A note on numbers with a large prime factor III

by

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§ 1. Introduction. Let us denote by $P(u, k)$ the largest prime factor of $(u + 1)(u + 2) \cdots (u + k)$ where $u$ and $k$ are natural numbers, $k \leq u$. From the well-known deep results of G. H. Halberstam and A. E. Ingham, it follows that, for $u \leq k^{3/2}$ and $k$ large, $P(u, k) \geq u + 1$. In other words one at least of the numbers $u + 1, \ldots, u + k$ is a prime number. In an earlier paper [5] we considered non-trivial lower bounds for $P(u, k)$ when $k \sim u^{1/2}$ and in a later paper [6] we considered non-trivial lower bounds for $P(u, k)$, roughly in the range $k^{3/2} \leq u \leq k\log k$.

Next (for fixed $k$ of course) let $Q(u) = \min P(u, k)$ as $u$ runs over all numbers $\geq v$. In [6] we pointed out, that as a consequence of our results and an argument due to P. Erdős follows

Theorem A. We have

$$\liminf_{k \to \infty} \frac{Q(k^{3/2})}{k \log k} \geq 1.$$ 

This was an improvement of an inequality due to Erdős where in place of 1 stood $\frac{1}{2}$. In both his paper [2] and in a letter to me (dated 6. 10. 1969), Erdős has expressed the opinion that any further improvement, however slight, beyond 1, would be considerably difficult. So far, I have not succeeded in getting an improvement; but I have succeeded in proving the following theorem, whose proof is the main object of this paper (for another result see Theorem 6 and the remark below, of this paper).

Theorem B. Let $Q_1(v) = \min P(u, k)$ as $u$ runs over all numbers $\geq v$ with the exception of numbers $u$ in the range

$$k^{(\log k)(\log \log k)^{-1}} \leq u \leq k^{(\log k)\log \log k}.$$ 

Then

$$\liminf_{k \to \infty} \frac{Q_1(k^{3/2})}{k \log k} \geq 2.$$ 

To prove this theorem we have to use some results of H. Halberstam and K. F. Roth on $k$-free integers [3], to cover the range $u \leq k^{(\log k)(\log \log k)^{-1}}$. 

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To cover the range \( u \geq \beta^{(\log \beta)^{1+\frac{1}{k_0}} \log^2 \log \log \beta} \), we have to use some famous results of A. Baker [1] and in this connection I use the presentation of Baker’s theory set forth in my paper [4]. I have also to use a somewhat interesting lemma (Lemma 5 of this paper) due to me. In addition I have to use some results of [5], but this paper is self contained as far as possible.

In concluding the introduction, I record with pleasure my indebtedness to Professor Erdős for the enthusiasm shown by him in my work at its various stages. It is also a pleasure to thank Professor A. Schinzel for his interest in this work.

§ 2. The following theorem is implicitly contained in the work of Halberstam and Roth [3].

**Theorem 1.** Let \( u, k \) and \( l \) be natural numbers such that \( u \geq k \), \( l \geq 2 \).

Let \( n_1, \ldots, n_l \) \((1 \leq n \leq k)\) be any \( I \) integers which are divisible by the \( l \)-th power of some prime (may be different prime for different numbers \( n_i \)) greater than \( k \) and further satisfying \( u < n_1 < n_2 < \ldots < n_l \leq u + k \). Let \( U \) be the maximum of the numbers \( d(n_i) \) \((i = 1 \rightarrow I)\). Then for every fixed \( s > 0 \),

\[
I = O \left( \frac{U}{k} \log \frac{u + l}{k} \right)
\]

where the constant implied by \( O \) depends only on \( s \). (Hereafter we write \( O \) to mean this fact.)

To prove this theorem however, we need a simple lemma.

**Lemma 1.** Let \((1 - x)^{n_k-1} = P(x) - x^kQ(x)\), where \( l \geq 2 \) is an integer, \( P(x) \) and \( Q(x) \) are polynomials of degree at most \( l \) \((l \geq 1)\) (with integer coefficients). Then the resultant \( R_1 \) of these polynomials is an integer and

\[
d([-R_1]) = \Omega(x^{l-1}d(Q(1))
\]

Proof. Now \( P \) is a monic polynomial and \( R_1 = \prod P(\alpha) \) where \( \alpha \) runs over the zeros of \( Q(x) \). The defining relation between \( P \) and \( Q \) shows that \( P(\alpha) = (1 - \alpha)^{n_k-1} \). Write \( \beta = 1 - \alpha \) so that \( \beta \) is a zero of \( Q(1 - x) = 0 \) and \( R_1 = (Q(1))^{n_k-1} \). (Actually \( Q(1) = (-1)^k \sqrt{2l-1} \), though we do not need this.) If we write \( Q(x) = \prod P(\alpha) \), then

\[
d([-R_1]) = \prod (1 + \alpha_\alpha(2l_\alpha - 1)) \leq (2k_1)^{2l-1} \frac{\prod \alpha_\alpha(1 + \alpha_\alpha)}{\prod \alpha_\alpha}
\]

where in the exponent \( p \) runs over the prime divisors of \( Q(1) \). Using \(|Q(1)| < 2^{2l_1} \) this is easily seen to be \( \Omega(x^{l-1}d(Q(1))) \). (For, \( \sum \frac{1}{\log \alpha} \leq \frac{2\log 2}{\log x} + \frac{1}{2} \log 2 \) and we need only set \( x = k_1^{-1} \) to get the result.)

Next we state two lemmas contained in their paper (our statement differs only slightly from theirs).

**Lemma 2.** Let \( \eta = 4^{-\left(1+\frac{1}{k_0}\right)} \) where \( \epsilon > 0 \) is fixed, \( p, p_0 \) primes, \( p > k \), \( p_0 > k \) and \( k = k_0(\epsilon) \).

Suppose that

\[
P, \eta \Rightarrow \lfloor \eta^{1-p_0} \rfloor \leq \frac{1}{2} p_0
\]

and

\[
\eta^{1-p_0} \leq \eta^{p_0},
\]

where \( \eta \) and \( \eta_0 \) are natural numbers. Then

\[
\Omega \left( \eta^{1-p_0} \right) = \Omega \left( \eta^{1-p_0} \right)
\]

**Lemma 3.** The number of pairs \((m, p)\) satisfying (2) and (3) is

\[
\Omega \left( \eta^{1-p_0} \right)
\]

Lemmas 2 and 3 follow from their argument. However in proving Lemma 3 we have to use Lemma 1 above.

The next lemma is also theirs.

**Lemma 4.** Let \( \epsilon > 0 \) be arbitrary small and fixed, \( k \geq k_0(\epsilon) \) and \( 0 < k \leq \eta \). Then

\[
\Omega \left( \eta^{1-k} \right) = \sum_{n<p, m<n+k, x<p, x < p_0} 1 = \Omega \left( \eta^{1-p_0} \right) \eta_{1,2}^{2l-1} \frac{1}{2} (kX^{-\eta^{1-k}} + 1),
\]

where, in the sum prime denotes that we restrict \( p^\alpha n \) \((p > k)\) to some representation of each of the numbers \( n_1, \ldots, n_l \). (This convention shall be adopted unless we complete the proof of Theorem 1.) Here \( \eta \) is the number already introduced in Lemmas 2, and \( X > 1 \).

Proof. Suppose \( I \) and \( J \) are sub-intervals of \([1, 2w] \) and \([X, 2X] \) respectively and that \( |I| \leq \eta X \) and

\[
|J| \leq \frac{1}{2} (2X)^{2l-1} - 1.
\]

It follows as in their paper (if \( N(I, J) \) defined below is \( \neq 0 \) so that \( X < 2n^{1/k_0} \))

\[
N(I, J) = \sum_{n<p, m<n+k, x<p, x < p_0} 1 = \Omega \left( \eta^{1-p_0} \right) \eta_{1,2}^{2l-1} \frac{1}{2} (kX^{-\eta^{1-k}} + 1),
\]

and again that

\[
\Omega \left( \eta^{1-k} \right) = \Omega \left( \eta^{1-k} \right) \eta_{1,2}^{2l-1} \frac{1}{2} (kX^{-\eta^{1-k}} + 1).
\]

Also \( \Omega \left( \eta^{1-k} \right) = 0 \) if \( X > 2n^{1/k_0} \). This proves the lemma.
Proof of Theorem 1. Now if \( k \geq k_0(c) \)

\[
\sum_{\eta=0}^{\infty} 1 = \sum_{n=0}^{\infty} M(u, k; 2^n) = O(n^{\frac{1}{2}+\epsilon}) \left( \eta^{-1} + \sum_{n=0}^{\infty} 2^{2n+1} \right).
\]

This proves the theorem since \( \eta^{-1} = 4^{t+\epsilon} \) and the infinite sum over \( n \) is \( O(n) \). Thus the theorem is proved if \( k \geq k_0(c) \). But if \( k < k_0(c) \) the theorem is trivially true since then \( I = \Omega(1) \). This completes the proof of Theorem 1.

As in our paper [6] we define \( P(u, k) \) to be the maximum prime factor of \( u + 1 \)\( \ldots \), and the function \( Q(X, Y) = \min \{ P(u, k) \} \) taken over all \( u \) satisfying \( u \leq u \leq X \) (we assume \( Y \geq X+1 \)), and \( Q(X) = Q(X, \infty) \).

We now proceed to prove

**Theorem 2.** We have, for any positive constant \( c_1 \),

\[
\liminf_{X \to \infty} \left\{ \frac{\left( \log k \right)^2}{8 \log (1+c_1)} \right\} \left( \log \log k \right)^{1/2} \geq 1 + \left( \frac{c_1}{1+c_1} \right)^{1/2}.
\]

For proving this theorem as well as for further use, we need a general lemma. Let \( k \geq k_0 \) which may depend on finitely many constants.

**Lemma 3.** Let \( u < n < u + k \) (where \( u \) and \( k \) are natural numbers), \( c > 0 \) a constant, \( f_{0}^0(n) \) (l and \( u \) natural numbers) be the number of primes not exceeding \( (1+c)k \) whose \( l \)-th power divides \( n \). Let \( f_{0}^0(n) \) be the number of distinct prime factors of \( n \) which do not exceed \( \exp(\log k - (\log \log k)^2) \). For our applications, \( (\log \log k)^{2} \) may also be replaced by \( g(k) \log \log k \), where \( g(k) \to \infty \) and there will be a slight change in the inequality which follows, and \( f_{0}^0(n) \) the number of remaining prime factors which do not exceed \( (1+c)k \log k \).

Let \( s > 0 \) be a small constant, \( k \geq k_0(c) \), \( l \geq 2 \), \( s \geq c \) \( \sum \frac{1}{s} = A \), and \( j \) natural numbers.

Then if \( n \) runs over \( K(\leq k) \) of the numbers in the interval mentioned we have

\[
\sum_{n} \left\{ \left( f_{0}^0(n) + 2 \right)^{1/2} \right\} \left( f_{0}^0(n) + 2 \right)^{1/2} \left( f_{0}^0(n) + 2 \right)^{1/2}
\]

\[
\leq (Kk)^{1/2} \left( \frac{k}{K} \right)^{1/2} \left( (1 + 2e) \log \log k \right)^{1/2} \left( (1 + 2e) \log \log k \right)^{1/2} \times
\]

\[
\left( (1 + e + 2s)^{1/2} \right)^{1/2}.
\]

Proof. By using Hölder's inequalities (with an obvious notation),

\[
\sum_{n} \left\{ a_{n}^{1/2} b_{n}^{1/2} c_{n}^{1/2} \right\} \leq \left( \sum_{n} a_{n}^{1/2} b_{n}^{1/2} \right)^{1/2} \left( \sum_{n} c_{n}^{1/2} \right)^{1/2} \leq \left( \sum_{n} a_{n}^{1/2} b_{n}^{1/2} c_{n}^{1/2} \right)^{1/2} \leq \left( \sum_{n} a_{n}^{1/2} \right)^{1/2} \left( \sum_{n} b_{n}^{1/2} \right)^{1/2} \left( \sum_{n} c_{n}^{1/2} \right)^{1/2}
\]

where \( \frac{1}{a} + \frac{1}{b} = \frac{1}{c} \) and \( 1 + \frac{1}{a} = 1 + \frac{1}{b} = 1 + \frac{1}{c} \). Applying this we see that the sum in question does not exceed

\[
\left( \frac{1}{a} \right)^{1/2} \left( \frac{1}{b} \right)^{1/2} \left( \frac{1}{c} \right)^{1/2} \left( \frac{1}{K} \right)^{1/2} \left( \frac{1}{K} \right)^{1/2} \left( \frac{1}{K} \right)^{1/2} \left( \frac{1}{K} \right)^{1/2} \left( \frac{1}{K} \right)^{1/2} \left( \frac{1}{K} \right)^{1/2}
\]

\[
\times \left( (1 + e + 2s)^{1/2} \right)^{1/2}
\]

Here we have used the well-known result \( \sum_{P \leq x} \frac{1}{P} = \log \log x + C + O\left( \frac{1}{\log x} \right) \),

and the final inequality proves the lemma.

We deduce a useful corollary from Lemma 5, which we state as

**Theorem 4.** Let \( P(u, k) \leq (1 + e) \log k \), \( c_2 \) a constant satisfying \( 0 < c_2 < 1, k \geq k_0(c_2) \), and \( K = [c_2, k] \). Then there exists a natural number \( K \) depending only on \( c_2 \) and \( c_2 \), and \( K_1 = k - K + 1 \) distinct integers \( n_1, \ldots, n_{K_1} \) in \( [u, u + k] \) with the properties

\[
f_{0}^0(n_i) \leq (\log \log k)^{c_2}, \quad f_{0}^{1} (n_i) \leq (\log \log k)^{c_2}
\]

and

\[
f_{0}^{(i)} (n_i) \leq [c_2^{-1} (1 + e)] \quad (i = 1, 2, \ldots, K_1).
\]

Here \( c_2 \) and \( c_1 \) are positive constants depending only on \( c_2 \) and \( c_2 \). Consequently for such \( n_i \) we have firstly (assuming \( k \geq k_0 \))

\[
d(n_i) \leq (\log \log k)^{c_2} \left( 1 + \frac{\log (u + k)}{\log 2} \right)^{c_2^{-1} (1 + e)}
\]

and secondly, the maximum \( l \)-th power free factor of \( n_i \) does not exceed

\[
(\exp \exp \exp \exp ((\log \log k)^{2} + c_2 \log \log k)) \frac{1}{2} (\log \log \log \log k)^{1/2}
\]

\[
\leq k (\log \log k)^{1/2}.
\]
One may also note that if \( P_k(n) \) denotes the number of prime factors of \( n \) which exceed \( k \), then \( P_k(n_1) + \ldots + P_k(n_{K_1}) \leq (1 + c + o(1))k \).

**Proof.** In Lemma 5, choose \( j = [4\log\log\log k] \) where \( A_4 \) is a large constant and \( A = [4\log\log\log\log k] \) where \( A_4 \) is a large constant. The minimum term of the sum in Lemma 5 does not exceed

\[
\frac{1}{k} \left( \frac{1}{2} + \frac{1}{2k} \right) (1 + 2e) \log k \log \left( \frac{k}{\log k} \right) \left( 1 + 2e \right) \log(k + 1) \left( 1 + e + e^0 + 2e^0 + 2e^1 \right) - \frac{1}{k} \left( \frac{1}{2} + \frac{1}{2k} \right) (1 + 2e) \log k \log \left( \frac{k}{\log k} \right) \left( 1 + 2e \right) \log(k + 1) \left( 1 + e + e^0 + 2e^0 + 2e^1 \right).
\]

It follows that \( e^{1/2} \) times \( f^{(0)}(n) + e \) does not exceed the above expression and this gives our assertion regarding \( f^{(0)}(n) \). The other assertions follow similarly. The last statement follows from the fact that the g.c.d. of any two of the numbers \( n_m \) cannot exceed \( k \).

We are now in a position to prove Theorem 2. In view of the results of my earlier note [9], we can assume that \( n \gg \log\log k \). We start with the fundamental formula (Lemma 1 of [6])

\[
\left( \sum_{\sigma} \sum_{u < n < \frac{k}{u}} \left( \frac{w + k}{n} \right)^{1/2} \phi \left( \frac{n}{k} \right) \right) = \log \frac{k}{k} + O\left( \frac{k \log u}{\log\log k} \right)
\]

where \( j_u = \frac{\log u}{\log k} \) if \( u \) is a power of \( k \) and otherwise \( j_u = \left[ \frac{\log u}{\log k} \right] \).

We set \( j_u = \log \frac{u}{k} \) where \( g \) is a constant satisfying \( \frac{1}{2} < g < 1 \).

We assume that \( P(w, k) \ll (1 + c) \log k \log k \) where \( 0 < c < 1 \). The contribution from \( a \gg \sqrt{u} \) is easily seen to be

\[
\sum_{a \gg \sqrt{u}} \sum_{w < u < \frac{k}{u}} \sum_{w < p < \frac{k}{u} \log k} \log p \ll (1 - g) \frac{\log u}{\log k} \log k + o(1) \sum_{w < p < \frac{k}{u} \log k} 1.
\]

Let \( K_3 \) be the number of integers in \( \{u, u + k\} \) divisible by \( p^k \) for some \( p \) in the range \( k < p \ll (1 + c) \log k \). Then if \( I \) is the number of integers common to these and also the \( K = k - K + 1 \) integers of Theorem 3, we have \( K_3 + (k - K + 1) - I \ll k \). Hence \( K_3 + (k - K + 1) \ll k \). Consider these common integers \( I \) in number. Then by Theorem 1 (taking \( t_u \) for \( t \))

\[
I = O\left( \frac{\log(u + k) \log k}{\log(2 + k)} \left( 1 + \frac{\log(u + k)}{\log(2 + k)} \right) \right)^{1/2} \frac{1}{2^{2 - \epsilon} - 1} \sum_{w < p < \frac{k}{u} \log k} \frac{\log(p)}{\log(2 + k)}.
\]

Let us confine to \( u \ll \frac{\log k}{\log(2 + k)} \) so that

\[
I = O\left( \frac{\log(u + k) \log k}{\log(2 + k)} \left( 1 + \frac{\log(u + k)}{\log(2 + k)} \right) \right)^{1/2} \frac{1}{2^{2 - \epsilon} - 1} \sum_{w < p < \frac{k}{u} \log k} \frac{\log(p)}{\log(2 + k)}.
\]

If we choose \( 4(\log 2) e g \ll 1 \). In fact we define \( c_3 \) by \( \frac{1}{2g} + 4(\log 2) e g \ll 1 \). Choosing \( c_3 \) thus we have

\[
I = O\left( \frac{1}{2g} \right).
\]

This gives \( K_4 \ll K + I - 1 \ll c_3 k(1 - c + o(1)) \) and thus the contribution from \( a \gg t_u \) is

\[
\ll \left( 1 - g \right) c_3 + o(1)) e 4k \log u.
\]

But, since the intervals \( \left( \frac{u}{n}, \frac{u + k}{n} \right) \) for \( n < u^{-a} \) are disjoint,

\[
\sum_{a = 1}^{t_u - 1} \sum_{n < u^{-c}} \left( \frac{1}{u} \right) \left( \frac{u + k}{n} \right)^{1/2} + \phi \left( \frac{u}{n} \right) \right) \ll \sum_{a = 1}^{t_u - 1} (1 + c_3 + o(1)) e k \log k \ll (1 + c_3 + o(1)) e k \log k.
\]

If we secure that \( g(1 + c_3) + (1 - g) c_3 < 1 \) it follows now from (5) that with \( c_3 = \min(c, c_3) \), 

\[
P(w, k) > 1 + c_3 e k \log k.
\]

Obviously \( c_3 < 1 \) and we could take \( c = c_3 = c_3 \). Now any \( c_3 < 1 - (2g - 1) \left( 4(\log 2) e 2^{-1} \right) \) will do. Here the R. H. S. increases from \( - \infty \) to \( 1 \) in \( [0, 1] \) and so we could choose \( c_3 \) to be any constant less than \( e 8\log 2 \). Let us fix \( c_3 = \frac{1}{8\log 2} \). Then \( g \) will be determined by \( \frac{1}{2} + \frac{g}{1 + c_3} = 2 - 6e \), i.e. by \( \frac{1}{2} = \frac{2 - 6e}{g} + \frac{1}{1 + c_3} = 0 \). Now \( 1 + c = 1 + c_3 \) can be chosen to be any constant less than \( 1 + c_3 \) since \( c_3 \) is arbitrary. This proves Theorem 2.

\section{§ 3. Next we shall apply Baker's method (I follow my paper [4] which is slightly more convenient for my purposes) to prove

**Theorem 4.** We have for every \( \epsilon > 0 \),

\[
\lim_{k \to \infty} \left( Q\left( \frac{\log k}{\log(2 + k)^{1/2}} \right) \log k \right) = 2.
\]

In view of Theorem 2, we may now confine to \( u \gg \log(k) \log(\log(2 + k)) \). For simplicity we shall suppose, in Theorem 3, that \( 0 < c < 1 \). We record a special case of Theorem 3.

**Lemma 6.** Let \( u \gg \log(k) \log(\log(2 + k)) \). Then there exist a positive constant
Lemma 7. Suppose that \( u < p_1^m m_1 < p_2^m m_2 < \ldots < p_r^m m_r \leq u+k \) where \( a_1, m_1, \ldots, m_r, u, k \) are natural numbers, \( p_1, \ldots, p_r \) are primes, each of the \( m_r \) are \( B \)-free where \( B \) is a positive integral constant independent of \( u, k, a_1 \), \( p_1^m, m_1 \) does not divide \( m_i \), \( k < p_i \leq k^2 \) and finally \( u > k^{\log_{k} u/2} \). Then \( \frac{p_1}{p_1}, \frac{m_1}{m_1}, \frac{p_2}{p_2}, \frac{m_2}{m_2}, \ldots, \frac{p_r}{p_r}, \frac{m_r}{m_r} \) are multiplicatively independent.

Proof. It suffices to confine our attention to the ratio \( \frac{m_{r+1}}{m_r} \) where \( u \) for \( p_r^r m_r \). Since \( m_r \neq m_r \), the claim is true for \( r = 2 \). Assume now that \( \prod_{i=1}^{r-1} \left( \frac{p_i^{m_i}}{p_i^{m_i-1}} \right)^{k_i} = 1 \), where \( k_i \) are integers none zero and \( b_1 \ldots b_{r-1} = 1 \). Now \( k_i \) are determined by (finitely many) linear homogeneous equations (uniquely up to a constant multiple, for uniqueness follows by assuming the truth of the lemma for \( r-1 \) in place of \( r \)) with bounded coefficients, and so bounded. We have

\[
\frac{b_1 \log \frac{u_1}{u_2} + b_2 \log \frac{u_2}{u_3} + \ldots + b_{r-1} \log \frac{u_{r-1}}{u_r}}{a_1 b_1 \log \frac{p_1}{p_2} + \ldots + b_{r-1} \log \frac{p_{r-1}}{p_r}}
\]

Here the denominator has absolute value at least \( k^{-A_1} (A_1 \text{ constant}) \), and so \( a = O\left( k^{-A_1} \right) \) which is a contradiction since \( u > k^{\log_{k} u/2} \) and \( u \geq 1 \).

Suppose we prove that, under the condition \( a < (\log k)^{\alpha} \),

\[
L > C(a) e^{-\left(\log k\right)^{\alpha+1}}
\]

for every fixed \( \epsilon > 0 \). Then since \( L < k/u \), Theorem 4 follows. We concentrate therefore on proving

Theorem 5. Let \( a_1, a_2 \) be positive multiplicatively independent rational numbers with sizes \( (i.e. sizes of a \text{ rational number } a/b) \) defined to be \( |a|, |b| \) provided \( a, b \) are integers satisfying \( (a, b) = 1 \). Not exceeding \( S_k \), and \( \beta_1, \beta_2, \alpha_1, \alpha_2 \) are integers whose size does not exceed \( (\log S_k)^{\alpha} \) for every fixed \( \epsilon > 0 \),

\[
|\beta_1 \log a_1 - \log a_2| > A(\epsilon) e^{-\left(\log S_k\right)^{\alpha+\epsilon}}
\]

where \( A = A(\epsilon) \) is a constant depending only on \( \epsilon \) and not on \( a_1, a_2, \beta_1, \beta_2 \).

Remark. Until we prove the proof of Theorem 5, which shall be along the lines of my paper [4], we shall ignore the other notations in the paper.
of the present paper and follow the notation of [1]. It may also be mentioned that I follow my paper instead of Baker’s [1, I] since my paper is more convenient for my purposes.

Proof. In the notation of my paper $s = f = d = 1$. We can take $C_1 = S_1^d$, $C_2 = 2$ ($S$ will be assumed to exceed an absolute constant without loss of generality; note also that all the constants $C_1, C_2, \ldots, C_{11}$ of my paper may be replaced also by bigger constants). However we make a small change. We write $W = |\log a_1^2|$ and we see that it cancels out ultimately. We take instead of our original $C_2, C_3 = C_1^2$ and instead of the estimate (4) on page 3, we write down $|p(2)| \leq C_1^{12} C_2^{27} (S/L)^{12}$. Next we take $C_1 = 3 \log S$, and instead of the estimate $β C_1^{18} C_2^{37} (S/L)^{27}$ (on page 4 line 9 from the bottom) we write down the estimate

$$\leq β (L+1)^{3} C_1^{29} C_2^{27} (S/L)^{23} C_1^{17} C_2^{27} (C_2 S/L)^{27} W C_1^3.$$

Hence we get

$$|f^{(m)}(r)| \leq \frac{β (28 C_1 C_2) C_2^{18} (C_2 S/L)^{23} C_1^{17} C_2^{27} (C_2 S/L)^{27} W}{|r|^2},$$

where $C_1 = S_1^d$, $C_2 = 2$, $C_3 = C_1^2$ and $C_4 = 3 \log S$.

Next we have

$$\text{max}_{|r|=1} |f^{(m)}(r)| \leq (L+1)^{3} C_1^{29} C_2^{27} (S/L)^{23} C_1^{17} C_2^{27} (C_2 S/L)^{27} W C_1^3,$$

where $C_1 = C_1^2$ and $C_2 = (4 C_1 C_2) C_1^{17} C_2^{27} (C_2 S/L)^{27} W$.

On page 5, line 9 from the top, we change the estimate to

$$\leq β (L+1)^{3} C_1^{29} C_2^{27} (S/L)^{23} C_1^{17} C_2^{27} (C_2 S/L)^{27} W,$$

where $C_1 = C_1^2$ and $C_2 = (28 C_1 C_2) C_1^{17} C_2^{27} (C_2 S/L)^{27}$.

We can take for some integer $C_1 = C_1^2 S^{27}$ where $C_1 \geq C_1$, and it would follow that $|r|$ $\geq A^{-1}$ and so we have

$$|W^{-1} f(0)| \geq C_1^{17} C_2^{27} S^{-\beta} \geq β (28 C_1 C_2) C_1^{17} C_2^{27} (C_2 S/L)^{27}.$$
where \( \delta > 0 \) is an arbitrarily small constant and the constant \( C_{17} \) depends only on \( \delta_1, \beta, A, B, \delta_2, 0 \). Suppose now that

\[
(10) \quad \beta < \exp \left( -C_{17} \delta_2 \log \log \log S_1 \right).
\]

Then we have to choose \( C_{16} \) and \( C_{18} \) such that

\[
(11) \quad \delta, \beta > 200 \left( \log \log \log S_1 \right)^2 \log \log \log S_1 \quad \left( r = 1, 2, \ldots, \bar{r} - 1 \right).
\]

It is plain now that it suffices to satisfy

\[
(12) \quad \delta \geq C_{16} \left( \log \log S_1 \right)^{\delta_1 \log \log \log S_1}
\]

where \( C_{16} \) depends only on \( \delta, \beta, A, B, \delta_2, 0 \). It is plain that this inequality can be secured by first fixing \( C_{18} \) and then a large \( C_{16} \) (provided that \( 2 \delta_2 < \log \log \log S_1 \)). This gives immediately that \( (10) \) is false (see the last sentence of this section). It is plain that \( \delta \geq \delta_1 \) is a sufficient condition for avoiding \( (2C_{17})^{\delta_1 \log \log \log S_1} \). We have

\[
(13) \quad \delta \geq C_{18} \left( \log \log \log S_1 \right)^{\delta_1 \log \log \log S_1}
\]

The condition immediately above allows us to choose \( \delta \geq C_{18} \) and makes both \( A \) and \( b \) small, gives

\[
(14) \quad \beta > C_{19} \exp \left( \log \log \log S_1 \right)^{\delta_1 \log \log \log S_1}
\]

where \( C_{19} \) depends only on \( \delta_1 \).

We have still to check the inequality \( (L + 1)^2 < h_7 \). This requires

\[
(15) \quad (2h_7)^{\delta_1 \log \log \log S_1} < \left( \log \log \log S_1 \right)^{\delta_1 \log \log \log S_1}
\]

i.e., something like

\[
2 + 2E \leq 1 + (\bar{E} - 1) \delta_2.
\]

We are compelled to choose \( b = B - A_1^{-1} \) and making both \( A_1 \) and \( b \) small, gives

\[
(16) \quad \beta > C_{20} \exp \left( \log \log \log S_1 \right)^{\delta_1 \log \log \log S_1}
\]

where \( C_{20} \) depends only on \( \delta_1 \).

We have still to check the inequality \( (L + 1)^2 < h_7 \). This requires

\[
(17) \quad (2h_7)^{\delta_1 \log \log \log S_1} < \left( \log \log \log S_1 \right)^{\delta_1 \log \log \log S_1}
\]

i.e.,

\[
2 + 2E \leq 1 + \left( (1 + 2E + \delta) (B - A_1^{-1}) + 1 \right) \delta_2.
\]

We are compelled to choose \( b = B - A_1^{-1} \).

This completes the proof of Theorem 5, since \( h_7 > (L + 1)^2 \) contradicts easily the multiplicativity of \( \frac{\omega}{\omega - 1} \).

\section{4.}

Thus we have proved Theorem 4. We now resume the notation of this paper. In view of Theorems 2 and 3 we can now confine the gap

\[
\exp \left( (\log \log \log \log \log \log \log \log S_1)^{\delta_1} \right) \lesssim u \lesssim \exp \left( (\log \log \log \log \log \log \log \log S_1)^{\delta_1} \right)
\]

where \( \delta > 0 \) is an arbitrarily small constant, but fixed constant. (When \( u \) does not lie in \( S \), we know that by Theorems 2 and 3, \( P(u, k) \) exceeds \( (3 - \epsilon_1) \exp \left( (\log \log \log \log \log \log \log \log \log S_1)^{\delta_1} \right) \), where \( \epsilon_1 > 0 \) is any constant and \( k \geq k(\epsilon_1, \omega) \).)

In this gap we prove Theorem 6 (below) which is not quite satisfactory. The proof of Theorem 6, which is quite simple, is based on the following

\[
\text{Lemma 8. Let } l_2 \geq 2 \text{ be a natural number, } X \geq 1 \text{ and } m, n \text{ natural numbers. Then}
\]

\[
S = S_{l_2}(X) = \sum_{x < \log \log \log \log \log \log \log \log \log S_1} \left( \frac{X}{x^5} \right) = O(\log \log \log \log \log \log \log \log \log \log S_1).
\]

Proof. We have easily

\[
S = \sum_{x < \log \log \log \log \log \log \log \log \log \log S_1} \left( \frac{X}{x^5} \right) \leq \frac{X}{\log \log \log \log \log \log \log \log \log \log S_1}
\]

and this gives Lemma 8.

We now take \( X = u \), write \( k = \min \left( \log \log \log \log \log \log \log \log \log S_1, \omega - 1 \right) \) and we get

\[
S = o \left( \frac{1}{k} \right).
\]

It follows that there exists a gap of length \( [k] \), viz. \( w \leq \exp \left( (\log \log \log \log \log \log \log \log \log S_1)^{\delta_1} \right) \).

This proves the gap theorem of Theorem 6 (below) which is not quite satisfactory. The proof of Theorem 6, which is quite simple, is based on the following

\[
\text{Lemma 9. Let } l_2 \geq 2 \text{ be the least integer satisfying } k \text{ and let } k = \min \left( \log \log \log \log \log \log \log \log \log S_1, \omega - 1 \right).
\]

Then there exists an \( x \) satisfying \( u < x < u + h \) such that \( \exp \left( \log \log \log \log \log \log \log \log \log S_1 \right) \).

We can now use Lemma 6 to prove Theorem 6 (below). Suppose \( P(u, k) = \exp \left( (\log \log \log \log \log \log \log \log \log S_1)^{\delta_1} \right) \) for some \( \epsilon > 0 \). Lemma 6 now gives at least \( K_1 \) distinct integers with \( P(u, k) = \exp \left( (\log \log \log \log \log \log \log \log \log S_1)^{\delta_1} \right) \).

\[
K_1 \geq \frac{\log \log \log \log \log \log \log \log \log S_1}{\log \log \log \log \log \log \log \log \log S_1}
\]

i.e.,

\[
K_1 \geq \frac{\log \log \log \log \log \log \log \log \log S_1}{\log \log \log \log \log \log \log \log \log S_1}
\]

If we take \( l_2 \) to be the least integer \( K_1 \), which occurs, then we have

\[
K_1 \geq \exp \left( (\log \log \log \log \log \log \log \log \log S_1)^{\delta_1} \right)
\]

and

\[
u \sim 1 = \exp \left( (\log \log \log \log \log \log \log \log \log S_1)^{\delta_1} \right).
\]

Thus we have (by contradiction to Lemma 9)
THEOREM 6. Let \( \exp \left( \frac{\log k}{\log \log k} \right) \leq u \leq \exp \left( \frac{1}{\log k} \right) \), \( \epsilon > 0 \) arbitrarily small but fixed constant and

\[
h = \exp \left( \left( \log k \left( 1 - \frac{\log \log k}{\log k} + \frac{\log k}{\log \log k} \left( \frac{1}{\log u} \right) \right) \right) \right)
\]

with a certain positive constant \( G \).

Then there exists an \( x \) satisfying \( \frac{1}{2} x \leq 2n \) such that for every integer \( n \in [x, x + h] \) we have

\[
P(n, k) > (2 - \epsilon) k \log k
\]

where \( \epsilon \) is an arbitrarily small positive constant (the constant \( G \) in \( h \) may depend on \( \epsilon \), but certainly does not depend on \( r \)).

Remark. We can make slight improvements on this theorem and we do not wish to state them here. We may also remark that in Theorem A of the introduction we can improve the R.H.S. to 2 if we can prove something like (for \( k \geq 100 \))

\[
|\log a_1 - \log a_2| + |\log a_3 - \log a_4| + |\log a_5 - \log a_6| > G \cdot \frac{\log \log k}{\log k}
\]

where \( a_1, a_2, a_3, a_4, a_5, a_6 \) are multiplicatively independent positive rational numbers with height at most \( k^{\log \log k} \), \( a \) is a positive integer not exceeding \( (\log k)^2 \) and \( C \) is a positive absolute constant.

Added in proof.

A Corollary to Theorem 2. Let \( k \gg h, u_1, u_2, \ldots \) the sequence of all natural numbers whose largest prime factors exceed \( k \). Then

\[
\log k \leq \frac{7k}{(\log k)^2}
\]

\((k = 1, 2, \ldots)\)

References


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**Quotientbasen und \((R)\)-dichte Mengen**

von

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In der Arbeit [1], an welche die vorliegende Arbeit anknüpft, sind die Quotientenmengen \( R(A) \) für die Mengen \( A \),

\[
A = \{1, 2, 3, \ldots\} = N
\]

so definiert: \( R(A) \) bedeutet die Menge aller rationalen Zahlen der Form \( c/d \), wo \( c, d \in A \). Diese Definition kann man in folgender natürlicher Weise allgemeinieren.

**Definition 1.** Wenn \( A, B \subset N \), dann bedeutet \( R(A, B) \) die Menge aller rationalen Zahlen der Form \( c/d, c \in A, d \in B \). \( R(A, B) \) nennt man die Quotientenmenge der Mengen \( A, B \).

Es gilt im allgemeinen \( R(A, B) \neq R(B, A) \). Weiter offensichtlich

\[
R(A, A) = R(A).
\]

Es sei für die weiteren Bedürfnisse bemerkt, daß für \( A \subset N \) das Symbol \( \delta_+(A) \) (\( \delta_+(A) \)) das Zahl \( \liminf_{n \to \infty} \frac{A(n)}{n} \) (\( \limsup_{n \to \infty} \frac{A(n)}{n} \)) bezeichnet, wo \( A(n) = \sum_{c \in A, c \leq n} 1 \) ist. Wenn der Grenzwert \( \lim_{n \to \infty} \frac{A(n)}{n} \) existiert, dann setzen wir \( \delta(A) = \lim_{n \to \infty} \frac{A(n)}{n} \). Die Zahlen \( \delta_+(A), \delta_-(A) \) bzw. \( \delta(A) \) nennt man die untere, obere asymptotische Dichte von \( A \) bzw. die asymptotische Dichte von \( A \).

Es bedeute im weiteren \( R^+ \) die Menge aller positiven rationalen Zahlen. Es ergibt sich die Frage, unter welchen Voraussetzungen über die Mengen \( A, B \) die Gleichheit \( R(A, B) = R^+ \) gilt.

**SATZ 1.** Die Mengen \( A, B \subset N \) sollen wenigstens einer der folgenden Bedingungen erfüllen:

(a) \( \delta(A) = 1, \delta(B) = 1 \);
(b) \( \delta(A) = 1, \delta(B) = 1 \).

Dann existiert zu jedem \( r \in R^+ \) eine unendliche Anzahl von Paaren \( (a, b) \in A \times B \), so daß \( r = a/b \).