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On a density problem of Erdös

by

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Let the integers $a_1, a_2, \ldots, a_k$ all be greater than unity. Erdös (private communication) conjectured that the asymptotic density of the set of natural numbers divisible by exactly one a_i could not exceed $1/2$. We shall prove it.

The letters A, B, C, D, and E, with or without subscripts, shall denote finite sequences of positive integers. For such a sequence A, let $\Delta(A)$ and $\Gamma(A)$ denote the sets of natural numbers divisible by *exactly one* and by *at least one* of the terms of A, respectively. For any set S, let δS be the asymptotic density of S.

THEOREM. *Let the terms of A all be greater than unity. Then*

$$\delta\Delta(A) \leqslant \tfrac{1}{2}.$$

The proof is by induction on the number $v(A)$ of distinct prime factors of the product of all the terms of A. If $v(A) = 1$, then the terms of A are all powers of some prime p, so that

$$\delta\Delta(A) \leqslant \Gamma(A) \leqslant p^{-1} \leqslant \tfrac{1}{2}.$$

Assume, therefore, that the theorem has been verified for all C with $v(C) < v(A)$. We define several auxiliary sequences. For $n = 0, 1, 2, \ldots$ let B_n denote the subsequence of A consisting of those a_i such that $p^n \| a_i$, and let A_n denote the union of $B_0, B_1, \ldots, B_n$. Let D_n be the sequence obtained by dividing each term of B_n by p^n, and let C_n denote the union of $D_0, D_1, \ldots, D_n$. Thus $v(C_n) < v(A)$.

Now A_n is the union of A_{n-1} and B_n, so $\Delta(A_n)$ is the union of the set of numbers divisible by exactly one term of A_{n-1} and by no term of B_n with the set of numbers divisible by exactly one term of B_n and by no term of A_{n-1}. Therefore, since the two sets are disjoint,

$$(1)\qquad \delta\Delta(A_n) = \delta\{\Delta(A_{n-1}) - \Gamma(B_n)\} + \delta\{\Delta(B_n) - \Gamma(A_{n-1})\}$$
$$= \delta\Delta(A_{n-1}) - \delta\{\Delta(A_{n-1}) \cap \Gamma(B_n)\} + \delta\Delta(B_n) - \delta\{\Delta(B_n) \cap \Gamma(A_{n-1})\}.$$

Similarily,

$$\delta\Delta(C_n) = \delta\Delta(C_{n-1}) - \delta\{\Delta(C_{n-1}) \cap \Gamma(D_n)\} + \delta\Delta(D_n) - \delta\{\Delta(D_n) \cap \Gamma(C_{n-1})\}. \tag{2}$$

But the set $\Delta(A_{n-1}) \cap \Gamma(B_n)$ is precisely the set of all $p^n\chi$ with χ in $\Delta(C_{n-1}) \cap \Gamma(D_n)$ and the set $\Delta(B_n) \cap \Gamma(A_{n-1})$ is precisely the set of all $p^n\chi$ with χ in $\Delta(D_n) \cap \Gamma(C_{n-1})$. From these considerations and from the definition of D_n, it follows that

$$\delta\{\Delta(A_{n-1}) \cap \Gamma(B_n)\} = p^{-n}\delta\{\Delta(C_{n-1}) \cap \Gamma(D_n)\},$$

$$\delta\{\Delta(B_n) \cap \Gamma(A_{n-1})\} = p^{-n}\delta\{\Delta(D_n) \cap \Gamma(C_{n-1})\},$$

$$\delta\Delta(B_n) = p^{-n}\delta\Delta(D_n).$$

Upon substituting these values in (1) and comparing with (2), we obtain the recursion

$$\delta\Delta(A_n) = \delta\Delta(A_{n-1}) + p^{-n}\{\delta\Delta(C_n) - \delta\Delta(C_{n-1})\}.$$

Summing over $n = 1, 2, \ldots,$ we arrive at

$$\begin{aligned}\delta\Delta(A) &= \delta\Delta(A_0) + \sum_{n=1}^{\infty}\{\delta\Delta(A_n) - \delta\Delta(A_{n-1})\} \\ &= \delta\Delta(C_0) + \sum_{n=1}^{\infty} p^{-n}\{\delta\Delta(C_n) - \delta\Delta(C_{n-1})\} \\ &= (1-p^{-1})\sum_{n=0}^{\infty} p^{-n}\delta\Delta(C_n).\end{aligned} \tag{3}$$

Note that since A is a finite sequence, the first two infinite series each have only a finite number of non-vanishing terms, while the third series is ultimately geometric and therefore convergent.

Two cases arise. Either no sequence C_n contains the number 1, or some C_n does. In the first case, the induction hypothesis applies to each C_n, so that (3) implies

$$\delta\Delta(A) \leqslant (1-p^{-1})\sum_{n=0}^{\infty} p^{-n}(\tfrac{1}{2}) = \tfrac{1}{2}.$$

In the second case, C_0 cannot contain the number 1, since $C_0 = A_0$ which is a subsequence of A. Hence there exists $N \geqslant 0$ such that C_n does not contain 1 for $0 \leqslant n \leqslant N$ and C_{N+1} does contain 1. Inasmuch as C_n is a subsequence of C_{n+1} it follows that C_n contains 1 for $n > N$. For $n > N$ let E_n be the sequence obtained by deleting all occurrences of 1 from C_n. Then for $n > N$,

$$\delta\Delta(C_n) = \begin{cases} 1-\delta\Gamma(E_n) & \text{if 1 occurs just once in } C_n, \\ 0 \text{ if 1 occurs more than once in } C_n. \end{cases} \tag{4}$$

But C_m is a subsequence of E_n for $m \leqslant N < n$, so that

$$\delta\Gamma(E_n) \geqslant \delta\Gamma(C_m) \geqslant \delta\Delta(C_m).$$

Therefore if $n > N \geqslant m$, (4) implies in all cases,

$$\delta\Delta(C_n) \leqslant 1 - \delta\Delta(C_m). \tag{5}$$

Letting μ denote the greatest of the quantities $\delta\Delta(C_0), \delta\Delta(C_1), \ldots, \delta\Delta(C_N)$, we have according to (5), for all n,

$$\delta\Delta(C_n) \leqslant \begin{cases} \mu & \text{for } n \leqslant N, \\ 1-\mu & \text{for } n > N. \end{cases}$$

Substituting these estimates in (3),

$$\begin{aligned}\delta\Delta(A) &\leqslant (1-p^{-1})\Big\{\sum_{n=0}^{N} p^{-n}\mu + \sum_{n=N+1}^{\infty} p^{-n}(1-\mu)\Big\} \\ &= (1-p^{-N-1})\mu + p^{-N-1}(1-\mu) = p^{-N-1} + \mu(1-2p^{-N-1}).\end{aligned}$$

But C_m satisfied the induction hypothesis for $m \leqslant N$, so that $\mu \leqslant \frac{1}{2}$. Therefore

$$\delta\Delta(A) \leqslant p^{-N-1} + (\tfrac{1}{2})(1-2p^{-N-1}) = \tfrac{1}{2}.$$

Afterword. Let $\Delta_k(A)$ denote the set of positive integers divisible by *exactly* k terms of A. Erdős has conjectured that if A has distinct terms, then $\delta\Delta_k(A)$ is bounded by a quantity which approaches zero as $k \to \infty$. Our method fails in this case, since C_n does not inherit the property of having distinct terms.

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