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On the speed of convergence of the Oppenheim series

by

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1. Summary. In this paper I continue my investigation concerning expansions, by the algorithm recommended by A. Oppenheim, of real numbers into infinite series of rationals. Putting p_n/q_n for the sum of the first n terms in an expansion of x, I shall give two sided estimates and asymptotic expressions for both $(x-p_n/q_n)$ and $\log(x-p_n/q_n)$, valid almost surely (Lebesgue measure), which provide means to compare several well known expansions covered by the Oppenheim algorithm.

2. Introduction. It was recommended by A. Oppenheim (see [5] and [1]) to expand real numbers $0 < x < 1$ by the algorithm

(1) $$x = x_1, \quad d_n = [1/x_n]+1, \quad x_n = 1/d_n+(a_n/b_n)x_{n+1}$$

where $a_n = a_n(d_1, \ldots, d_n)$ and $b_n = b_n(d_1, \ldots, d_n)$ are given positive integer valued functions and $[y]$ denotes the integer part of y. This yields the infinite series associated with x:

(2) $$x \sim \frac{1}{d_1}+\frac{a_1}{b_1}\frac{1}{d_2}+\ldots+\frac{a_1a_2\ldots a_n}{b_1b_2\ldots b_n}\frac{1}{d_{n+1}}+\ldots$$

By (1)

(3) $$\frac{1}{d_n} < x_n < \frac{1}{d_n-1}$$

and hence by the last equality in (1)

(4a) $$d_{n+1} > \frac{a_n}{b_n}d_n(d_n-1).$$

The expansion defined by (1) and (2) is convergent and its sum is equal to x. A sufficient condition for a series on the right hand side in (2) to be the expansion of its sum by the algorithm (1) is

(4b) $$d_{n+1} \geqslant \frac{a_n}{b_n}d_n(d_n-1)+1.$$

I called in [1] the expansion (2), obtained by the algorithm (1), restricted Oppenheim expansion (ROE) of x, if a_n and b_n depend on the last denominator d_n only and if the function

$$h_n(j) = \frac{a_n(j)}{b_n(j)} j(j-1) \tag{5}$$

is integer valued. Here again I deal with ROE only. Note that for ROE (4a) and (4b) are equivalent.

In order to quote the results of [1] which are needed in this paper, I introduce some notations. The probability space $(\Omega, \mathscr{R}, P)$ is chosen as $\Omega = (0, 1)$, $\mathscr{R}$ the set of Lebesgue measurable subsets of $(0, 1)$ and Lebesgue measure as P. Put

$$z_j = -\log\frac{h_j(d_j)}{d_{j+1}-1} \quad (j \geqslant 1). \tag{6}$$

Note that by (4b), $z_j \geqslant 0$. I quote Theorems 2, 5 and 6 and Corollary 2 of [1] as Lemmas 1–4.

Lemma 1. *If $h_n(j) \geqslant j-1$ $(n = 1, 2, \ldots)$ then, with probability unity, for all but a finite number of values of n, strict inequality occurs in* (4b).

Lemma 2. *Under the assumption of $h_n(j) \geqslant j-1$, for $n = 1, 2, \ldots$*

$$P\left(\lim_{n=+\infty} \frac{1}{n}\sum_{j=1}^{n} z_j = 1\right) = 1.$$

Lemma 3. *Assume that there exist constants $t > 1$, $1 \leqslant K_1 < K_2$ such that*

$$K_1 \leqslant h_n(j)/j^t \leqslant K_2 \quad \text{for all } n, j \text{ large},$$

then

$$P(\lim_{n=+\infty} t^{-n}\log d_n \text{ exists}) = 1.$$

Lemma 4. *If $h_n(j) = h_n$ constant for all n, then the d_n are independent random variables with distribution*

$$P(d_n = k) = \frac{h_{n-1}}{k(k-1)} \quad \text{for} \quad k > h_{n-1},$$

and 0 otherwise, with the convention $h_0 = 1$.

Put

$$\frac{p_n}{q_n} = \frac{1}{d_1} + \frac{a_1}{b_1}\frac{1}{d_2} + \ldots + \frac{a_1 a_2 \ldots a_{n-1}}{b_1 b_2 \ldots b_{n-1}}\frac{1}{d_n}. \tag{7}$$

In this paper I investigate the magnitude of $x - p_n/q_n$.

3. Estimates in terms of $d_1, d_2, \ldots$ Two simple but very useful estimates are immediate. By repeated applications of (1) we get

$$x = p_n/q_n + A(a, b, n)x_{n+1}$$

where

$$A(a, b, n) = (a_1 a_2 \ldots a_n)/(b_1 b_2 \ldots b_n) \tag{8}$$

and hence by (3)

$$A(a, b, n)/d_{n+1} < x - p_n/q_n < A(a, b, n)/(d_{n+1}-1). \tag{9}$$

From (9) we immediately have that if $d_n \to +\infty$, then

$$x - \frac{p_n}{q_n} = \left(1 + \frac{c}{d_{n+1}}\right)\frac{A(a, b, n)}{d_{n+1}-1}, \quad |c| < 2. \tag{10}$$

We are now in the position to prove

Theorem 1. *If $h_n(j) \geqslant j-1$, then for almost all x,*

$$\log\left(x - \frac{p_n}{q_n}\right) = -\{1 + o(1)\}\sum_{k=1}^{n}\log d_k.$$

Proof. Lemma 1 and (4b) imply that, under the assumption $h_n(j) \geqslant j-1$, $d_n \to +\infty$, hence (10) is applicable, and since

$$\log(1 + c/d_{n+1}) = O(1/d_{n+1}) = o(1),$$

by (8) and (10) we have

$$\log\left(x - \frac{p_n}{q_n}\right) = \sum_{k=1}^{n}\log\frac{a_k}{b_k} - \log(d_{n+1}-1) + o(1)$$

which, by the notations (5) and (6), can be rewritten as

$$\begin{aligned} \log\left(x - \frac{p_n}{q_n}\right) &= -\sum_{k=1}^{n} z_k + \sum_{k=1}^{n}\log\frac{d_{k+1}-1}{d_k(d_k-1)} - \log(d_{n+1}-1) + o(1) \\ &= -\sum_{k=1}^{n} z_k - \sum_{k=1}^{n}\log d_k - \log(d_1 - 1) + o(1). \end{aligned} \tag{11}$$

Lemma 1 asserts that there is an integer valued function $n_0(x)$, defined for almost all x, such that for $n \geqslant n_0$,

$$d_{n+1} > h_n(d_n) + 1 \geqslant d_n$$

the last inequality having been obtained from $h_n(j) \geqslant j-1$. Hence $d_n \geqslant n-n_0$, i.e. for almost all x,

(12) $$\liminf n^{-1} d_n \geqslant 1 \quad (n \to +\infty).$$

(12) yields that for almost all x, as $n \to +\infty$,

$$0 \leqslant \liminf \left(\sum_{k=1}^{n} \log \frac{d_k}{k} + O(n)\right) = \liminf \left(\sum_{k=1}^{n} \log d_k - n\log n + O(n)\right),$$

i.e. for almost all x and for n large,

$$\sum_{k=1}^{n} \log d_k > \tfrac{1}{2} n \log n.$$

Since, by Lemma 2, for almost all x,

$$\sum_{k=1}^{n} z_k = \{1+o(1)\} n,$$

we have got that

$$\sum_{k=1}^{n} z_k = o\left(\sum_{k=1}^{n} \log d_k\right)$$

which, in view of (11), terminates the proof.

Theorem 1 shows an interesting character of the Oppenheim series. Namely, if $h_n(j) \geqslant j-1$, the choices of a_n and b_n do not effect directly the convergence of $\log(x-p_n/q_n)$ to $-\infty$, and the calculation of a single additional denominator d_{n+1} results in a decrease of $\log(x-p_n/q_n)$ by $\log d_{n+1}$. In view of (4b), however, we can expect that the larger a_n/b_n are the larger are the denominators, hence by the choice of large a_n/b_n we can improve on the approximation of x. This fact will be proved for a large class of expansions, covering all the well known ones, in the next section.

4. The case of $h_n(j)$ being polynomials. In this section I shall discuss the cases when the $h_n(j)$ are polynomials. These cases cover the Engel and Sylvester series and the Cantor and Oppenheim products; for these see Examples 1–5 of [1], and also [6], pp. 116–127, and [4]. As it turns out, these are comperatively ‘slow’ approximations (much faster, though, than by continued fractions), which fact would justify to work out more details of number theoretical character of some special Oppenheim series.

I first investigate the case when $h_n(j)$ are linear for all n.

THEOREM 2. *If $h_n(j) = Aj+B$ with $A>0$, $B>-2$ integers, for $n=1,2,\ldots$, then, for almost all x,*

(13) $$\log\left(x-\frac{p_n}{q_n}\right) = -\tfrac{1}{2}\{1+\log A + o(1)\} n(n+1).$$

Proof. We first deduce from Lemma 2 that, as $n \to +\infty$,

(14) $$n^{-1}\log d_n \to 1+\log A.$$

As a matter of fact, by (6)

(15) $$\frac{1}{n}\sum_{k=1}^{n} z_k = -\frac{1}{n}\sum_{k=1}^{n} \log\frac{Ad_k+B}{d_{k+1}-1}$$
$$= -\log A - \frac{1}{n}\sum_{k=1}^{n}\log\left(1+\frac{A+B}{A(d_k-1)}\right) - \frac{1}{n}\log(d_{n+1}-1).$$

Since $d_k \to +\infty$ (Lemma 1 and (4b)) and $\log(1+y) = cy$ with $|c|<2$ as $y \to 0$, the last but one term in (15) is $o(1)$ and since

$$\log(d_{n+1}-1) = \log d_{n+1} + \log(1-1/d_{n+1}) = \log d_{n+1} + o(1)$$

we have got that

(16) $$\frac{1}{n}\sum_{k=1}^{n} z_k = -\log A - \frac{1}{n}\log d_{n+1} + o(1).$$

By Lemma 2 and by (16) we get (14). Hence, by Theorem 1, for almost all x,

$$\log\left(x-\frac{p_n}{q_n}\right) = -\{1+o(1)\}\sum_{k=1}^{n}\log d_k$$
$$= -\{1+o(1)\}\sum_{k=1}^{n} k\,\frac{\log d_k}{k} = -\{1+o(1)\}(1+\log A)\sum_{k=1}^{n} k$$
$$= -\tfrac{1}{2}\{1+o(1)\}(1+\log A)n(n+1)$$

what was to be proved.

If $A=-B=1$, we get the Engel series. Theorem 2 shows that an increase in A results in faster approximation, i.e. among all those Oppenheim series for which $h_n(j) = Aj+B$, the convergence of the Engel series is the slowest.

THEOREM 3. *If there exist constants $t>1$, $1 \leqslant K_1 < K_2$ such that*

$$K_1 \leqslant h_n(j)/j^t \leqslant K_2 \quad \textit{for all } n, j \textit{ large},$$

then for almost all x

(17) $$\lim_{n=+\infty} t^{-n}\log\left(x-\frac{p_n}{q_n}\right) = -D(x)t(t-1)^{-1}$$

where

$$D(x) = \lim_{n=+\infty} t^{-n} \log d_n. \tag{18}$$

Proof. The statement follows immediately from Lemma 3 and Theorem 1. Namely, by Lemma 3, $D(x)$ in (18) exists for almost all x, and hence

$$\sum_{k=1}^{n} \log d_k = \sum_{k=1}^{n} t^k \frac{\log d_k}{t^k}$$

$$= \{1+o(1)\} D(x) \sum_{k=1}^{n} t^k = \{1+o(1)\} D(x) t^{n+1} (t-1)^{-1}.$$

In view of Theorem 1, Theorem 3 is established.

For the Sylvester series and the Cantor and Oppenheim products $h_n(j)$ are quadratic polynomials, hence Theorem 3 is applicable to these cases, which shows that $x - p_n/q_n$ tends to 0 as fast as $\exp(-c2^n)$, where c depends on x. To obtain better means for comparisons, I shall deduce inequalities for $x-p_n/q_n$, the estimates being independent of x.

The idea of these estimates is to solve the difference equation $D_{n+1} = h_n(D_n)+1$ (or to estimate its solution) with the initial condition $D_1 = 2$. By (4b), $d_n = d_n(x) \geqslant D_n$ for all x, hence we can estimate $D(x)$ in (17) and (18) in terms of the constant sequence D_n (constant as a function of x).

(i) The Sylvester series. If $h_n(j) = j(j-1)$, (1) and (2) reduce to the Sylvester series. In this case the difference equation for D_n is $D_{n+1} = D_n(D_n-1)+1$, for the solution of which we have that

$$D_{n+1} - 1 > (D_n - 1)^2. \tag{19}$$

Since $D_1 = 2, D_2 = 3, D_3 = 7, D_4 = 43$, we get by induction from (19)

$$\log(D_n - 1) > 2^{n-2.5}, \quad n \geqslant 4, \tag{20}$$

therefore, in view of (4b), (17), (18) and (20), for almost all x,

$$\lim 2^{-n} \log(x - p_n/q_n) \leqslant -1/2\sqrt{2}. \tag{21}$$

(ii) The Cantor product. The Cantor product of $1+x$ is obtained by $h_n(j) = (j+1)(j-1)$. The difference equation is $D_{n+1} = D_n^2$ with $D_1 = 2$. Thus $\log D_n = 2^{n-1}\log 2$, hence, as before, we get from Theorem 3 that

$$\lim 2^{-n} \log(x - p_n/q_n) \leqslant -\log 2 \tag{22}$$

is valid for almost all x.

(iii) The Oppenheim product. Let us estimate the speed of convergence of the product expansion, introduced in [4], for the special case $h_n(j) = (j+4)(j-1)$, in which case the series (2) is equivalent to the product expansion of $1+4x$ by the algorithm of [4]. The difference equation is $D_{n+1} = D_n^2 + 3D_n - 3$ with $D_1 = 2$. Thus $D_2 = 7$, $D_3 = 67$. Since $\log 67 > 4$, we get by induction that $\log D_n \geqslant 2^{n-1}$, which yields, as before, that for almost all x,

$$\lim 2^{-n} \log(x - p_n/q_n) \leqslant -1. \tag{23}$$

From Theorem 3 we have that by choosing a_n and b_n so that $h_n(j)$ be polynomials of degree higher than 2, $x - p_n/q_n$ will tend to 0 much faster than in the cases which have been investigated earlier.

As an interesting corollary to Theorems 2 and 3 I settle a question raised by A. Oppenheim (oral communication). It is known that there is an infinite sequence of real numbers (quadratic irrationals) for which the expansion (1) and (2) is both the Engel and the Sylvester series at the same time. Professor Oppenheim asked whether the measure of the set with the property that the Engel and Sylvester series coincide is zero or positive. Theorems 2 and 3 yield immediately

COROLLARY. *Let A denote the set of real numbers for which the Engel and the Sylvester series are identical. Then $P(A) = 0$.*

Indeed, if $x \in A$, then x should belong to the exceptional set in one of Theorems 2 and 3, since both results can not hold for the same x, hence the Corollary is established.

5. The Lüroth type expansions. If $h_n(j) = 1$ for all n and j, we get the expansion known as that of Lüroth. I call an Oppenheim expansion Lüroth type if $h_n(j) = h_n$ do not depend on j. By Lemma 4, the denominators d_n are independent random variables. Considering the case when $h_n = h$ for all n, we get from (5), (8) and (9)

$$\left|\log(x - p_n/q_n) - n\log h + \sum_{k=1}^{n} \log d_k(d_k - 1)\right| < \log d_{n+1}. \tag{24}$$

By Lemma 4, $\log d_k(d_k-1)$ are identically distributed random variables with expectation

$$E(h) = h \sum_{k=h+1}^{+\infty} \frac{\log k(k-1)}{k(k-1)}$$

hence by the strong law of large numbers (see [3], p. 238) and by (24) we have

THEOREM 4. *If $h_n(j) = h$ for all n and j, then for almost all x,*

$$\log(x - p_n/q_n) = -\{1+o(1)\}\big(E(h) - h\big)n.$$

Theorem 4 gives that among the Lüroth type expansions the Lüroth expansion is the slowest in convergence. Also, among the well known expansions, for almost all x, the Lüroth expansion requires the largest number of terms to provide the same accuracy.

The case $h = 1$ was recently investigated in detail in [2].

References

[1] J. Galambos, *The ergodic properties of the denominators in the Oppenheim expansion of real numbers into infinite series of rationals*, Quart. J. of Math. (Oxford) (2), 21 (1970), pp. 177–191.

[2] H. Jager and C. de Vroedt, *Lüroth series and their ergodic properties*, Nederl. Akad. Wet., Proc., Ser. A., 72 (1969), pp. 31–42.

[3] M. Loéve, *Probability theory*, 3rd ed, Princeton, N. J., 1963.

[4] A. Oppenheim, *On the representation of real numbers by products of rational numbers*, Quart. J. of Math. (Oxford) (2), 4 (1953), pp. 303–307.

[5] — *Representations of real numbers by series of reciprocals of odd integers*, Acta Arith. 18 (1971), pp. 115–124.

[6] O. Perron, *Irrationalzahlen*, 2nd ed., New York 1948.

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On the simultaneous diophantine approximation of values of certain algebraic functions*

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INTRODUCTION

In a recent paper [3] the present author obtained a result, which is sketched immediately below, about the simultaneous diophantine approximation of the values of $(N+s_1)^{k_1k^{-1}}, \ldots, (N+s_n)^{k_1k^{-1}}$, where $0 = s_1 < s_2 < \ldots < s_n$ were $n \geqslant 2$ integers, $k \geqslant 2$ was an integer, $1 \leqslant k_1 \leqslant k$ was an integer satisfying $(k_1, k) = 1$, N was a sufficiently large positive integer, and the kth roots above were the positive real kth roots.

Let ε denote a positive real number, $(p_1, \ldots, p_n)$ denote any nonzero vector of nonnegative integers, C denote a real number, and q denote a positive integer. Then three functions $\psi = \psi(s_1, \ldots, s_n, k, k_1, \varepsilon, N)$, $\varphi = \varphi(s_1, \ldots, s_n, k, k_1, \varepsilon, N)$ and $\Lambda = \Lambda(s_1, \ldots, s_n, k, k_1, N)$ were given explicitly(1). It was shown that if $\varepsilon < (2n-4)^{-1}$, $N \geqslant \psi$, $q \geqslant \varphi$, and $0 \leqslant C \leqslant 1$, then

$$\max_{1\leqslant j\leqslant n} \{|C(N+s_j)^{k_1k^{-1}} - p_j q^{-1}|\} \geqslant \frac{1}{n}(2q)^{-\left(1+\frac{1+\varepsilon}{\Lambda}\right)}. \tag{1}$$

Further, as $N \to +\infty$ (and all of the other parameters were held constant) Λ increased to $n-1$.

In this paper we shall prove results allowing us to make statements analogous to (1) about a larger class of algebraic functions. In these statements the auxiliary functions corresponding to φ and ψ above are not given explicitly; however, it is shown that they are effectively computable.

Let Q denote the rational field, C the complex field, $Q(i)$ the Gaussian field, Z the integers, and $Z[i]$ the Gaussian integers. In what follows N will always denote a Gaussian integer.

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(1) The present notation differs slightly from that used in [3].