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# Polynomial values with small prime divisors

by

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**1. Introduction.** Let $f(x) = a_g x^g + \ldots + a_1 x + a_0$ be an irreducible polynomial with integral coefficients and suppose $a_g \geqslant 1$, $(a_g, \ldots, a_0) = 1$. Let $p(n)$ $(q(n))$ be the largest (smallest) prime divisor of $n$ $(p(1) = q(1) = 1)$. Write, for $2 \leqslant y \leqslant x$,

$$\Psi(x, y) = \Psi(f, x, y) = \sum_{\substack{n \leqslant x \\ p(f(n)) \leqslant y}} 1, \quad u = \frac{\ln x}{\ln y}. \tag{1}$$

Hmyrova ([5]) proved

$$\Psi(x, y) < c(f) x \exp\left(-u \ln \frac{u}{e}\right), \quad \text{if} \quad e \leqslant u \leqslant \frac{\ln x}{\ln\ln x}. \tag{2}$$

A similar inequality was shown for the function

$$\Psi_\Pi(x, y) = \sum_{\substack{r \leqslant x, r \text{ prime} \\ p(f(r)) \leqslant y}} 1. \tag{3}$$

It is the purpose of this paper to improve Hmyrova's results as follows.

THEOREM 1. (i) *For* $(\ln x)^g < y \leqslant (\ln x)^A$ $(A = 80 g^2)$ *we have*

$$\Psi(x, y) \ll x \exp\big(-gu \ln u + O(u \ln\ln u)\big).$$

(ii) *For* $(\ln x)^A < y \leqslant x$ *we have*

$$\Psi(x, y) \ll x \exp\left(-gu \ln\left(\frac{u}{e^4}\right)\right).$$

*(The constants implied by the symbols* $O$ *and* $\ll$ *may depend on* $f$.)

THEOREM 2. *For* $(\ln x)^g < y \leqslant x$ *and* $\varepsilon > 0$ *the inequality*

$$\Psi_\Pi(x, y) \leqslant c(f, \varepsilon) \Pi(x) \exp\big(-(1-\varepsilon) gu \ln u\big)$$

*holds. ($\Pi(x)$ is the number of primes* $\leqslant x$.)

By the same method one can prove results of the following kind:

$$\Psi(x+x^a, y)-\Psi(x, y) \leqslant c(f, a)x^a \exp\left(-gu\ln u+O(u\ln\ln u)\right), \tag{4}$$
$$\text{if} \quad 0 < a < 1 \quad \text{and} \quad (\ln x)^{g/a} < y \leqslant x^{e^{-1}}.$$

$$\Psi(x, y, k, l) = \sum_{\substack{n\leqslant x\\ p(nk+l)\leqslant y}} 1 \leqslant c(a)x\exp\left(-u_k\left(\ln u_k+O(\ln\ln u_k)\right)\right), \tag{5}$$

if $0 < a,\ k \leqslant x^a,\ 0 \leqslant l < k,\ u_k = \dfrac{\ln(kx)}{\ln y}$, and $(\ln x)^{a+1} < y \leqslant x^{e^{-1}}$.

**2. Some Lemmas.** Let $\varrho(n)$ be the number of solutions of the congruence $f(x) \equiv 0 \bmod n$.

LEMMA 1.

(i) *$\varrho(n)$ is a multiplicative function,*

(ii) *$\varrho(p^a) \ll 1$ for all primes $p$ and $a = 1, 2, \ldots$,*

(iii) *$\varrho(p^{a+1}) = \varrho(p^a)$ for $p \geqslant p_0(f)$ and $a = 1, 2, \ldots$,*

(iv) $\displaystyle\sum_{p\leqslant x}\frac{\varrho(p)\ln p}{p} = \ln x\left(1+o(1)\right)$,

(v) $\displaystyle\sum_{p\leqslant x}\varrho(p) = \frac{x}{\ln x}\left(1+O\left(\frac{1}{\ln x}\right)\right)$.

For a proof, see Erdös, [4].

LEMMA 2. *For $d \leqslant x^{1/2}$ we have, uniformly in $d$,*

$$\sum_{\substack{n\leqslant x\\ d\mid f(n)\\ q\left(\frac{f(n)}{d}\right)>p(d)}} 1 \ll x\frac{1}{\varphi(d)}\prod_{p\mid d}\frac{p\varrho(p)}{p-\varrho(p)}\prod_{p\leqslant p(d)}\left(1-\frac{\varrho(p)}{p}\right)+x^{0.1}.$$

This is proved by Selberg's sieve (see Barban, [1]).

**3. Proof of Theorem 1, (ii).**

3.1. In the following let $x$ be sufficiently large. Secondly, we may assume $y \leqslant x^{0.1}$. Write

$$\Psi'(x, y) = \sum_{\substack{x/2<n\leqslant x\\ p(f(n))\leqslant y}} 1 \tag{6}$$

and

$$I_\nu = (y^{\frac{\nu-1}{2g}}, y^{\frac{\nu}{2g}}] = (y_{\nu-1}, y_\nu] \quad (\nu = 1, \ldots, 2g). \tag{7}$$

For $n \in (x/2, x]$ we have

$$a_g/2^{g+1}\cdot x^g < f(n) \leqslant 2a_g x^g. \tag{8}$$



We may write

$$f(n) = d_1' \ldots d_{2g}' \quad \text{where} \quad p\mid d_\nu' \text{ implies } p \in I_\nu.$$

Because of (8) we have

$$d_\nu' \geqslant Cx^{1/2} \quad (1 > C = C(f) > 0) \quad \text{for at least one } \nu \leqslant 2g. \tag{9}$$

Let $\mu = \mu(n)$ be the smallest number $\nu$ for which (9) holds. Hence

$$\Psi'(x, y) = \sum_{\mu=1}^{2g}\ \sum_{\substack{x/2<n\leqslant x\\ \mu(n)=\mu,\ p(f(n))\leqslant y}} 1 = \sum_{\mu=1}^{2g}\Psi_\mu, \text{ say}. \tag{10}$$

3.2. Let us start with the case $\mu = 1$. For $n$ counted in $\Psi_1$, we have

$$f(n) = d_1'h \quad \text{where} \quad p(d_1') \leqslant y_1,\ q(h) > y_1,\ d_1' \geqslant Cx^{1/2}.$$

Put

$$d_1 = d_1' \quad \text{if} \quad d_1' \leqslant x^{1/2}$$

and, if $d_1' > x^{1/2}$,

$$d_1' = d_1d_1''$$

where $d_1 \leqslant x^{1/2}$, $p(d_1) < q(d_1'') = q$, and $d_1q^b > x^{1/2}$ $(q^b\mid d_1',\ q^{b+1}\nmid d_1')$.

We distinguish two cases

$$\text{I. } d_1 < C\frac{x^{1/2}}{y_1}, \qquad \text{II. } d_1 \geqslant C\frac{x^{1/2}}{y_1}. \tag{11}$$

In the first case we have, because of (9),

$$f(n) \equiv 0 \bmod p^a, \quad p \leqslant y_1, \quad p^a > y_1.$$

The contribution of these $n$ to the sum $\Psi_1$ is

$$\leqslant \sum_{\substack{n\leqslant x\\ f(n)\equiv 0\bmod p^a,\\ p\leqslant y_1,\, x\geqslant p^a>y_1}} 1 \leqslant 2x\sum_{\substack{y_1<p^a\leqslant x\\ p\leqslant y_1}}\frac{\varrho(p^a)}{p^a}.$$

By Lemma 1, (ii) and the assumption on $y$, this is

$$\ll x^{1-(1/5g)} \leqslant xe^{-gu\ln u}.$$

Hence, if we write

$$x_1 = \frac{Cx^{1/2}}{y_1}, \tag{12}$$

we obtain

$$\Psi_1 \ll xe^{-gu\ln u} + \sum_{\substack{x/2<n\leqslant x\\ x_1<d_1(n)\leqslant x^{1/2}}} 1 \leqslant \tag{13}$$

$$\ll xe^{-gu\ln u}+\sum_{\substack{x_1<d\leqslant x^{1/2}\\ p(d)\leqslant y_1}}\ \sum_{\substack{n\leqslant x\\ d\mid f(n),\, q\left(\frac{f(n)}{d}\right)>p(d)}} 1 = xe^{-gu\ln u}+\Psi_{1,1},\ \text{say.}$$

3.3. We have

$$\Psi_{1,1}\leqslant \sum_{\substack{x_1<d\leqslant x^{1/2}\\ p(d)\leqslant(\ln x)^{10}}}\ \sum_{\substack{n\leqslant x\\ d\mid f(n)}} 1+ \tag{14}$$

$$+\sum_{\substack{i\geqslant 0\\ y_1^{2^{-i}}\geqslant(\ln x)^{10}}}\ \sum_{\substack{x_1<d\leqslant x^{1/2}\\ p(d)\in(y_1^{2^{-(i+1)}},\, y_1^{2^{-i}}]}}\ \sum_{\substack{n\leqslant x\\ d\mid f(n)\\ q\left(\frac{f(n)}{d}\right)>p(d)}} 1 = \Psi'_{1,1}+\sum_i \Psi_{1,1,i},\ \text{say.}$$

Using $\varrho(d)\ll d^{0.01}$ and a crude upper bound for the function $\sum_{\substack{n\leqslant x\\ p(n)\leqslant(\ln x)^{10}}} 1$ (see de Bruijn, [3]) we get

$$\Psi'_{1,1}\ll x^{1.01}\sum_{\substack{d>x^{1/4}\\ p(d)\leqslant(\ln x)^{10}}} d^{-1}\ll xe^{-gu\ln u}. \tag{15}$$

For fixed $i\geqslant 0$ $(y_1^{2^{-i}}\geqslant(\ln x)^{10})$ we write $z=y_1^{2^{-i}}$. Lemma 2 and Lemma 1 give

$$\Psi_{1,1,i}\ll\frac{x}{\ln z}\sum_{\substack{d>x_1\\ p(d)\leqslant z}}\frac{h(d)}{d}+xe^{-gu\ln u}, \tag{16}$$

where

$$h(d)=\frac{d}{\varphi(d)}\prod_{p\mid d}\frac{p\,\varrho(p)}{p-\varrho(p)}. \tag{17}$$

Obviously,

$$h(p)=\varrho(p)+O(1/p)\quad\text{and}\quad h(n)\leqslant(\tau(n))^{c(f)}\quad (\tau(n)=\sum_{d\mid n}1). \tag{18}$$

We now estimate the sum

$$H=\sum_{\substack{d>x_1\\ p(d)\leqslant z}}\frac{h(d)}{d} \tag{19}$$

by using an idea of Rankin (see Rankin, [7]). For $\delta$ with $0\leqslant\delta\leqslant 1/4$ and because of (18) the following holds

$$H\leqslant\sum_{\substack{d>x_1\\ p(d)\leqslant z}}\frac{h(d)}{d}\left(\frac{d}{x_1}\right)^{\delta}\leqslant x_1^{-\delta}\sum_{\substack{d\\ p(d)\leqslant z}}\frac{h(d)}{d^{1-\delta}}\ll \tag{20}$$



$$\ll x_1^{-\delta}\ln z\prod_{p\leqslant z}\left(1+\varrho(p)\left(\frac{1}{p^{1-\delta}}-\frac{1}{p}\right)\right)$$

$$\ll \ln z\exp\left(-\delta\ln x_1+\sum_{p\leqslant z}\varrho(p)\left(\frac{1}{p^{1-\delta}}-\frac{1}{p}\right)\right).$$

By Lemma 1 (iv), the sum in the exponential is

$$=\sum_{p\leqslant z}\frac{\varrho(p)}{p}\sum_{\nu=1}^{\infty}\frac{(\delta\ln p)^{\nu}}{\nu!}\leqslant\sum_{\nu=1}^{\infty}\frac{\delta^{\nu}(\ln z)^{\nu-1}}{\nu!}\sum_{p\leqslant z}\frac{\varrho(p)\ln p}{p}$$

$$\leqslant 2\sum_{\nu=1}^{\infty}\frac{\delta^{\nu}(\ln z)^{\nu}}{\nu!}\leqslant 2e^{\delta\ln z}.$$

Thus, by (20), $H\ll\ln z\exp(-\delta\ln x_1+2z^{\delta})$. Put

$$\delta=\frac{1}{\ln z}\ln\left(\frac{\ln x_1}{\ln z}\right).$$

The condition $z\geqslant(\ln x)^{10}$ implies $\delta\leqslant 1/4$. Some computation gives

$$H\ll\ln z\exp\left(-\frac{\ln x_1}{\ln z}\ln\left(\frac{\ln x_1}{\ln z}\right)+2^{i+1}gu\right). \tag{21}$$

Putting (14), (15), (16), and (21) together, we arrive at

$$\Psi_{1,1}\ll xe^{-gu\ln u}+ \tag{22}$$

$$+x\sum_{\substack{i\geqslant 0\\ y_1^{2^{-i}}\geqslant(\ln x)^{10}}}\exp\left(-\frac{\ln x_1}{\ln(y_1^{2^{-i}})}\ln\left(\frac{\ln x_1}{\ln(y_1^{2^{-i}})}\right)+2^{i+1}gu\right).$$

A simple computation shows

$$\exp\left(-\frac{\ln x_1}{\ln(y_1^{2^{-i}})}\ln\left(\frac{\ln x_1}{\ln(y_1^{2^{-i}})}\right)\right)\leqslant\exp(-g2^{i}u\ln u+2u).$$

Hence

$$\Psi_1\ll xe^{-gu\ln u}+x\sum_{i\geqslant 0}e^{-2^{i}gu\ln u}e^{2(2^{i}g+1)u}\ll xe^{-gu\ln(u/e^4)}. \tag{23}$$

3.4. The case $\mu\geqslant 2$ is much easier. By (9) and because of $y\leqslant\sqrt{x}$ we have

$$\Psi_{\mu}\leqslant\sum_{\substack{n\leqslant x\\ f(n)\equiv 0\bmod d\\ Cx^{1/2}<d\leqslant x\\ p\mid d\to p\in I_{\mu}}}1\leqslant 2x\sum_{\substack{Cx^{1/2}<d\leqslant x\\ p\mid d\to p\in I_{\mu}}}\frac{\varrho(d)}{d}\leqslant 2x\sum_{\substack{d>Cx^{1/2}\\ p\mid d\to p\in I_{\mu}}}\frac{\varrho(d)}{d}.$$

The last sum can be estimated in the same manner as above. Thus

$$\Psi_\mu \ll x e^{-gu\ln(u/e^4)} \qquad (2 \leqslant \mu \leqslant 2g). \tag{24}$$

3.5. (6), (10), (23), and (24) give

$$\Psi(x, y) \ll x^{1/2} + \sum_{\substack{i\geqslant 0\\ 2^i \leqslant x^{1/2}}} \frac{x}{2^i} e^{-gu_i \ln(u_i/e^4)} \quad \text{where} \quad u_i = \frac{\ln(x/2^i)}{\ln y}. \tag{25}$$

One easily sees that for $y \geqslant (\ln x)^A$ and $2^i \leqslant x^{1/2}$

$$u_i \ln\left(\frac{u_i}{e^4}\right) \geqslant u \ln\left(\frac{u}{e^4}\right) - \frac{i\ln 4}{A}$$

holds. This and (25) imply statement (ii).

**4. Proof of Theorem 1 (i) and Theorem 2.** By $d_\nu$ $(1 \leqslant \nu \leqslant g)$ we denote integers for which $p \mid d_\nu$ implies $y^{(\nu-1)/g} < p \leqslant y^{\nu/g}$. As in part 3.1 one can see

$$\Psi'(x, y) \leqslant \sum_{\nu=1}^{g} \sum_{\substack{x/2 < n \leqslant x\\ d_\nu \mid f(n)\\ Cx/4 < d_\nu \leqslant x}} 1 = \sum_{\nu=1}^{g} \overline{\Psi}_\nu, \text{ say}. \tag{26}$$

Further

$$\overline{\Psi}_\nu \leqslant \sum_{Cx/y < d_\nu \leqslant x} \sum_{\substack{n \leqslant x\\ d_\nu \mid f(n)}} 1 \ll x \sum_{Cx/y < d_\nu \leqslant x} \frac{\varrho(d_\nu)}{d_\nu} \ll y \sum_{d_\nu \leqslant x} \varrho(d_\nu). \tag{27}$$

Using Lemma 1 (v), we get, in the same manner as de Bruijn, [3],

$$\sum_{d_\nu \leqslant x} \varrho(d_\nu) \ll \exp\left(Z\left(1 + O\left(\frac{1}{\ln y}\right)\right)\right), \tag{28}$$

where

$$Z = \frac{\ln x}{\ln(y^{1/g})} \ln\left(1 + \frac{y^{1/g}}{\ln x}\right) + \frac{y^{1/g}}{\ln(y^{1/g})} \ln\left(1 + \frac{\ln x}{y^{1/g}}\right).$$

For $(\ln x)^g < y \leqslant (\ln x)^A$ we have

$$Z\big(1 + O(1/\ln y)\big) = \ln x - gu\ln u + O(u \ln\ln u),$$

which, together with (26), (27), and (28) implies

$$\Psi' \ll x\exp\big(-gu\ln u + O(u\ln\ln u)\big).$$

By summation over intervals of the kind $(t/2, t]$ the statement easily follows.



The proof of Theorem 2 is very similar to that of Theorem 1. Here one uses the prime number theorems of Brun–Titchmarsh (see Prachar, [6]) and Bombieri (see Bombieri, [2]).

**References**


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