\|\geqslant t^{-1}(2q)^{-\left(1+\frac{1+\varepsilon}{\Delta}\right)}.$$

Examination of the proof of the Lemma enables us to see that if the Δ_m have Gaussian integral entries, $V\neq0$ has complex entries, S is a general t by 1 matrix of Gaussian integers, and $q\neq0$ is a Gaussian integer then if $|q|\geqslant\frac12(f(m_1))^{\Delta\left(1-\frac{\varepsilon}{r_1}\right)}$ we have

$$\|V-Sq^{-1}\|>t^{-1}(2|q|)^{-\left(1+\frac{1+\varepsilon}{\Delta}\right)}.$$

We now apply this slight extension of the Lemma of [3] to the proof of Theorem V. Notice that if $\varepsilon<\frac12$, as we shall assume below, without loss of generality, then $r_1=4$ satisfies

$$\left(1+\frac{\varepsilon}{r_1}\right)^2\left(1-\frac{\varepsilon}{r_1}\right)<1+\varepsilon.$$

For each $m\geqslant1$ let Δ_m be any nonsingular $p(y)$ by $p(y)$ submatrix of the matrix of coefficients of the $T_{1,y}(N)L_{mn}^{\delta,j}(y(N))$, for $0\leqslant j\leqslant n(K+1)-1$ and $0\leqslant\delta\leqslant K$, when written in terms of the elements $T_{1,y}(N)U_{1,y}(N),\ldots,T_{1,y}(N)U_{p,y}(N)$. Such a submatrix exists by Lemma V since $T_{1,y}(N)T_y(N)\neq0$. Then from Lemma VIII if $m\geqslant1$

$$\|\Delta_m\|\leqslant(m\beta_{16})\beta_{17}^m|N|^{\gamma(m+\beta_{16}-1)}\leqslant|\beta_{21}N|^{\gamma(m+\beta_{16}-1)},$$

where β_{21} is effectively computable.

Set $f(m)=|\beta_{21}N^{\gamma}|^m$. Recall $|N|\geqslant K_1>2$. For some effectively computable m_1 depending only on β_{21} (which depends only on π and n), $\varepsilon<\frac12$, and γ (which depends only on π), parts (i) and (iii) of the hypotheses of the Lemma from [3] hold with $r_1=4$. Set V equal to the column matrix with entries $U_{1,y}(N)\ldots U_{p,y}(N)$ which corresponds to each matrix Δ_m.

Then, also from Lemma VIII,

$$\|\Delta_m V\| \leqslant t\beta_{18}^m |N|^{-(n-1)m+\beta_{12}+\beta_{19}+r} \leqslant \big(f(m)\big)^{-(n-1)\gamma^{-1}(1-\varepsilon/4)}$$

if N is larger than some effectively computable number which is dependent only on π, n, r, and ε. This establishes part (ii) of the hypotheses of the Lemma of [3].

We notice that $p(y) \leqslant (K+1)n(2K+1)$ by consideration of the generators of $P_1(y)$. Then applying the extension of the Lemma of [3] with $\Delta = \frac{n-1}{\gamma}$ and $|N|$ and $m \geqslant m_1$ as large as required above, we have that the left hand side of (17) is larger than

$$\big(n(K+1)^2\big)^{-1}(2|q|)^{-\left(1+\frac{\gamma(1+\varepsilon)}{n-1}\right)}$$

for all q's with

$$|q| > \tfrac{1}{2}\big(f(m_1)\big)^{\frac{n-1}{\gamma}\left(1-\frac{\varepsilon}{4}\right)} = |\beta_{22}N|^{(n-1)m_1\left(1-\frac{\varepsilon}{4}\right)}.$$

Since $|N| > 2$ we see that the lower bound on $|q|$ may be replaced by $|N|^{\beta_{20}}$ where β_{20} is effectively computable and depends only on π, n, r, and ε. Substituting $\varepsilon/2$ for ε above and requiring that

$$|q|^{\frac{\varepsilon\gamma}{2(n-1)}} > 2^{1+\left(\gamma+\frac{\varepsilon}{2}\right)(n-1)^{-1}} n(2K+1)^2$$

(which does not affect the form of our lower bound on $|q|$) we have proven Theorem V.

Section II

Suppose that $w_1(z), \ldots, w_k(z)$, are the k (not necessarily distinct) algebraic functions defined by $q(w,z) = 0$ where $q(w,z) \in Z[i, w, z]$, $q(w,z)$ has degree exactly $k \geqslant 1$ in w, and the coefficient of w^k in $q(w,z)$ is one. We recall (see [2], p. 118) that each $w_j(z)$, $1 \leqslant j \leqslant k_1$, is analytic except (possibly) at a finite number of points in the extended plane. Renumbering if necessary suppose that $w_1(z), \ldots, w_d(z)$ $(d \leqslant k)$ is a maximal linearly independent subset of $w_1(z), \ldots, w_k(z)$. For any functions $f_1, \ldots, f_d$ analytic in a domain D let $W(f_1, \ldots, f_d)$ denote the Wronskian of $f_1, \ldots, f_d$. If $f_1, \ldots, f_d$ are linearly independent then $W(f_1, \ldots, f_d) \neq 0$ and

$$W(f_1, \ldots, f_d, w)\big(W(f_1, \ldots, f_d)\big)^{-1} = 0$$

is a linear homogeneous differential equation of order d satisfied by $f_1, \ldots, f_d$ (see [4], Chapter 4); further, the above equation is the only linear homogeneous differential equation of order d in $p(w)$ with the coefficient of $w^{(d)}$ equal to one which is satisfied by $f_1, \ldots, f_d$, since if we had two such equations we could subtract them and obtain a linear homogeneous differential equation of order less than d satisfied by $f_1, \ldots, f_d$. The equation

$$W(w_1, \ldots, w_d, w)\big(W(w_1, \ldots, w_d)\big)^{-1} = 0 \tag{18}$$

has coefficients in $Q(i, z, w_1, \ldots, w_d)$. Suppose that σ is any automorphism of the Galois Group of the least normal extension of $Q(i, z, w_1, \ldots, w_d)$. Then $W(\sigma w_1, \ldots, \sigma w_d, w)\big(W(\sigma w_1, \ldots, \sigma w_d)\big)^{-1} = 0$ is satisfied by σw_j, $1 \leqslant j \leqslant k$, and so by the w_j, $1 \leqslant j \leqslant d$. From our uniqueness result for (18) each σ must leave every coefficient of (18) unchanged. Thus the coefficients of (18) are in $Q(i, z)$.

Note that if σ is as above,

$$\sigma W(w_1, \ldots, w_d) = W(\sigma w_1, \ldots, \sigma w_d) = \varrho W(w_1, \ldots, w_d)$$

for some complex ϱ since $\sigma(w_1), \ldots, \sigma(w_d)$ are d linearly independent solutions of (18) (see [4]). The order of σ divides $k!$; hence,

$$W(w_1, \ldots, w_d) = \sigma^{k!} W(w_1, \ldots, w_d) = \varrho^{k!} W(w_1, \ldots, w_d).$$

Thus for each σ,

$$\sigma\big(W(w_1, \ldots, w_d)\big)^{k!} = \big(W(w_1, \ldots, w_d)\big)^{k!}$$

and it follows that $\big(W(w_1, \ldots, w_d)\big)^{k!}$ belongs to $Q(i, z)$.

Let $a(z)$ be any element of $Z[i, z, w_j]$ such that each $a(z)w_j'(z)$ belongs to $Z[i, z, w_j]$, for each $1 \leqslant j \leqslant k$. Then each $\big(a(z)D\big)^l w_j(z)$ belongs to $Z[i, z, w_j]$ for every integer $l \geqslant 0$. It follows by induction on l that each $\big(a(z)\big)^l D^l w_j(z)$ belongs to $Z[i, z, w_j]$. We shall show that the coefficients of

$$\big(a(z)\big)^{(k-1)k(k!)}\big(W(w_1, \ldots, w_d)\big)^{k!-1} W(w_1, \ldots, w_d, w) = 0 \tag{19}$$

are in $Z[i, z]$. The coefficients of (19) are in

$$Z[i, w_1(z), \ldots, w_k(z)] \cap Q(i, z).$$

The desired result will follow from:

LEMMA IX. *If $w_1(z), \ldots, w_j(z), \ldots, w_k(z)$ are the roots of a monic polynomial over $Z[i, z]$ then where I is $Z[i, z, w_1(z), \ldots, w_k(z)] \cap Q(i, z)$ we have $I = Z[i, z]$.*

If we know that Lemma IX is true, then one can easily conclude from what was shown above that:

LEMMA X. *Suppose that $q(w, z)$ belongs to $Z[i, w, z]$, has degree $k \geqslant 1$ in w, and as a polynomial in w is monic. Suppose that $w_1(z), \ldots, w_k(z)$ are the k (not necessarily distinct) solutions of $q(w, z) = 0$. Let $r(X)$ be some element of $Z[i, X]$ and θ be a nonnegative integer. Suppose that for some $1 \leqslant d \leqslant k$ the functions $D^\theta r\big(w_1(z)\big), \ldots, D^\theta r\big(w_d(z)\big)$ are a maximal linearly independent subset over C from among the collection of analytic* *continuations of* $D^{\theta} r(w_1(z))$. *Let* $a(z)$, *a nonzero element of* $Z[i, z]$, *be chosen such that each* $a(z) w_j'(z)$, $1 \leqslant j \leqslant k$, *belongs to* $Z[i, z, w]$. *Then*

$$(20)\qquad \big(a(z)\big)^{(\theta+k-1)k(k!)} W\Big(D^{\theta} r\big(w_1(z)\big), \ldots, D^{\theta} r\big(w_d(z)\big)\Big)^{k!-1} \times$$
$$\times W\Big(D^{\theta} r\big(w_1(z)\big), \ldots, D^{\theta} r\big(w_d(z)\big), w\Big) = 0$$

is a linear homogeneous differential equation of order exactly d *with coefficients in* $Z[i, z]$ *which is satisfied by each* $D^{\theta} r\big(w_j(z)\big)$, $1 \leqslant j \leqslant k$.

Proof of Lemma IX. Because the $w_j(z)$ are roots of a monic polynomial over $Z[i, z]$ it follows that each $w_j(z)$ is bounded in every bounded region. Thus we see

$$I = Z[i, z, w_1(z), \ldots, w_k(z)] \cap Q[i, z].$$

Since each $w_j(z)$ is integral over $Z[i, z]$ (i.e. is the root of a monic polynomial with coefficients in $Z[i, z]$) we have that each element of $Z[i, z, w_1, \ldots, w_k]$ is integral over $Z[i, z]$. We shall next show that $Z[i, z]$ is integrally closed in $Q[i, z]$, from which the desired result follows.

Suppose that $b^{-1} c(z)$ is integral over $Z[i, z]$ where $b \neq 0, \pm 1$, or $\pm i$ and $c(z)$ belongs to $Z[i, z]$ and is primitive (i.e. the only common divisors of its coefficients are ± 1 and $\pm i$). Then for some positive integer n, $b^{-1}\big(c(z)\big)^n$ belongs to $Z[i, z]$, which is impossible since by Gauss's Lemma $\big(c(z)\big)^n$ must be a primitive polynomial. This contradiction proves Lemma IX.

Section III

Consider the function of two variables $p(w) - z$ where, for some $k \geqslant 2$, $p(w) = w^k + a_{k-1} w^{k-1} + \ldots + a_1 w^1$ and each a_l, $1 \leqslant l \leqslant k-1$ is a Gaussian integer. By the local mapping theorem each root $w_j(z)$ of $p(w) - z = 0$ is analytic everywhere except at $z = \infty$ and points z such that $p'\big(w_j(z)\big)$ vanishes. Finally we may take $a(z)$, see Lemma X, to be $\prod_{j=1}^{k} p'\big(w_j(z)\big)$.

Lemma XI. *Let* $p(w)$ *be as above, let* N *be a Gaussian integer, and suppose that each* $|a_l| \leqslant (\frac{2}{3} k^{-1})^{k^3} |N|^{1-lk^{-1}}$. *Then if* $|z| \geqslant \frac{1}{2}|N|$:

(i) $|w_j(z)| \leqslant (1 + \frac{2}{3} k^{-1})^k |z|^{k^{-1}}$ $(1 \leqslant j \leqslant k)$; *and*

(ii) *each* $\big|p'\big(w_j(z)\big)\big| > 0$ $(1 \leqslant j \leqslant k)$.

Proof. Set $U = (1 + \frac{2}{3} k^{-1})$. If $|t| \geqslant U^k |z|^{k^{-1}}$ then

$$|p(t)| \geqslant \Big(U^{k^2} - (\tfrac{2}{3} k^{-1})^{k^3} \Big(\sum_{l=1}^{k-1} U^{lk}\Big)\Big) |z|$$
$$= \big(U^{k^2} - (U-1)^{k^3-1} (U^{k^2} - U^k)\big) |z|$$
$$> \big(1 - (U-1)^{k^3-1}\big) U^{k^2} |z| > (\tfrac{2}{3}) U^{k^2} |z| > (\tfrac{2}{3})(2) |z| > |z|.$$

This proves part (i).

Now set $V = 2^{k^{-1}} (1 + \frac{2}{3} k^{-1})^{k^2} = 2^{k^{-1}} U^{k^2}$. Note that if $|z| \geqslant \frac{1}{2}|N|$ each
$$|w_j(z)| \geqslant |z| (U^k |z|^{k^{-1}})^{-(k-1)} > 2^{-k^{-1}} (1 + \tfrac{2}{3} k^{-1})^{-k^2} |N|^{k^{-1}} = V^{-1} |N|^{k^{-1}}.$$
If $|t| \geqslant V^{-1} |N|^{k^{-1}}$ then
$$|p'(t)| \geqslant k(|N| V^{-k})^{\frac{(k-1)}{k}} \Big(1 - \sum_{l=1}^{k-1} (U-1)^{k^3} V^l\Big)$$
and
$$1 - \sum_{l=1}^{k-1} (U-1)^{k^3} V^l) > 1 - 2k(U-1)^{k^3} U^{k^3}.$$

Since $(U-1) U \leqslant \frac{1}{3}(\frac{4}{3}) = (\frac{2}{3})^2$ and $2k(\frac{2}{3})^{2k^3} < \frac{1}{2}$ if $k \geqslant 2$ we have $|p'(t)| > 0$. This proves Lemma XI.

Above there must exist a positive integer a with $1 \leqslant a \leqslant k$ such that $w_1(z^a)$ is single valued in every sufficiently small neighborhood of $z = \infty$. Now $\lim_{z \to \infty} |w_1(z^a)| = \infty$ but for large $|z|$ we see that $|w_1(z^a)|$ is less than $2^{k^3} |z|^{ak^{-1}}$. If $a < k$ we do have a pole at $z = \infty$ but it has order less than one, which is impossible. Thus $a = k$ and $w_1(z) = \sum_{m=-1}^{\infty} b_m (z^{-k^{-1}})^m$, for a sequence of complex numbers b_m, where the above series converges for at least all $|z| \geqslant \frac{1}{2}|N|$. We see that each $w_j(z)$, $1 \leqslant j \leqslant k$, is an analytic continuation of $w_1(z)$ about $z = \infty$ a certain number of times. This implies that $p(w) - z$ is irreducible over $Q(i, z)$ as is $k^k \prod_{j=1}^{k} (wk^{-1} - w_j(z) - a_{k-1} k^{-1})$. This latter polynomial has coefficients in $Z[i, z]$ by Lemma IX.

Lemma XII. *Let the* $w_j(z)$ *be as in this section. For some choice of* $\pm$ *signs the analytic continuations* $y_j(z)$ *of*

$$y_1(z) \overset{\text{def}}{=} \Big(k w_1^{k-1}(z) - \sum_{t=1}^{k} w_t^{k-1}(z)\Big) \pm$$
$$\pm \Big(k w_1^{k-2}(z) - \sum_{t=1}^{k} w_t^{k-2}(z)\Big) \pm \ldots \pm \Big(k w_1(z) - \sum_{t=1}^{k} w_t(z)\Big)$$

obtained by substituting $w_j(z)$, $1 \leqslant j \leqslant k$, *for* $w_1(z)$ *in* $y_1(z)$ *are all distinct; the* $y_1(z), \ldots, y_k(z)$ *span a vector space over* C *of dimension exactly* $k-1$; *and, for each* $1 \leqslant j \leqslant k$, 1 *and the* $y_j^{(l)}(z)$, *for* $0 \leqslant l \leqslant k-2$ *are linearly independent over* $Q(i, z)$.

Proof. Notice that $w_1^l(z)$ for $l = 1, 2, \ldots$ has an expansion about $z = \infty$ which begins with $z^{lk^{-1}}$ times a nonzero coefficient. Thus one of $w_1^{k-1}(z) \pm w_1^{k-2}(z)$ has an expansion about $z = \infty$ beginning with a nonzero coefficient times $z^{k^{-1}(k-1)}$ and then a nonzero coefficient times $z^{k^{-1}(k-2)}$. Continuing we see that one can construct, in the above manner,

$$w_1^{k-1}(z) \pm w_1^{k-2}(z) \pm \ldots \pm w_1(z) = v_1(z) = b_{k-1} z^{k^{-1}(k-1)} +$$
$$+ \ldots + b_l z^{k^{-1} l} + \ldots + b_1 z^{k^{-1}} + (\text{terms of lower degree in } z),$$

where each b_l above is nonzero. It is clear that $v_1(z)$ has at most k distinct analytic continuations since it is a polynomial in $w_1(z)$; also $v_1(z)$ has at least k distinct analytic continuations since $(k, k-1) = 1$. Relabeling the $v_j(z)$ if necessary we have that where $\varrho = \exp(2\pi i k^{-1})$

$$v_j(z) = b_{k-1}(\varrho^{k-1})^j z^{k^{-1}(k-1)} + \ldots + b_l(\varrho^l)^j z^{k^{-1}l} + \\ + \ldots + b_1 \varrho^j z^{k^{-1}} + \ldots = w_j^{k-1}(z) \pm w_j^{k-2}(z) \pm \ldots \pm w_j(z).$$

By the nonvanishing of the Vandermonde determinant $c_1 v_1(z) + \ldots + c_{k-1} v_{k-1}(z) = 0$ for a collection of complex numbers $c_1, \ldots, c_{k-1}$ implies that $c_1 = c_2 = \ldots = c_{k-1} = 0$. Thus the dimension of the vector space over C spanned by the $v_1(z), \ldots, v_k(z)$ is at least $k-1$. Setting

$$y_1(z) = kv_1(z) - \sum_{l=1}^{k-1}\Big(\pm \sum_{j=1}^{k} w_j^l(z)\Big)$$

for any choice of $\pm$ signs gives us a function with exactly k distinct analytic continuations $y_1(z), \ldots, y_j(z), \ldots, y_k(z)$ of which at least $k-1$ are linearly independent over C. If we choose the $\pm$ signs to agree with the signs in the definition of $v_1(z)$ then

$$y_1(z) + \ldots + y_j(z) + \ldots + y_k(z) = 0,$$

so the vector space spanned by the $y_j(z)$'s has dimension exactly $k-1$.

Given any linear differential equation with coefficients and nonhomogeneous term in $Q(i, z)$ which is satisfied by any $y_j(z)$ we see that it is satisfied by every $y_l(z)$, $1 \leqslant l \leqslant k$, and by $k^{-1}\sum_{j=1}^{k} y_j(z) \equiv 0$. Thus the equation is homogeneous and has order at least $k-1$. It follows that for each $1 \leqslant j \leqslant k$ the $y_j^{(l)}(z)$, $0 \leqslant l \leqslant k-2$, and 1 are linearly independent over $Q(i, z)$. This proves Lemma XII.

LEMMA XIII. *Let $w_1(z), \ldots, w_k(z)$ be as in this section and $y_1(z), \ldots, y_k(z)$ be as in Lemma XII. Then there exists $b(z) \in Z[i, z]$ such that*

(i) $b(z)$ does not vanish at any z where every $w_j(z)$ is analytic and

(ii) each $b(z)[kw_j^l(z) - \sum_{t=1}^{k} w_t^l(z)]$, with $1 \leqslant l \leqslant k-1$, may be written as a linear combination over $Z[i, z]$ of the $y_j^{(l)}(z)$ for $0 \leqslant l \leqslant (k!)k^3$.

Proof. We notice from equation (20) in Lemma X that, after setting

$$a(z) = \prod_{j=1}^{k} p'\big(w_j(z)\big), \quad \theta = 0, \quad \text{and} \quad r\big(w_j(z)\big) = y_j(z)$$

the coefficient of $w^{(d)}$ is the nonzero quantity

$$\Big(\prod_{j=1}^{k} p'\big(w_j(z)\big)\Big)^{(k!)k(k-1)}\Big(W\big(y_1(z), \ldots, y_{k-1}(z)\big)\Big)^{k!} \tag{21}$$

in $Z[i, z]$. The degree of this polynomial is at most

$$k!\left(k(k-1)^2 + \frac{k-1}{k}\right) < k!k^3.$$

Some derivative of (21) less than or equal to $k!k^3$ will be a constant. Using the formula for differentiation of a determinant we see that for any $z_1 \in C$ such that each $w_j(z)$ is analytic at $z = z_1$ there exist $0 \leqslant \theta_1 < \theta_2 < \ldots < \theta_{k-1} < (k!)k^3$ such that, where $1 \leqslant j, l \leqslant k-1$, $\det\big(y_j^{(\theta_l)}(z_1)\big) \neq 0$.

We may write for each θ_l $\big(\prod_{t=1}^{k} p'\big(w_t(z)\big)\big)^{\theta_l} y_j^{(\theta_l)}(z)$ as a linear combination over $Z[i, z]$ of 1, $kw_j(z), \ldots, kw_j^{k-1}(z)$. Since $k^{-1}\sum_{j=1}^{k} y_j(z) = 0$ we may write, for each $1 \leqslant l \leqslant k-1$, $\big(\prod_{t=1}^{k} p'\big(w_t(z)\big)\big)^{\theta_l} y_j^{(\theta_l)}(z)$ as a linear combination over $Z[i, z]$ of the $kw_j^l(z) - \sum_{t=1}^{k} w_t^l(z)$. Notice that the matrix of coefficients generated above is square. Denote the determinant of this matrix by $\Delta_{z_1}(z)$. We notice that $\Delta_{z_1}(z) \in Z[i, z]$ and $\Delta_{z_1}(z_1) \neq 0$, since $\Delta_{z_1}(z_1) = 0$ would imply that $\det\big(y_j^{(\theta_l)}(z_1)\big) = 0$ contrary to the choice of $\theta_1, \ldots, \theta_{k-1}$. Now set $b(z)$ equal to the greatest common divisor of the polynomials $\Delta_{z_1}(z)$ for all $z_1 \in C$ such that each $w_j(z)$ is analytic at $z = z_1$. This proves Lemma XIII.

DEFINITION. For each nonnegative integer θ and polynomial $q(X) \in Z[i, X]$ we define

$$E^{\theta}\Big(kq\big(w_j(z)\big) - \sum_{t=1}^{k} q\big(w_t(z)\big)\Big)$$

to be

$$k\big((\theta-1)!\big)^{-1}\Big(\int_0^{w_j(z)} \big(z - p(u)\big)^{\theta-1} q(u) p'(u)\, du\Big) - \\ - \big((\theta-1)!\big)^{-1}\sum_{t=1}^{k}\Big(\int_0^{w_t(z)} \big(z - p(u)\big)^{\theta-1} q(u) p'(u)\, du\Big).$$

Obviously $\dfrac{d}{dz}E^{\theta} = E^{\theta-1}$. Each function $f(z)$ for which E^{θ} is defined may be written as a series of the form

$$\sum_{\substack{m=-c \\ m \not\equiv 0 \pmod k}} b_m z^{-k^{-1}m}$$

for sufficiently large $|z|$.

Since $E^{\theta} f(z)$ is a θ-fold integral it equals

$$\sum_{\substack{m=-c \\ m \not\equiv 0 \pmod k}} b_m (1 - k^{-1}m)^{-1} \ldots (\theta - k^{-1}m)^{-1} z^{\theta - k^{-1}m}$$

plus a polynomial of degree at most $\theta-1$. By construction the sum of the first k analytic continuations of $E^\theta f(z)$ about ∞ is zero. Thus the above-mentioned polynomial is zero. This uniquely describes E^θ.

DEFINITIONS. Where $p(w)-z$ is the irreducible polynomial satisfied by the $w_j(z)$ and $p(w) = \sum_{l=1}^{k} a_l w^l$ set

$$\mu = \max_{1\leqslant l\leqslant k-1} \{|a_l|^{k(k-l)^{-1}}\}.$$

Let $a_1, \ldots, a_n$ denote any n distinct Gaussian integers. Set

$$\mu_1 = \max\{\mu;\ |a_r| \text{ for } 1\leqslant r\leqslant n\}.$$

Let K denote a positive integer.

LEMMA XIV. *For each* $0\leqslant\theta\leqslant m+K+1$ *and* $0\leqslant\varphi\leqslant K$, *each*

$$\Big(\prod_{r=1}^{n}\prod_{j=1}^{k} p'\big(w_j(z+a_r)\big)\Big)^K \times$$
$$\times \big(\text{l.c.m. }\{1, \ldots, k(m+K+1)\}\big)\big((\theta-1)!\big)E^\theta y_j(z+a_r)$$

and each

$$\Big(\prod_{r=1}^{n}\prod_{j=1}^{k} p'\big(w_j(z+a_r)\big)\Big)^K \big(\text{l.c.m. }\{1, 2, \ldots, k(m+K+1)\}\big)D^\varphi y_j(z+a_r)$$

may be written as a linear combination over $Z[i, z]$ *of the*

$$kw_j^l(z+a_r) - \sum_{t=1}^{k} w_t^l(z+a_r), \qquad 1\leqslant l\leqslant k-1,$$

with coefficient polynomials

$$\ll \beta_{23}^m (z^{m+\beta_{24}} - \mu_1^{m+\beta_{24}})(z-\mu_1)^{-1},$$

for effectively computable positive integers β_{23} *and* β_{24} *depending on* K, k, *and* n *but independent of* θ, φ, *the* a_l*'s, and the* a_r*'s.*

Proof. A result due to Rosser [5] says that

$$\text{l.c.m. }\{1, 2, \ldots, k(m+K+1)\} < 2^{2^{\frac{3}{2}k(m+K+1)}}.$$

Let $a(z) = \prod_{t=1}^{k} p'\big(w_t(z)\big)$ and note that it is effectively computable as a polynomial in the elementary symmetric functions in $w_1(z), \ldots, w_k(z)$. Thus $a(z)$ is effectively computable as an element of $Z[i, z, a_1, \ldots, a_{k-1}]$. Since each $a(z)\big(Dw_j(z)\big)$ may be effectively computed as a polynomial in the elementary symmetric functions in $w_1(z), \ldots, w_{j-1}(z), w_{j+1}(z), \ldots, w_k(z)$ (hence in the coefficients of $\big(p(w)-z\big)\big(w-w_j(z)\big)^{-1}$) we see that each $a(z)\big(Dw_j(z)\big)$ may be effectively computed as an element of the module over $Z[i, z, a_1, \ldots, a_{k-1}]$ spanned by 1 and the $w_j^{l'}(z)$ for $1\leqslant l'\leqslant k-1$. Analogously, we can effectively compute each $a(z)Dw_j^l(z)$ for $1\leqslant l\leqslant k-1$ as an element of the module over $Z[i, z, a_1, \ldots, a_{k-1}]$ spanned by 1 and the $w_j^{l'}(z)$; by induction, we may do the same for each $\big(a(z)\big)^\varphi D^\varphi w_j^l(z)$, for $1\leqslant l\leqslant k-1$. We note that each $|a_l|\leqslant\mu^{k-1}$. Then one may effectively choose positive integers β_{23} and β_{24} so that when each

$$\text{l.c.m. }\{1, 2, 3, \ldots, k(m+K+1)\}\Big(\prod_{r=1}^{n} a(z+a_r)\Big)^K D^\varphi y_j(z+a_r),$$

for $0\leqslant\varphi\leqslant K$ and $1\leqslant j\leqslant k$, is written as a linear combination over $Z[i, z, a_1, \ldots, a_{k-1}]$ of the

$$kw_j^l(z+a_r) - \sum_{t=1}^{k} w_t^l(z+a_r), \quad \text{for} \quad 1\leqslant l\leqslant k-1 \text{ and } 1\leqslant r\leqslant n,$$

the coefficients are

$$\ll \beta_{23}^m (z^{m+\beta_{24}} - \mu_1^{m+\beta_{24}})(z-\mu_1)^{-1},$$

and β_{23} and β_{24} are independent of z and the a_l's, $1\leqslant l\leqslant k-1$.

For possibly larger values of β_{23} and β_{24} the statement about the different

$$\Big(\prod_{r=1}^{n} a(z+a_r)\Big)^K\big(\text{l.c.m. }\{1, 2, \ldots, k(m+K+1)\}\big)\big((\theta-1)!\big)E_{y_j}^\theta(z+a_r)$$

will now be shown to hold also. First we notice that in each case all denominators introduced by integration have been cancelled so that we are dealing with an element of $Z[i, z, w_j(z+a_r), a_1, \ldots, a_{k-1}]$. Then using $z+a_r = p\big(w_j(z+a_r)\big)$ repeatedly we may write by this "reduction scheme" our polynomial in $z, w_j(z+a_r), a_1, \ldots, a_{k-1}$ as a linear combination over $Z[i, z, a_1, \ldots, a_{k-1}, a_1, \ldots, a_n]$ of the

$$kw_j^l(z+a_r) - \sum_{t=1}^{k} w_t^l(z+a_r), \quad \text{for} \quad 1\leqslant l\leqslant k-1,$$

and so as a linear combination over $Z[i, z, a_1, \ldots, a_{k-1}, a_1, \ldots, a_n]$ of 1 and the $w_j^l(z+a_r)$. It would suffice to prove the desired inequalities for this latter set of coefficients. We introduce a weighted degree, d_w, on polynomials in $z, w_j(z+a_r), a_1, \ldots, a_{k-1}, a_1, \ldots, a_n$. Set d_w of a polynomial equal to the maximum over all monomials of the expression k^{-1}(degree in $w_j(z+a_r)$ + degree in a_{k-1}) + $2k^{-1}$(degree in a_{k-2}) + ... + + $(k-1)k^{-1}$(degree in a_1) + (degree in z) + (degree in a_1) + ... + (degree in a_n). Notice that in the "reduction scheme" above d_w does not increase. Therefore one may effectively compute from the original polynomial times the integral a positive integer β_{24} independent of m such that $m+\beta_{24}-1$ is larger than the weighted degree d_w of the coefficients of 1 and the $w_j^l(z+a_r)$ obtained at the end of the "reduction scheme". We note that if b is a power product of $a_1, \ldots, a_{k-1}, \alpha_1, \ldots, \alpha_n$ with weighted degree at most s then $|b| \leqslant \mu_1^s$. Thus, not taking into account in the coefficients of 1 and $w_j^l(z+a_r)$, $1 \leqslant l \leqslant k-1$, their own coefficients in $Z[i]$ we would have that the coefficients of 1 and $w_j^l(z+a_r)$, $1 \leqslant l \leqslant k-1$, are

$$\ll (z^{m+\beta_{24}} - \mu_1^{m+\beta_{24}})(z-\mu_1)^{-1},$$

where β_{24} is independent of m, the a_l's, and the a_r's. For an effectively computable β_{23} independent of m, the a_l's and the a_r's, the factor β_{23}^m bounds the absolute values of the coefficients in $Z[i]$. This proves Lemma XIV.

Section IV

We shall prove below Theorem I, using the results in Sections I, II, and III, and assuming that the $kw_j^l(z+a_r) - \sum_{s=1}^{k} w_s^l(z+a_r)$, for $1 \leqslant l \leqslant k-1$ and $1 \leqslant r \leqslant n$, are linearly independent. (The reader should have no difficulty in showing that this is the case when no a_r equals a difference of two distinct singular points of any $w_j(z)$.) If the $kw_j^l(z+a_r) - \sum_{s=1}^{k} w_s^l(z+a_r)$ are not independent we may nonconstructively extend the $kw_j^l(z+a_1) - \sum_{s=1}^{k} w_s^l(z+a_1)$, $1 \leqslant l \leqslant k-1$, to a maximal linearly independent proper subset of the $kw_j^l(z+a_r) - \sum_{s=1}^{k} w_s^l(z+a_r)$, say $v_{1,j}(z), \ldots, v_{p,j}(z)$. Our method of proof below will be to apply Theorem V to $y(z) = \sum_{j=1}^{k} c_j y_j(z)$ where each $y_j(z)$ is obtained as in Lemma XII and the $U_{1,y}(z), \ldots, U_{p,y}(z)$ are the

$$\sum_{j=1}^{k} c_j \Big(kw_j^l(z+a_r) - \sum_{s=1}^{k} w_s^l(z+a_r)\Big), \quad \text{for} \quad 1 \leqslant l \leqslant k-1 \text{ and } 1 \leqslant r \leqslant n.$$

If the $kw_j^l(z+a_r) - \sum_{s=1}^{k} w_s^l(z+a_r)$ are dependent we may use the $\sum_{j=1}^{k} c_j v_{1,j}(z), \ldots, \sum_{j=1}^{k} c_j v_{p,j}(z)$ for the $U_{1,y}(z), \ldots, U_{p,y}(z)$, and obtain noneffectively the same lower bound on simultaneous diophantine approximation — but this time for the $\sum_{j=1}^{k} c_j v_{1,j}(z), \ldots, \sum_{j=1}^{k} c_j v_{p,j}(z)$. Also, from the upper bound on $\|\Delta_m V\|$ from the proof of Theorem V, we see that for each pair $r, \varepsilon > 0$, if $|N|$ is sufficiently large (the bound is not computable) and N is fixed, then there exist an infinite number of nonzero p-tuples of Gaussian integers A_s such that

$$\Big|\sum_{s=1}^{p} A_s \Big(\sum_{j=1}^{k} c_j v_{s,j}(N)\Big)\Big| < (\max_s \{|A_s|\})^{-(n-1)+\varepsilon},$$

for all $c_1, \ldots, c_k$ with each $|c_j| < |N|^r$.

Somewhat later in this paper, after using a transference theorem argument to show that Theorem II follows from Theorem I, we shall be able to observe by a closely related argument that the two ineffective inequalities just described above are inconsistant. Thus, the $kw_j^l(z+a_r) - \sum_{s=1}^{k} w_s^l(z+a_r)$ will be shown to always be linearly independent.

We shall first verify that Conditions A and B hold for the class π of all functions $\sum_{j=1}^{k} c_j y_j(z)$ $(1 \leqslant j \leqslant k)$, shown to exist in Lemma XII, corresponding to all algebraic equations of the form $p(w) - z = 0$ where $p(w) \in Z[i, w]$ has degree k, for some fixed $k \geqslant 2$.

Using Lemma X one can see that there exists a differential equation of type (20) of order $k-1$ with coefficients bounded from above effectively in degree which is satisfied by $y_1(z), \ldots, y_k(z)$. Thus there exists an equation of type (15) with effectively bounded order which is satisfied by $y_1(z), \ldots, y_k(z)$. In our equation of type (15), $q_0(t)$ will only vanish at nonnegative integers and roots of the indicial polynomial, corresponding to $z = \infty$, of our differential equation of type (20). By the nonvanishing of the Vandermonde determinant and the fact that each $y_j(z)$, $1 \leqslant j \leqslant k$, has powers of z congruent to $(k-1)k^{-1}, \ldots, 2k^{-1}, k^{-1}$ (modulo one) in its expansion about ∞ (and no other powers of z in this expansion) we see that the roots of the above mentioned indicial polynomial must be exactly $(k-1)k^{-1}, \ldots, 2k^{-1}, k^{-1}$. Therefore we have that any $K \geqslant 1$ will satisfy the condition here. We set $\eta = \frac{1}{2}$, $K_1(y, a_1, \ldots, a_n) = \max_r \{(1 + 2\max + \{|a_r|\}), N_1\}$, $K_2(y) = 0$, and $K_3(y) = +\infty$, where N_1 is the smallest positive integer such that each $w_j(z)$ is N_1 admissible. From Lemma XI and the definition of the $y_j(z)$ we see that we have satisfied Condition A, for $\sum_{j=1}^{k} c_j y_j(z)$.

For Condition B take

$$\{U_{1,y}(z), \ldots, U_{p_1,y}(z)\} = \Big\{\sum_{j=1}^{k} c_j \Big(kw_j^l(z+a_r) - \sum_{t=1}^{k} w_t^l(z+a_r)\Big), 1 \leqslant l \leqslant k-1 \text{ and } 1 \leqslant r \leqslant n\Big\}.$$

These latter functions are all linearly independent over $Q(i, z)$ by assumption. Set $p(y) = p_1(y)$, $T_{1,y}(z) = \prod_{r=1}^{n} b(z+a_r)$ where $b(z)$ is the polynomial mentioned in Lemma XIII and require now that $K \geqslant (k!)k^3$. (Then by Lemmas II and XIII each of the $T_{1,y}(z)U_{1,y}(z), \ldots, T_{1,y}(z)\times \times U_{p_1,y}(z)$ belongs to $P_1(y)$, since by Lemma II every $D^l y(z+a_r)$, for $0 \leqslant l \leqslant K$, belongs to $P_1(y)$ and by Lemma XIII each

$$b(z+a_r)\Big(kw_j^l(z+a_r) - \sum_{t=1}^{k} w_t(z+a_r)\Big)$$

is a linear combination over $Z[i, z]$ of the $D^l y(z+a_r)$ for $0 \leqslant l \leqslant (k!)k^3$.) If, where N_1 is as above, $|z| \geqslant \frac{1}{2}N_1$ then $b(z)$ does not vanish, by Lemma XI. Since by assumption we have each $|a_r| < \frac{1}{2}K_1(y, a_1, \ldots, a_n)$ and $K_1(y, a_1, \ldots \ldots, a_n) \geqslant N_1$ it follows that $T_{1,y}(z)$ does not vanish if $|z| \geqslant K_1(y)$.

Set

$$T_y(z) = \prod_{r=1}^{n}\prod_{j=1}^{k}\big(p'(w(z+a_r))\big)^K,$$

$$S_y(m) = \text{l.c.m.}\{1, 2, \ldots, k(m+K+1)\},$$

and let $E^\theta y$ be as in the definition appearing before Lemma XIV. Observe that $\mu_1 \leqslant |N| \leqslant K_1(y, a_1, \ldots, a_n)$. Then the requisite estimates on $S_y(m)$, $\deg T_y(z)$, and the absolute values of the coefficients of $T_y(z)$ follow from Lemma XIV with $\gamma = 1$. This establishes Condition B with $\gamma = 1$.

We then obtain a statement of approximation about the

$$\sum_{j=1}^{k} c_j\Big(kw_j^l(z+a_r) - \sum_{t=1}^{k} w_t^l(z+a_r)\Big),$$

where we set $\sum_{j=1}^{k} c_j = 0$; therefore, about the $k\sum_{j=1}^{k} c_j w_j^l(z+a_r)$, or the $\sum_{j=1}^{k} c_j w_j^l(z+a_r)$ if one substitutes $k^{-1}c_j$ for each c_j above. Setting $\psi_1 = \max\{\psi_5, 1+2\max_r\{|a_r|\}\}$ we have established Theorem I.

LEMMA XV. *Suppose that $w_1(z), \ldots, w_k(z)$ are the $k \geqslant 2$ different roots of $q(w, z)+r(z) = 0$ where $q(w, z)$ belongs to $Z[i, w, z]$, is monic in w, and has degree $k \geqslant 2$ in w; $r(z) \in Z[i, z]$ has degree at least one; and $q(0, z) \equiv 0$. Suppose, further, that in the expansion of the $w_j(z)$ about $z = \infty$ in terms of descending fractional powers of z the initial terms (i.e. dominant terms as $|z| \to \infty$) in any two distinct expansions do not agree in both power of z and coefficient. Consider now the equation $q(w, z)+r(z) = u$. Make any cut in the z plane between $z = 0$ and $z = \infty$; then there exist k distinct roots $w_j(u, z)$, $1 \leqslant j \leqslant k$, of $q(w, z)+r(z) = u$ and they are each analytic in both u and z, and take on distinct values, at all points (u, z) such that z does not belong to the cut, $|z| > \beta_{25}$, and $|\mu| < \eta|r(z)|$, for effectively computable β_{25} and η independent of both z and u. If $|z| \geqslant \beta_{25}$ and $|u| < \eta|r(z)|$ then each*

$$|w_j(u, z)| < \beta_{26}|u - r(z)|^{\beta_{27}}$$

for effectively computable constants β_{26} and β_{27} independent of both z and u. Finally, for every $u \in C$ and $1 \leqslant j \leqslant k$ the functions $w_j(u, z)$ each have in their expansions about $z = \infty$ the same dominant term.

Proof. Using the assumption that no two of the $w_j(z)$ have the same dominant term in their expansions about $z = \infty$ and that $r(z) \not\equiv 0$ we see that there exist three not necessarily effectively computable positive constants θ, θ_1 and θ_2 such that if $|z_1| > \theta$ and we set, for each $1 \leqslant j \leqslant k$, $c_j = \{w \mid |w - w_j(z)| = \theta_1|w_j(z_1)|\}$, then (i) the c_j are disjoint (hence each c_j encloses one and only one root of $q(w, z_1)+r(z_1) = 0$) and (ii) on each set c_j the inequality

$$\prod_{j=1}^{k}|w - w_j(z_1)| = |q(w, z_1)+r(z_1)| > \theta_2|r(z_1)|$$

holds. By Rouché's Theorem it follows then that, for each u with

$$|u| < \frac{\theta_2}{2}|r(z_1)| \overset{\text{def}}{=} \theta_3|r(z_1)|$$

every c_j encloses one and only one root of $q(w, z_1)+r(z_1) = u$. This says that $q(w, z)+r(z)-u = 0$ has no multiple roots if $|z| > \theta$ and $|u| < \theta_3|r(z)|$.

We shall show later that one may effectively calculate values of θ and θ_3 such that this last statement holds. (It is necessary to use the existence of such θ and θ_3 to prove the effective computability.) Assuming that θ and θ_3 above have been effectively computed we shall show how the Lemma follows. First we verify that $q(w, z)+r(z)-u = 0$ has k distinct solutions, each analytic in u and z, about every point

$$(u_1, z_1) \in R = \{(u, z) \mid |z| > \theta \text{ and } |u| < \theta_3|r(z_1)|\}.$$

We know that the algebraic equation $q(w, z_1)+r(z_1)-u_1 = 0$ has k distinct roots, which we denote by $r_1, \ldots, r_k$. Consider the integrals

$$(2\pi i)^{-1}\int_{d_j} w\,\frac{\partial q(w, z)}{\partial w}\big(q(w, z)+r(z)-u\big)^{-1}dw$$

where each d_j is a circular path about r_j which does not encircle or pass through any r_{j_1} for $j \neq j_1$. For all points (u, z) in the closure of some simply connected region $N(u_1, z_1) \subseteq R$ which contains (u_1, z_1) every

$$\big|\big(q(w, z_1)+r(z_1)-u_1\big) - \big(q(w, z)+r(z)-u\big)\big| < |q(w, z_1)+r(z_1)-u_1|$$

for every w on every d_j, $1 \leqslant j \leqslant k$, by the continuity of $q(w,z)+r(z)-u$ and the compactness of the d_j. Thus by Rouché's Theorem and the residue theorem each function

$$w_j(u,z) \overset{\text{def}}{=} (2\pi i)^{-1} \int_{d_j} w \frac{\partial q(w,z)}{\partial w} \big(q(w,z)+r(z)-u\big)^{-1} dw \qquad (1 \leqslant j \leqslant k),$$

is defined for all $(u,z) \in N(u_1,z_1)$, is a solution of $q(w,z)+r(z)-u=0$, and each is analytic on $N(u_1,z_1)$ by uniform convergence. Note that the $w_j(u,z)$ have different functional values at (u_1,z_1); hence, they are distinct functions. Since the meromorphic functions on $N(u_1,z_1)$ form a field these are the only analytic roots of our equation on $N(u_1,z_1)$. Further, on any subregion of $N(u_1,z_1)$ the restrictions of the $w_j(u,z)$ are k different solutions of $q(w,z)+r(z)-u=0$; hence, the only analytic solutions of the equation on the subregion.

We wish to see that given any path $\gamma(t)$, $0 \leqslant t \leqslant 1$ in R one may analyticly continue the k distinct solutions of $q(w,z)+r(z)-u=0$ which exist about the point $\gamma(0)$ to the point $\gamma(1)$. Suppose that for some path $\gamma(t)$ we are unable to construct an analytic continuation to $\gamma(1)$. Then there will exist a real number $0<\delta<1$ such that we may construct an analytic continuation of these roots to any point $\gamma(t)$ with $0 \leqslant t < \delta < 1$ but we can not construct an analytic continuation to $\gamma(\delta)$. Setting $\gamma(\delta) = (u_1,z_1)$ we recall that we may define k different solutions $w_1(u,z), \ldots, w_k(u,z)$ of our equation on a simply connected region $N(u_1,z_1)$ which contains (u_1,z_1). For some $t_1<\delta$ each point $\gamma(t)$ with $t \in (t_1,\delta]$ belongs to $N(u_1,z_1)$. Choose $\tau \in (t_1,\delta)$ and set $\gamma(\tau) = (u_2,z_2)$; then, choose $N(u_2,z_2)$, a polydisk containing (u_2,z_2) with $N(u_2,z_2) \subset N(u_1,z_1)$, where k distinct analytic roots of $q(w,z)+r(z)-u=0$ are defined. As we have seen the different roots of our equation on $N(u_2,z_2)$ must be the restrictions of our functions $w_1(u,z), \ldots, w_k(u,z)$. Thus we have analyticly extended our roots from $\gamma(\tau)$ to $\gamma(\delta)$ and beyond. This contradiction proves that we may continue the roots analyticly along any path $\gamma(t)$ in R. Making a cut in the z plane from 0 to ∞ and throwing out all elements of R which project onto this cut, we are left with a simply connected region on which, by the two variable version of the monodromy theorem, we may define k analytic roots. For each root there must exist $1 \leqslant k' \leqslant k$ such that if one continues the root k' times around $z=\infty$ (i.e. the projection of the path winds k' times about $z=\infty$) we return to the original branch of the root.

Now to see that θ and θ_3 can be effectively computed. Set

$$D(z,u) = \prod_{j=1}^{k} \left(\frac{\partial q(w(u,z),z)}{\partial z}\right)^2 = \prod_{j_1<j_2} \big(w_{j_1}(u,z)-w_{j_2}(u,z)\big)^2.$$

One may effectively compute $D(u,z)$ as an element of $Z[i,u,z]$ by writing it effectively in terms of the elementary symmetric functions in $w_1(u,z), \ldots, w_k(u,z)$. Now substitute $tr(z)$ for u. We define $E(t,z)$ and $E_1(t,z)$, respectively, by

$$E(t,z) \overset{\text{def}}{=} D\big(tr(z),z\big) \overset{\text{def}}{=} tE_1(t,z)+E(0,z).$$

Note that $E(t,z)$ and $E_1(t,z)$ each belong to $Z[i,t,z]$. We wish to see that

$$\deg_z E(t,z) = \deg_z E(0,z).$$

Suppose that this is not the case; one possibility, then, is that $E(0,z) \equiv 0$. Then, however, for each z some $w_j(u,z)$ is not analytic at $u=0$, contrary to the existence of θ and θ_3. If $\deg_z E(t,z) > \deg_z E(0,z)$ but $E(0,z) \not\equiv 0$ then choose c_n, a sequence of circles about $z=0$ on which $E(0,z)$ does not vanish, with radii $r_n \to \infty$, and a sequence of real numbers $t^{(n)} \to 0$ such that, for each n, $\deg_z E(t^{(n)},z) = \deg_z E(t,z)$ and $|E(t^{(n)},z)-E(0,z)| < |E(0,z)|$ on c_n. An application of Rouché's Theorem shows that there exists, in each case, a root $z^{(n)}$ of $E(t^{(n)},z)$ with $|z^{(n)}| > r_n$.

Thus we would have $E(t^{(n)},z^{(n)}) = 0$ for each n and $(t^{(n)},z^{(n)}) \to (0,\infty)$. Again this would violate the existence of θ and θ_3. Thus $\deg_z E(0,z) = \deg_z E(t,z)$.

Given for $n \geqslant 1$, $z^n + a_{n-1}(t)z^{n-1}+\ldots+a_1(t)z+a_0(t)$ where the $a_j(t)$ are complex valued functions of t, it is clear that there are no zeros (t,z) of the above function satisfying

$$|z| \geqslant 2\max\{1; |a_j(t)|, \text{ for } 0 \leqslant j \leqslant n-1\}.$$

Thus if

$$E(t,z) = b_n(t)z^n + b_{n-1}(t)z^{n-1}+\ldots+b_1(t)z+b_0(t)$$

where each $b_j(t) \in Z[i,t]$ we see that there are no zeros (t,z) of $E(t,z)$ with

$$|z| \geqslant 2|b_n(t)|^{-1}\big(\max\{1; |b_n(t)|^{n-1}|b_j(t)|, \text{ for } 0 \leqslant j \leqslant n\}\big).$$

Since $b_n(0) \neq 0$, above, $|b_n(0)| \geqslant 1$. One may bound $\max\{1, |b_n'(t)|\}$ from above effectively on $|t| \leqslant 1$. Set θ_3 equal to one over twice this bound. Then if $|t| \leqslant \theta_3$ and $|z_1|$ is larger than or equal to an effectively computable number θ, we have that $E(t,z_1) \neq 0$. This is equivalent to $D(u,z)$ does not vanish if $|z| = \theta$ and $|u| \leqslant \theta_3|r(z)|$. Set $\eta = \theta_3$. This shows part of the lemma.

For some effectively computable $\beta_{25} \geqslant \theta$, $r(z)$ does not vanish if $|z| \geqslant \beta_{25}$. Suppose that

$$q(z,w) = w^k + a_{k-1}(z)w^{k-1}+\ldots+a_1(z)w^1.$$

Then if $|z| \geqslant \beta_{25}$ and $|u| \leqslant \eta |r(z)|$ we must have

$$|u - r(z)| \geqslant (1-\eta)\,|r(z)|$$

and

$$|w_j(u, z)| \leqslant 2\max\{1;\ |a_j(z)|,\ 1 \leqslant j \leqslant k-1;\ (1+\eta)\,|r(z)|\} \leqslant \beta_{26}\,|u - r(z)|^{\beta_{27}}$$

for effectively computable β_{26} and β_{27} independent of u and z.

From all that has been shown so far it is clear that about the point $(u, z) = (0, \infty)$ each function $w_j(u, z)$ possesses an expansion in terms of ascending powers of u and descending powers of $z^{(k')^{-1}}$, for some $1 \leqslant k' \leqslant k$, with the property that given any $c > 0$ there exists a $d > 0$ such that the double series converges absolutely for all $|u| < c$ and $|z| > d$. Thus we may write each double series as an expansion in terms of descending powers of $z^{(k')^{-1}}$ with coefficients which are functions of u defined for all u with $|u| < c$. Let $w_j(u, z)$ denote the algebraic function such that $w_j(0, z) = w_j(z)$. Let $r_1, \ldots, r_j, \ldots, r_k$ denote the degrees of the dominant terms in the expansions of the k different $w_j(u, z)$ about $z = \infty$. Let $r_1', \ldots, r_j', \ldots, r_k'$ denote the degrees of the dominant terms in the expansions of the $w_j(z)$ about $z = \infty$. Clearly each $r_j \geqslant r_j'$. However,

$$\sum_{j=1}^{k} r_j = \deg_z\big(u - r(z)\big) = d = \deg_z\big(-r(z)\big) = \sum_{j=1}^{k} r_j',$$

so each $r_j = r_j'$. We shall now show that the coefficient $\gamma_j(u)$ of z^{r_j} in the expansion of $w_j(u, z)$ is a constant, for each $1 \leqslant j \leqslant k$, and this will complete the proof of Lemma XV. Substitute $\gamma_j(u) z^{r_j} + \ldots$ into the polynomial $q(w, z) + r(z) - u = 0$. We see that then $\gamma_j(u)$ satisfies a polynomial equation with constant coefficients; hence, $\gamma_j(u)$ is a constant. This proves Lemma XV.

Now we wish to prove Theorem III part (i). Let $a_1, \ldots, a_n \in Z[i, z]$ be given. Let $p(w, z)$ in Theorem III be given and consider $p(w, z) = u$. Let $\pi' = \{w_j(u, N)$ satisfying $p(w, N) = u$ for each $1 \leqslant j \leqslant k$ and each $N \in Z[i]\}$ and let $\pi = \{\sum_{j=1}^{n} c_j y_j(u, N)$ derived from the $w_j(u, N)$ in π by the construction of Lemma XII$\}$. We shall show that Conditions A and B hold for every $y \in \pi$. We assume that for each u the $k w_j^l(u, z+a_r) - \sum_{s=1}^{k} w_s^l(u, z+a_r)$ are linearly independent over $Q(i, z)$, for $1 \leqslant l \leqslant k-1$ and $1 \leqslant r \leqslant n$, as will be shown after the proof of Theorem II.

Using (20) we obtain for every N a differential equation with coefficients in $Z[i, u]$ of order $k-1$ which is satisfied by each of $y_1(u, N), \ldots, y_k(u, N)$. Since each $|y_j(u, N)|$, $1 \leqslant j \leqslant k$, is less as $|z| \to \infty$ than some constant times $|u|^{(k-1)/k}$ we may obtain a uniform upper bound on the degrees of the coefficient polynomials (in u) of our differential equation of type (20). Thus we have a uniform upper bound on the orders of the equations of type (15) satisfied by our functions $y \in \pi$.

As in the proof of Theorem I the roots of $q_0(t)$ include no negative integers. Set η equal to the value of η obtained from Lemma XV with $q(w, z) + r(z)$ equal to $w^k + \sum_{l=0}^{k-1} a_l(z) w^l$, set $t = u - a_0(N)$, and set $K_1\big(y(N), a_1, \ldots, a_n\big) = \max\{|a_0(N)|, M\}$ where $M > 1 + (2+\eta^{-1})\max_r\{|a_r|\}$ and M is such that if $|a_0(N)| \geqslant M$ then $|N| \geqslant \beta_{25}$. Set $K_2(y) = -a_0(N)$ and set $K_3(y) = 0$. The remaining parts of Condition A follow by Lemma XV.

Next consider Condition B. In analogy with the proof of Theorem I set $K \geqslant (k!)k^3$ and

$$\{U_{1,y}(u), \ldots, U_{p_1,y}(u)\} = \Big\{k w_j^l(u+a_r, N) - \sum_{i=1}^{k} w_i^l(u+a_r, N)\Big|\ 1 \leqslant l \leqslant k-1 \text{ and } 1 \leqslant r \leqslant n\Big\}.$$

Also set

$$T_{1,y}(u) = \prod_{r=1}^{n} b(u+a_r)$$

where $b(u)$ is as in Lemma XIII. Note that by Lemmas XIII and XV if $|N| > \beta_{25}$ and $|u| < \eta |a_0(N)|$ we have that $b(u) \neq 0$. Since each $\eta^{-1}|a_r| < M$ if $|K_2(y)| \geqslant M$, then $T_{1,y}(0) \neq 0$. Let $E^\theta y$ be defined as before Lemma XIV. Let $T_y(u)$ be

$$\prod_{r=1}^{n} \prod_{j=1}^{k} \left(\frac{\partial q\big(w_j(u+a_r, N), N\big)}{\partial w_j(u+a_r, N)}\right)^K$$

which can be effectively computed as an element of $Z[i, N, u, a_1, \ldots, a_n]$. Thus we may effectively bound $T_y(u)$ in degree and bound the height of $T_y(u)$ by

$$K_1(y, a_1, \ldots, a_n) \geqslant \max\{|a_0(N)|, |a_r|\}$$

to an effectively computable power. Let

$$S_y(m) = \text{l.c.m.}\{1, 2, \ldots, k(m+K+1)\} < 2^{2^{\frac{3}{2}k(m+K+1)}}.$$

Applying Lemma XIV we see that Condition B holds with

$$\gamma(M) = \sup\{1;\ k(k-l)^{-1}\big(\ln|a_l(N)|\big)\big(\ln|a_0(N)|\big)^{-1},\quad 1 \leqslant l \leqslant k-1 \text{ and } N \in Z[i] \text{ with } |N| \geqslant M\}.$$

As $M \to +\infty$,

$$\gamma(M) \to \max\{1; k(k-1)^{-1}(\deg a_l(z))(\deg a_0(z))^{-1}, 1 \leqslant l \leqslant k-1\} = \gamma.$$

Thus given $\varepsilon > 0$ we may choose M effectively and apply Theorem V (with an effectively computable value substituted for ε in Theorem V) to obtain, when $|a_0(N)| \geqslant \psi_5$ and $|N| \geqslant \max\{1+(2+\eta^{-1})(\max_r\{|a_r|\}), M\}$ a statement of approximation about the numbers

$$\sum_{j=1}^{k} c_j\Big[kw_j^l(a_r, N) - \sum_{t=1}^{k} w_t^l(a_r, N)\Big],$$

for $1 \leqslant r \leqslant n$ and $1 \leqslant l \leqslant k-1$, with exponent $\gamma(1+\varepsilon)(n-1)^{-1}$. Since $\sum_{j=1}^{k} c_j = 0$, if we substitute $k^{-1}c_j$ for c_j above we obtain the numbers $\sum_{j=1}^{k} c_j w_j^l(a_r, N)$. Theorem III part (i) follows immediately.

Section V

In this section we shall see that Theorems II and III (ii) follow from Theorems I and III (i), respectively, and that Theorem IV follows from Theorem III (ii). We begin by proving Theorem II and, by essentially the same proof, one may also obtain Theorem III (ii). [We shall place comments in square brackets which will aid in carrying through the proof of Theorem III (ii).]

Let ψ_1 and φ_1 be as in Theorem I. Let $|N| > \psi_1$. Given any matrix $(c_{j,h})$ of complex numbers with each $|c_{j,h}| \leqslant (2k^2)^{-1}|N|^{\alpha}$, for each $1 \leqslant j \leqslant k-1$ and $h \leqslant k-1$, and any nonzero $(k-1)$-tuple of Gaussian integers $(A_1, \ldots, A_h, \ldots, A_{k-1})$ with $\max_{1\leqslant h\leqslant k-1}\{|A_h|\} \geqslant |N|^{\varphi_1}$ we have, using Theorem I, that

$$(22) \qquad \max_{\substack{1\leqslant r\leqslant n\\ 1\leqslant l\leqslant k-1}} \Big\{\Big\|\sum_{j=1}^{k-1}\Big(\sum_{h=1}^{k-1} c_{j,h}A_h\Big)\Big[kw_j^l(N+a_r) - \sum_{t=1}^{k} w_t^l(N+a_r)\Big] - p_{r,l}\Big\|\Big\} \geqslant (\max_r\{|A_h|\})^{-\frac{1+\varepsilon}{n-1}},$$

$1 \leqslant l \leqslant k-1$, are a nonzero $n(k-1)$-tuple of Gaussian integers. We shall see that we may choose the $c_{j,h}$ such that

$$(c_{j,h})\Big(kw_j^l(N) - \sum_{t=1}^{k} w_t^l(N)\Big)$$

equals the identity, where the column parameters are j and l respectively, and

$$\max_{j,h}\{|c_{j,h}|\} \leqslant (2k^2)^{-1}|N|^{\beta_{28}}$$

for some effectively computable number β_{28} independent of N.

Notice that $\det(w_j^l(N))$, where $1 \leqslant j \leqslant k$ and $0 \leqslant l \leqslant k-1$, equals

$$(-1)^k \det(w_j^l(N) - w_k^l(N)) \quad \text{where} \quad 1 \leqslant j, l \leqslant k-1.$$

Also

$$\begin{pmatrix} k-1 & & -1 \\ \vdots & \ddots & \vdots \\ -1 & & k-1 \end{pmatrix} (w_j^l(N) - w_k^l(N)) = \Big(kw_j^l(N) - \sum_{t=1}^{k} w_t^l(N)\Big),$$

where the row parameters are j, the column parameters are l, and in the first matrix each diagonal entry is $k-1$ while every other entry is -1. Writing each row in the first matrix as $(0, \ldots, 0, k, 0, \ldots, 0)-(1, 1, \ldots, 1)$ and then writing the determinant of this first matrix as a sum of 2^{k-1} determinants, we see it equals k^{k-2}. If $|N|$ is (effectively computably) large enough then the $w_j(N)$ [or $w_j(0, N)$] are distinct by Lemma XI [or Lemma XV]. Using the product formula for the Vandermonde determinant we see that

$$|\det(w_j^l(N))| = \prod_{j=1}^{k} |p'(w_j(N))|^{1/2} \geqslant 1.$$

Using the upper bound on $|w_j(N)|$ [or on $|w_j(0, N)|$] from Condition A [or from Condition A and an upper bound on $|a_0(z)|$ which involves z and d] we may effectively compute constants β_{28} and β_{29} independent of N so that

$$\max_{j,h}\{|c_{j,h}|\} \leqslant (2k^2)^{-1}|N|^{\beta_{28}} \quad \text{if} \quad |N| \geqslant \beta_{29}.$$

Without loss of generality we take $a_1 = 0$. Let $\|$a complex number$\|$ denote the distance from the number to the nearest Gaussian integer. Set each $p_{1,l} = A_l, 1 \leqslant l \leqslant k-1$, in inequality (22). If $|N| \geqslant \max\{\beta_{29}, \psi_1(a_1, \ldots, a_n, \beta_{28}, k, \varepsilon)\}$ and $\max_{1\leqslant l\leqslant k-1}\{|A_l|\} \geqslant |N|^{\varphi_1}$, where $\varphi_1 = \varphi_1(n, \beta_{28}, k, \varepsilon)$ then

$$(23) \qquad \max_{\substack{2\leqslant r\leqslant n\\ 1\leqslant l\leqslant k-1}} \Big\{\Big\|\sum_{h=1}^{k-1}\Big(\sum_{j=1}^{k-1} c_{j,h}\Big[kw_j^l(N+a_r) - \sum_{t=1}^{k} w_t^l(N+a_r)\Big]\Big)A_h\Big\|\Big\} \geqslant \big(\max_{1\leqslant h\leqslant k-1}\{|A_h|\}\big)^{-\left(-\frac{1+\varepsilon}{n-1}\right)}.$$

Theorem II on page 77 of [1] states: "Let

$$L_j(\bar{x}) = \sum_i \theta_{j,i}x_i, \qquad M_i(\bar{u}) = \sum_j \theta_{j,i}u_j$$

where $1 \leqslant i \leqslant m', 1 \leqslant j \leqslant n'$. Suppose that there are integers $\bar{x} \neq \bar{0}$

such that $\|L_j(\bar{x})\| \leqslant C$, $|x_i| \leqslant X$ for some constants C and X where $0 < C < 1 < X$. Then there are integers $\bar{u} \neq \bar{0}$ such that $\|M_i(\bar{u})\| \leqslant D$, $|u_j| \leqslant U$, where

$$D = (l'-1)X^{(1-n')(l'-1)^{-1}}C^{n'(l'-1)^{-1}},$$

$$U = (l'-1)X^{m'(l'-1)^{-1}}C^{(1-m')(l'-1)^{-1}} \quad \text{and} \quad l' = m'+n'".$$

Our $M_i(\bar{u})$, $1 \leqslant i \leqslant 2(n-1)(k-1)$, are the $2(m-1)(k-1)$ linear forms obtained from breaking the

$$\sum_{h=1}^{k-1}\Big(\sum_{j=1}^{k-1} c_{j,h}\Big[kw_j^l(N+a_r) - \sum_{t=1}^{k} w_t^l(N+a_r)\Big]\Big) A_h$$

into real and imaginary parts with the u_j, $1 \leqslant j \leqslant 2(k-1)$, being the real and imaginary parts of the A_h. The $L_j(\bar{x})$ are the real and imaginary parts of the

$$(24) \qquad \sum_{r=2}^{n}\sum_{l=1}^{k-1}\Big(\sum_{j=1}^{k-1} c_{j,h}\Big[kw_j^l(N+a_r) - \sum_{t=1}^{k} w_t^l(N+a_r)\Big]\Big) B_{r,l}$$

$1 \leqslant h \leqslant k-1$ for Gaussian integer $B_{r,l}$ where the x_i, $1 \leqslant i \leqslant 2(n-1)\times(k-1)$ are the real and imaginary parts of the different $B_{r,l}$. Set $X = \max_{r,l}\{|B_{r,l}|\}$ and $C = X^{-(n-1)-\varepsilon_1}$ where $\varepsilon_1 > 0$. Then

$$U^{l'-1} = (l'-1)^{l'-1}\exp\big\{(\ln X)\big(2(n-1)(k-1) + [2(n-1)(k-1)-1](n-1) + \varepsilon_1[2(n-1)(k-1)-1]\big)\big\}$$

and

$$D^{l'-1} = (l'-1)^{l'-1}\exp\big\{(\ln X)\big(1-2(k-1)+2(k-1)(1-n-\varepsilon_1)\big)\big\}$$
$$= (l'-1)^{l'-1}\big((l'-1)^{-(l'-1)}U^{l'-1}\big)^{-\left(\frac{1+\theta}{n-1}\right)}$$

where

$$\theta = \varepsilon_1\big(-1+(n+\varepsilon_1)[2(n-1)(k-1)-1]\big)^{-1} > \varepsilon_1\big(2(n^2-1)(k-1)\big)^{-1}$$

if $\varepsilon_1 < 1$. Note $U > X$. One sees from the above with $\varepsilon_1 < 1$ and (23) with $\varepsilon = \varepsilon_1\big(2(n^2-1)(k-1)\big)^{-1}$ that for some effectively computable β_{30} independent of N, but dependent on n, k, ε_1, and φ_1

$$(25) \qquad \max_{1\leqslant h\leqslant k-1}\Big\{\Big|\sum_{r=2}^{n}\sum_{l=1}^{k-1} B_{r,l}\Big(\sum_{j=1}^{k-1} c_{j,h}\Big[kw_j^l(N+a_r) - \sum_{t=1}^{k} w_t^l(N+a_r)\Big]\Big) + B_{1,h}\Big|\Big\} \geqslant \big(\max_{l,2\leqslant r\leqslant n}\{|B_{r,l}|\}\big)^{-(n-1)-\varepsilon}$$

if $\max_{l,2\leqslant r\leqslant n}\{|B_{r,l}|\} > |N|^{\beta_{30}}$, where the $B_{1,h}$ are any $k-1$ Gaussian integers, and if $|N|$ is larger than an effectively computable lower bound depending

on n, k, ε_1, and Ψ_1. (The alternative is that (23) is violated for some N by a collection of forms $M_i(t\bar{u})$ where the M_i and $\bar{u}$ are as above and t is the least positive integer such that $\max_j\{|tu_j|\} > |N|^{\varphi_1}$.)

We note from the fact that $(c_{j,h})$ is a two sided inverse of $\big(kw_j^l(N) - \sum_{t=1}^{k} w_t^l(N)\big)$ that each

$$\sum_{h=1}^{k-1}\Big(kw_\theta^h(N) - \sum_{t=1}^{k} w_t^h(N)\Big)c_{j,h} = \delta_\theta^j$$

where δ_θ^j is the Kronecker delta. Thus, if one makes the forms on the left hand side of (25) entries in a column vector v with row parameter $1 \leqslant h \leqslant k-1$, then $\big(kw_j^l(N) - \sum_{j=1}^{k} w_t^l(N)\big)v$ equals a column vector w with entries

$$\sum_{r=1}^{n}\sum_{l=1}^{k-1} B_{r,l}\Big[kw_j^l(N+a_r) - \sum_{t=1}^{k} w_t^l(N+a_r)\Big],$$

$1 \leqslant j \leqslant k-1$. The maximum of the absolute values of the entries of w is larger than or equal to one over the maximum of the absolute values of the entries in $\big(kw_j^l(N) - \sum_{t=1}^{k} w_t^l(N)\big)^{-1}$ times $(k-1)^{-1}$ times $\big(\max_{r,l}\{|B_{r,l}|\}\big)^{-(n-1)-\varepsilon}$. Given a new $\varepsilon > 0$, if $\max_{r,l}\{|B_{r,l}|\}$ is larger than $|N|^{\beta_{31}}$, where β_{31} is effectively computable and independent of N

$$\max_{1\leqslant j\leqslant k-1}\Big\{\Big|\sum_{r=1}^{n}\sum_{l=1}^{k-1} B_{r,l}\Big[w_j^l(N+a_r) - k^{-1}\sum_{t=1}^{k} w_t^l(N+a_r)\Big]\Big|\Big\} \geqslant \max_{r,l}(\{|B_{r,l}|\})^{-(n-1)-\varepsilon}.$$

To conclude the proof notice that if $B_{0,0}$ is a Gaussian integer and

$$(26) \qquad \max_{1\leqslant j\leqslant k}\Big\{\Big|\sum_{r=1}^{n}\sum_{l=1}^{k-1} B_{r,l}w_j^l(N+a_r) + B_{0,0}\Big|\Big\} < \frac{1}{k}$$

then

$$B_{0,0} = \sum_{r=1}^{n}\sum_{l=1}^{k-1} B_{r,l}\Big(-k^{-1}\sum_{j=1}^{k} w_j^l(N+a_r)\Big)$$

so the left hand side of (26) is larger than $\max_{l,r}(\{|B_{r,l}|\})^{-(n-1)-\varepsilon}$. This proves Theorem II.

Suppose now that the $kw_j^l(z+a_r) - \sum_{s=1}^{k} w_s^l(z+a_r)$, for $1 \leqslant l \leqslant k-1$ and $1 \leqslant r \leqslant n$, are not linearly independent over $Q(i, z)$. Recall the

$v_{l,j}$, $1 \leqslant l \leqslant p$, from before the proof of Theorem I. Let $\varphi_{1,j}(z), \ldots, \varphi_{\theta,j}(z)$, where $\theta < (n-1)(k-1)$, denote those $v_{l,j}(z)$ which are not of the form $kw_j^l(z+a_1) - \sum_{s=1}^{k} w_s^l(z+a_1)$. If the $c_{j,h}$ are as in the proof of Theorem II then we have, by the comments before the proof of Theorem I and the proof of Theorem II, that for each $\varepsilon > 0$ and each $N \in Z[i]$ with sufficiently large absolute value (depending on ε)

$$\max_{1 \leqslant h \leqslant k-1} \left\{ \left\| \sum_{l=1}^{\theta} A_l \left(\sum_{j=1}^{k-1} c_{j,h} \varphi_{l,j}(N) \right) \right\| \right\} \leqslant (\max_l \{|A_l|\})^{-(n-1)+\varepsilon}$$

holds for infinitely many distinct choices in $Z[i]$ of the A_l.

Applying the transference theorem used in the proof of Theorem II to the above statement we see that, since $\theta < (n-1)(k-1)$, for each $\varepsilon > 0$ and every $N \in Z[i]$ with a sufficiently large absolute value (depending on ε),

$$\max_{1 \leqslant h \leqslant k-1} \left\{ \left\| \sum_{h=1}^{k-1} B_h \left(\sum_{j=1}^{k-1} c_{j,h} \varphi_{l,j}(N) \right) \right\| \right\} \leqslant (\max_h \{|B_h|\})^{-\theta_1+\varepsilon}$$

holds for infinitely many distinct choices of the B_h in $Z[i]$, where $\theta_1 > 1/(n-1)$.

Dividing through above by $\max_h \{|B_h|\}$ we have contradicted the noneffective result on the simultaneous diophantine approximation of the $v_{l,j}(N)$ obtained along with the proof of Theorem I. This shows that the $kw_j^l(z+a_r) - \sum_{s=1}^{k} w_s^l(z+a_r)$ are linearly independent over $Q(i, z)$.

Proof of Theorem IV. It will suffice to prove this result if $\max\{\deg B_{r,l}(z)\} \geqslant \varphi_4$ since if $\deg A(z) > \max_{r,l}\{\deg B_{r,l}(z)\}$ then the expression in (8) has degree equal to $k^n \deg A(z)$. Note also that

$$\deg \prod_{\{j(r)\}} \left(\sum_{r=1}^{n} \sum_{l=1}^{k-1} B_{r,l}(z) w_{j(r)}^l(\delta_r, z) + A(z) \right)$$
$$\geqslant \deg \prod_{\{j(r)\}} \left(\sum_{r=1}^{n} \sum_{l=1}^{k-1} B_{r,l}(z) \left(kw_{j(r)}^l(\delta_r, z) - \sum_{t=1}^{k} w_t^l(\delta_r, z) \right) \right),$$

so it suffices to prove that the degree of this latter expression is at least

$$\left(k^n - k^{n-1} - \sum_{l=1}^{n-1} k^l - \varepsilon \right) \left(\max_{r,l} \{\deg B_{r,l}(z)\} \right).$$

In analogy with the proof of Corollary I of Theorem I we may reduce to the case where each δ_r is a rational integer. We observe that since $\gamma = 1$ each $|w_j(\delta_r, z)|$ is less as $|z|$ goes to $+\infty$ than some constant times $|z|^{dk^{-1}}$. Since $\deg \left(\prod_{j=1}^{n} w_j(\delta_r, z) \right) = d$, the expansion of each $w_j(\delta_r, z)$ about $z = \infty$ must begin with a term involving $z^{dk^{-1}}$. Since $(k, d) = 1$ it follows that each $w_j(\delta_r, z)$ is the analytic continuation of any $w_{j_1}(\delta_r, z)$ about $z = \infty$ a finite number of times.

Reindexing the $w_j(\delta_r, z)$ if necessary, set

$$(27) \qquad kw_{r(j)}^l(\delta_r, z) - \sum_{t=1}^{k} w_t^l(\delta_r, z) = \sum_{t=1}^{k-1} z^{tk^{-1}} (\varrho^t)^{r(j)} g_{r,l,t}(z)$$

where $\varrho = \exp(2\pi i k^{-1})$ and where every $g_{r,l,t}(z)$ is meromorphic at $z = \infty$. Suppose that we know that for some $j(r)$

$$(28) \qquad \sum_{r=1}^{n} \sum_{l=1}^{k-1} B_{r,l}(z) \left(kw_{j(r)}^l(\delta_r, z) - \sum_{t=1}^{k} w_t^l(\delta_r, z) \right)$$

vanishes at $z = \infty$ to at least the order $-\infty < R < +\infty$, and not every $B_{r,l}(z)$ is zero. Using (27) in (28) and writing the final result as a sum of $k-1$ linear forms, one for each $1 \leqslant t \leqslant k-1$, in the $z^{tk^{-1}} g_{r,l,t}(z)$ $(1 \leqslant r \leqslant n, 1 \leqslant l \leqslant k-1)$ with coefficients in $Q[\varrho^t, z]$, we see that each linear form vanishes to at least the order R at $z = \infty$. From this we see, since k is a prime, that we may pick $n(k-1)$ integers $m_{r,l}$ such that each

$$\sum_{r=1}^{n} \sum_{l=1}^{k-1} m_{r,l} B_{r,l}(z) z^{tk^{-1}} g_{r,l,t}(z)$$

vanishes at $z = \infty$ to at least the order R and not every $m_{r,l} B_{r,l}(z)$ is zero. Thus, for each $1 \leqslant j \leqslant k$,

$$\sum_{r=1}^{n} \sum_{l=1}^{k-1} m_{r,l} B_{r,l}(z) \left[kw_j^l(\delta_r, z) - \sum_{t=1}^{k} w_t^l(\delta_r, z) \right]$$

vanishes to at least the order R at $z = \infty$ and some $m_{r,l} B_{r,l}(z)$ does not equal zero.

Analogously if (28) vanishes to at least the order R at $z = \infty$ for both $j(r) = j_1(r)$ and $j(r) = j_2(r)$, if $n' \geqslant 1$ is the cardinality of the set of r belonging to $\{1, \ldots, n\}$ such that $j_1(r) \neq j_2(r)$, if $j_1(r_1) \neq j_2(r_1)$ for some $1 \leqslant r_1 \leqslant n$, and if $B_{r_1,l_1}(z)$ is nonzero for some $1 \leqslant l_1 \leqslant k-1$, then there exist $n'(k-1)$ integers $m_{r,l}$ such that for each $1 \leqslant j \leqslant k-1$

$$\sum_{r \mid j_1(r) \neq j_2(r)} \sum_{l=1}^{k-1} m_{r,l} B_{r,l}(z) \left(kw_j^l(\delta_r, z) - \sum_{t=1}^{k} w_t^l(\delta_r, z) \right)$$

vanishes to at least the order R at $z = \infty$ and $m_{r_1,l_1} B_{r_1,l_1}(z)$ is nonzero.

In what follows it will be useful to be able to know that none of the vectors $(B_{r,1}(z), \ldots, B_{r,l-1}(z))$ is ever zero. To see that one may assume this without loss of generality note that if exactly $1 \leqslant m \leqslant n-1$ such vectors are zero, then applying the present Theorem for the case of $n-m$ distinct δ_r gives the desired inequality.

Now to prove the Theorem by induction on n. If $n = 1$ we must show that

$$\deg \Big(\prod_{j=1}^{k} \Big(\sum_{l=1}^{k-1} B_{1,l}(z)\Big[kw_j^l(\delta_1, z) - \sum_{t=1}^{k} w_t^l(\delta_1, z)\Big]\Big)\Big) > (k-1-\varepsilon)\big(\max_l \{\deg B_{1,l}(z)\}\big)$$

if $\max_l \{\deg B_{1,l}(z)\}$ is larger than some effectively computable constant. By the nonvanishing of $\det(w_j^l(\delta_1, z))$, $1 \leqslant j, l \leqslant k-1$, we see that each $B_{1,l}(z)$ equals a linear combination of the

$$\sum_{l=1}^{k-1} B_{1,l}(z)\Big[kw_j^l(\delta_1, z) - \sum_{t=1}^{k} w_t^l(\delta_1, z)\Big] \quad (1 \leqslant j \leqslant k-1)$$

with coefficients independent of the $B_{1,l}(z)$. Since the absolute values of these coefficients grow more slowly, as $|z|$ goes to $+\infty$, than some constant times $|z|^d$ to an effectively computable power depending on k we see that each

$$\sum_{l=1}^{k-1} B_{1,l}(z)\Big[kw_j^l(\delta_1, z) - \sum_{t=1}^{k} w_t^l(\delta_1, z)\Big]$$

grows faster in absolute value, as $|z|$ goes to $+\infty$ than $|z|$ to the power $\max_l \{\deg B_{1,l}(z)\}$ minus an effectively computable constant. This proves the Theorem if $n = 1$.

We suppose next that $n > 1$ and that no $(B_{r,1}(z), \ldots, B_{r,l-1}(z))$ is zero. Using Theorem III (ii) we see that there exists an effectively computable constant h such that if $\max_{r,l} \{\deg B_{r,l}(z)\} \geqslant h$ then

$$\sum_{r=1}^{n} \sum_{l=1}^{k-1} B_{r,l}(z)\Big[kw_j^l(\delta_r, z) - \sum_{t=1}^{k} w_t^l(\delta_r, z)\Big]$$

does not vanish to an order larger than $(n-1+\varepsilon)\big(\max_{r,l} \{\deg B_{r,l}(z)\}\big)$ at $z = \infty$ for any $1 \leqslant j \leqslant k$. (Look at the proofs of Corollaries I and II of Theorem II.) We next choose h to be sufficiently large that for any set $S \subset \{1, 2, \ldots, n\}$ of cardinality at most $2 \leqslant n' \leqslant n$ the expression

$$\sum_{r\in S} \sum_{l=1}^{k-1} B_{r,l}(z)\Big[kw_j^l(\delta_r, z) - \sum_{t=1}^{k} w_t^l(\delta_r, z)\Big]$$

does not vanish to an order larger than $(n'-1+\varepsilon)\big(\max_{r,l} \{\deg B_{r,l}(z)\}\big)$ at $z = \infty$, for any $1 \leqslant j \leqslant k$, and assume in what follows that $\max_{r,l} \{\deg B_{r,l}(z)\} \geqslant h$.

By what we have already seen it follows that each expression of type (28) can not vanish at $z = \infty$ to an order larger than

$$(n-1+\varepsilon) \max_{r,l} \{\deg B_{r,l}(z)\}.$$

Also no two distinct expressions of type (28) which correspond to functions $j(r)$ that disagree on at most $n' \geqslant 1$ points can both vanish to an order larger than $(n'-1+\varepsilon)\big(\max_{r,l} \{\deg B_{r,l}(z)\}\big)$ at $z = \infty$. Thus, for each $1 \leqslant m \leqslant n-2$, at most k^m distinct expressions of type (28) can vanish to an order larger than $(n-m-1+\varepsilon)\big(\max_{r,l} \{\deg B_{r,l}(z)\}\big)$ at $z = \infty$. Given any $k^{n-1}+1$ distinct expressions of type (28) we can find two such that the corresponding functions $j(r)$ agree except at $r = r_1$, where r_1 an arbitrary element of $\{1, \ldots, n\}$. Further, for any l_1 such that $B_{r,l_1}(z)$ is nonzero we may pick integers $m_{r_1,l}$ such that

$$(29) \qquad \sum_{l=1}^{k-1} m_{r,l} B_{r,l}(z)\Big[kw_j^l(\delta_{r_1}, z) - \sum_{t=1}^{k} w_t^l(\delta_r, z)\Big]$$

vanishes to an order larger than or equal to the minimal order of vanishing of the $k^{n-1}+1$ forms, and such that $m_{r_1,l} B_{r_1,l}(z)$ is nonzero. Pick $B_{r_1,l_1}(z)$ such that $\deg B_{r_1,l_1}(z) = \max_{r,l} \{\deg B_{r,l}(z)\}$. Then in analogy with our argument when $n = 1$, we see that (29) vanishes to an order less than or equal to $-\max_{r,l} \{\deg B_{r,l}(z)\}$ plus an effectively computable constant depending only on d and k, at $z = \infty$. Therefore the degree of the product mentioned in the statement of the Theorem is, for sufficiently large $\max_{r,l} \{\deg B_{r,l}(z)\}$ at least

$$(k^n - k^{n-1} - \varepsilon/2)\big(\max_{r,l} \{\deg B_{r,l}(z)\}\big) -$$
$$- [(k^{n-1} - k^{n-2}) + 2(k^{n-2} - k^{n-3}) + \ldots + (n-2)(k^2 - k) + (n-1)k^1 + \varepsilon/2],$$
$$\big(\max_{r,l} \{\deg B_{r,l}(z)\}\big) = \Big(k^n - k^{n-1} - \sum_{l=1}^{n-1} k^l - \varepsilon\Big)\big(\max_{r,l} \{\deg B_{r,l}(z)\}\big).$$

This proves Theorem IV.

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О представлении чисел положительными тернарными диагональными квадратичными формами, II*

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§ 4. В этом и следующих параграфах α, β, γ всюду обозначают неотрицательные целые числа; n — фиксированное натуральное число; m, u, v — нечетные натуральные числа; ω — бесквадратные числа.

Пусть $r(n; a_1, a_2, a_3)$ обозначает число представлений числа n формой $F = a_1x_1^2+a_2x_2^2+a_3x_3^2$, т.е. число решений уравнения

$$n = a_1x_1^2+a_2x_2^2+a_3x_3^2. \tag{4.1}$$

Без ограничения общности будем предполагать, что $(a_1, a_2, a_3) = 1$ и $2\nmid a_3$. Как и выше, $\Delta = a_1a_2a_3$ и a обозначает общее наименьшее кратное всех a_k. Далее, если положить $M = 4an$, то уравнение (4.1) примет вид:

$$M = \frac{a}{a_1}y_1^2+\frac{a}{a_2}y_2^2+\frac{a}{a_3}y_3^2, \quad y_k \equiv 0 \pmod{2a_k} \quad (k = 1, 2, 3). \tag{4.2}$$

Обозначим через $R(M; a_1, a_2, a_3)$ число решений уравнения (4.2). Очевидно, что

$$r(n; a_1, a_2, a_3) = R(M; a_1, a_2, a_3). \tag{4.3}$$

Из (2.3), (4.2) и (4.3) следует:

$$\prod_{k=1}^{3}\vartheta_{00}(\tau; 0, 2a_k) = 1+\sum_{M} R(M; a_1, a_2, a_3)e\left(\frac{M\tau}{4a}\right) = \tag{4.4}$$

$$= 1+\sum_{n=1}^{\infty} r(n; a_1, a_2, a_3)e(n\tau). \tag{4.5}$$

Далее, из (3.92) следует:

$$\theta(\tau; a_1, a_2, a_3) = \theta(\tau) = 1+\sum_{M} P(M; a_1, a_2, a_3)e\left(\frac{M\tau}{4a}\right), \tag{4.6}$$

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