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A note on numbers with a large prime factor III

by

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§ 1. Introduction. Let us denote by $P(u, k)$ the largest prime factor of $(u+1)(u+2)\dots(u+k)$ where u and k are natural numbers, $k \leqslant u$. From the well-known deep results of G. Hoheisel and A. E. Ingham, it follows that, for $u \leqslant k^{3/2}$ and k large, $P(u, k) \geqslant u+1$. In other words one at least of the numbers $u+1, \dots, u+k$ is a prime number. In an earlier paper [5] we considered non-trivial lower bounds for $P(u, k)$ when $k \sim u^{1/2}$ and in a later paper [6] we considered non-trivial lower bounds for $P(u, k)$, roughly in the range $k^{3/2} \leqslant u \leqslant k^{\log\log k}$.

Next (for fixed k of course) let $Q(v) = \min P(u, k)$ as u runs over all numbers $\geqslant v$. In [6] we pointed out, that as a consequence of our results and an argument due to P. Erdös follows

THEOREM A. *We have*

$$\liminf_{k\to\infty}\left(Q(k^{3/2})\,(k\log k)^{-1}\right) \geqslant 1.$$

This was an improvement of an inequality due to Erdös where in place of 1 stood $\frac{1}{3}$. In both his paper [2] and in a letter to me (dated 6. 10. 1969), Erdös has expressed the opinion that any further improvement, however slight, beyond 1, would be considerably difficult. So far, I have not succeeded in getting an improvement; but I have succeeded in proving the following theorem, whose proof is the main object of this paper (for another result see Theorem 6 and the remark below, of this paper).

THEOREM B. *Let $Q_1(v) = \min P(u, k)$ as u runs over all numbers $\geqslant v$ with the exception of numbers u in the range*

$$k^{(\log k)(\log\log k)^{-1}} \leqslant u \leqslant k^{(\log k)^{3+\frac{1}{1000}}}.$$

Then

$$\liminf_{k\to\infty}\left(Q_1(k^{3/2})\,(k\log k)^{-1}\right) \geqslant 2.$$

To prove this theorem we have to use some results of H. Halberstam and K. F. Roth on k-free integers [3], to cover the range $u \leqslant k^{(\log k)(\log\log k)^{-1}}$.

To cover the range $u \geqslant k^{(\log k)^{3+\frac{1}{1000}}}$, we have to use some famous results of A. Baker [1] and in this connection I use the presentation of Baker's theory set forth in my paper [4]. I have also to use a somewhat interesting lemma (Lemma 5 of this paper) due to me. In addition I have to use some results of [6], but this paper is self contained as far as possible.

In concluding the introduction, I record with pleasure my indebtedness to Professor Erdös for the enthusiasm shown by him in my work at its various stages. It is also a pleasure to thank Professor A. Schinzel for his interest in this work.

§ 2. The following theorem is implicitly contained in the work of Halberstam and Roth [3].

THEOREM 1. *Let u, k and l_1 be natural numbers such that $u \geqslant k$, $l_1 \geqslant 2$. Let $n_1, \ldots, n_I$ ($I \leqslant k$) be any I integers which are divisible by the l_1-th power of some prime (may be different prime for different numbers n_i) greater than k and further satisfying $u < n_1 < n_2 \ldots < n_I \leqslant u+k$. Let U be the maximum of the numbers $d(n_i)$ ($i = 1$ to I). Then for every fixed $\varepsilon > 0$,*

$$I = O\left(U\left(\frac{u}{k}\right)^{\frac{1}{2l_1-1}} 16^{(1+\varepsilon)l_1}\right) \tag{1}$$

where the constant implied by O depends only ε. (Hereafter we write O_ε to mean this fact.)

To prove this theorem however, we need a simple lemma.

LEMMA 1. *Let $(1-z)^{2l_1-1} = P(z) - z^{l_1}Q(z)$, where $l_1 \geqslant 2$ is an integer, $P(z)$ and $Q(z)$ are polynomials of degree at most l_1-1 (with integer coefficients). Then the resultant R_1 of these polynomials is an integer and*

$$d(|R_1|) = O_\varepsilon(4^{(1+\varepsilon)l_1}).$$

Proof. Now $Q(z)$ is a monic polynomial and $R_1 = \prod_a P(a)$ where a runs over the zeros of $Q(z)$. The defining relation between P and Q shows that $P(a) = (1-a)^{2l_1-1}$. Write $\beta = 1-a$ so that β is a zero of $Q(1-z) = 0$ and so $R_1^2 = \left(Q(1)\right)^{4l_1-2}$. (Actually $Q(1) = (-1)^{l_1-2}\binom{2l_1-1}{l_1-2}$, though we do not need this.) If we write $\left(Q(1)\right)^2 = \prod_p p^{2a_p}$, then

$$d(|R_1|) = \prod_p \left(1+a_p(2l_1-1)\right) < (2l_1)^{\sum_p 1} d\left(|Q(1)|\right)$$

where in the exponent p runs over the prime divisors of $Q(1)$. Using $|Q(1)| < 2^{2l_1}$ this is easily seen to be $O_\varepsilon(4^{(1+\varepsilon)l_1})$. (For, $\sum_p 1 = \sum_{p\leqslant x} 1 + \sum_{p>x} 1 \leqslant \frac{x(1+\varepsilon)}{\log x} + \frac{2l_1\log 2}{\log x}$ and we need only set $x = l_1^{1-\varepsilon}$ to get the result.)

Next we state two lemmas contained in their paper (our statement differs only slightly from theirs).

LEMMA 2. *Let $\eta = 4^{-(1+\varepsilon)l_1}$ where $\varepsilon > 0$ is fixed, p, p_0 primes, $p > k$, $p_0 > k$ and $k \geqslant k_0(\varepsilon)$. Suppose that*

$$m|p-p_0|^{2l_1-1} \leqslant \tfrac{1}{2}p_0^{l_1} \tag{2}$$

and

$$|p^{l_1}m - p_0^{l_1}m_0| \leqslant \eta p_0, \tag{3}$$

where m and m_0 are natural numbers. Then

$$mP(p/p_0) = m_0Q(p/p_0).$$

LEMMA 3. *The number of pairs (m, p) satisfying (2) and (3) is*

$$O_\varepsilon\left(d(m_0p_0^{l_1})4^{(1+\varepsilon)l_1}\right).$$

Lemmas 2 and 3 follow from their argument. However in proving Lemma 3 we have to use Lemma 1 above.

The next lemma is also theirs.

LEMMA 4. *Let $\varepsilon > 0$ be arbitrarily small and fixed, $k \geqslant k_0(\varepsilon)$ and $0 < k \leqslant u$. Then*

$$M(u, k; X) = \sideset{}{'}\sum_{\substack{m,p\\ u<p^{l_1}m\leqslant u+k,\, X<p\leqslant 2X}} 1 = O_\varepsilon\left(U4^{(1+\varepsilon)l_1}\left(\frac{u}{X}\right)^{\frac{1}{2l_1-1}}(kX^{-1}\eta^{-1}+1)\right),$$

where, in the sum prime denotes that we restrict $p^{l_1}m$ ($p > k$) to some representation of each of the numbers $n_1, \ldots, n_I$. (This convention shall be adopted untill we complete the proof of Theorem 1.) Here η is the number already introduced in Lemma 2, and $X \geqslant 1$.

Proof. Suppose I' and J are sub-intervals of $[u, 2u]$ and $[X, 2X]$ respectively and that $|I'| \leqslant \eta X$ and

$$|J| \leqslant \tfrac{1}{4}(u^{-1}X^{2l_1})^{\frac{1}{2l_1-1}}.$$

It follows as in their paper (if $N(I', J)$ defined below is $\neq 0$ so that $X < 2u^{1/l_1}$)

$$N(I', J) = \sideset{}{'}\sum_{p,m,p\in J,\, p^{l_1}m\in I'} 1 = O_\varepsilon(U4^{(1+\varepsilon)l_1})$$

and again that

$$M(u, k; X) = O_\varepsilon\left(U4^{(1+\varepsilon)l_1}(kX^{-1}\eta^{-1}+1)\left((uX^{-1})^{\frac{1}{2l_1-1}}+1\right)\right).$$

Also $M(u, k; X) = 0$ if $X \geqslant 2u^{1/l_1}$. This proves the lemma.

Proof of Theorem 1. Now if $k \geqslant k_0(\varepsilon)$

$$\sum_{\substack{m,p \\ u<p^{l_1}m\leqslant u+k,\, p>k}}' 1 = \sum_{n=0}^{\infty} M(u, k; 2^n k)$$

$$= O_\varepsilon\left(U4^{(1+\varepsilon)l_1}\left(\frac{u}{k}\right)^{\frac{1}{2l_1-1}}\left(\eta^{-1}+\sum_{n=0}^{\infty} 2^{-\frac{n}{2l_1-1}}\right)\right).$$

This proves the theorem since $\eta^{-1} = 4^{(1+\varepsilon)l_1}$ and the infinite sum over n is $O(l_1)$. Thus the theorem is proved if $k \geqslant k_0(\varepsilon)$. But if $k \leqslant k_0(\varepsilon)$ the theorem is trivially true since then $I = O_\varepsilon(1)$. This completes the proof of Theorem 1.

As in our paper [6] we define $P(u, k)$ to be the maximum prime factor of $(u+1)\dots(u+k)$ and $Q(X, Y) = \min P(u, k)$ taken over all u satisfying $X \leqslant u \leqslant Y$ (we assume $Y \geqslant X+1$), and $Q(X) = Q(X, \infty)$. We now proceed to prove

THEOREM 2. *We have, for any positive constant* c_1,

$$(4)\qquad \liminf_{k\to\infty}\left\{Q\left(k^{3/2}, \operatorname{Exp}\left(\frac{(\log k)^2}{8\log 2\,(1+c_1)}\right)\right)(k\log k)^{-1}\right\} \geqslant 1+\left(\frac{c_1}{1+c_1}\right)^{1/2}.$$

For proving this theorem as well as for further use, we need a general lemma. Let $k \geqslant k_0$ which may depend on finitely many constants.

LEMMA 5. *Let $u < n \leqslant u+k$ (where u and k are natural numbers), $c > 0$ a constant, $f_k^{(l)}(n)$ (l and n natural numbers) be the number of primes not exceeding $(1+c)k\log k$ whose l-th power divides n. Let $f_k'(n)$ be the number of distinct prime factors of n which do not exceed* $\operatorname{Exp}\operatorname{Exp}\left(\log\log k - (\log\log\log k)^2\right)$ *(for our applications, $(\log\log\log k)^2$ may also be replaced by $g(k)\log\log\log k$, where $g(k) \to \infty$ and there will be a slight change in the inequality which follows), and $f_k''(n)$ the number of remaining prime factors which do not exceed $(1+c)k\log k$. Let $\varepsilon > 0$ be a small constant, $k \geqslant k_0(\varepsilon)$, $l \geqslant 2$, $\varepsilon^{(l)} = \sum_p p^{-l}$ and A and j natural numbers.*

Then if n runs over K ($\leqslant k$) of the numbers in the interval mentioned we have

$$\sum_n \left\{\left(f_k'(n)+\varepsilon\right)^{\frac{1}{4Aj}}\left(f_k''(n)+\varepsilon\right)^{\frac{1}{4j}}\left(f_k^{(l)}(n)+\varepsilon\right)^{\frac{1}{2}}\right\}$$

$$\leqslant (Kk)^{\frac{1}{2}}\left(\frac{k}{K}\right)^{\frac{1}{4j}(1+\frac{1}{A})}\left((1+2\varepsilon)\log\log k\right)^{\frac{1}{4Aj}}\left((1+2\varepsilon)\,(\log\log\log k)^2\right)^{\frac{1}{4j}} \times$$

$$\times\,(1+c+\varepsilon^{(l)}+2\varepsilon)^{\frac{1}{2}}.$$

Proof. By using Hölder's inequalities (with an obvious notation),

$$\sum\left(a_i^{\frac{1}{4Aj}}\, b_i^{\frac{1}{4j}}\, c_i^{\frac{1}{2}}\right) \leqslant \left(\sum\left(a_i^{\frac{1}{2Aj}}\, b_i^{\frac{1}{2j}}\right)\right)^{\frac{1}{2}}\left(\sum c_i\right)^{\frac{1}{2}} \leqslant \left(\sum a_i^{\frac{1}{Aj}}\right)^{\frac{1}{4}}\left(\sum b_i^{\frac{1}{j}}\right)^{\frac{1}{4}}\left(\sum c_i\right)^{\frac{1}{2}}$$

$$\leqslant \left(\sum 1\right)^{\frac{1}{4q_1}}\left(\sum a_i\right)^{\frac{1}{4Aj}}\left(\sum 1\right)^{\frac{1}{4q_2}}\left(\sum b_i\right)^{\frac{1}{4j}}\left(\sum c_i\right)^{\frac{1}{2}}$$

where $\frac{1}{q_1}+\frac{1}{Aj} = \frac{1}{q_2}+\frac{1}{j} = 1$ and so $\frac{1}{q_1}+\frac{1}{q_2} = 2-\frac{1}{j}-\frac{1}{Aj}$. Applying this we see that the sum in question does not exceed

$$K^{\frac{1}{2}-\frac{1}{4j}(1+\frac{1}{A})}\, k^{\frac{1}{2}+\frac{1}{4j}(1+\frac{1}{A})}\left(\frac{1}{k}\sum\left(f_k'(n)+\varepsilon\right)\right)^{\frac{1}{4Aj}}\left(\frac{1}{k}\sum\left(f_k''(n)+\varepsilon\right)\right)^{\frac{1}{4j}} \times$$

$$\times\left(\frac{1}{k}\sum\left(f_k^{(l)}(n)+\varepsilon\right)\right)^{\frac{1}{2}}$$

$$\leqslant (kK)^{\frac{1}{2}}\left(\frac{k}{K}\right)^{\frac{1}{4j}(1+\frac{1}{A})}\left((1+2\varepsilon)\log\log k\right)^{\frac{1}{4Aj}}\left((1+2\varepsilon)\,(\log\log\log k)^2\right)^{\frac{1}{4j}} \times$$

$$\times\,(1+c+\varepsilon^{(l)}+2\varepsilon)^{\frac{1}{2}}.$$

Here we have used the well-known result $\sum_{p\leqslant x}\frac{1}{p} = \log\log x + C + O\left(\frac{1}{\log x}\right)$, and the final inequality proves the lemma.

We deduce a useful corollary from Lemma 5, which we state as

THEOREM 3. *Let $P(u, k) \leqslant (1+c)k\log k$, c_2 a constant satisfying $0 < c_2 < 1$, $k \geqslant k_0(c_1, c_2)$, and $K = [c_2 k]$. Then there exists a natural number l depending only on c and c_2, and $K_1 = k-K+1$ distinct integers $n_1, \dots, n_{K_1}$ in $(u, u+k]$ with the properties*

$$f_k'(n_i) \leqslant (\log\log k)^{c_3}, \qquad f_k''(n_i) \leqslant (\log\log\log k)^{c_4}$$

and

$$f_k^{(l)}(n_i) \leqslant [c_2^{-1}(1+c)] \qquad (i = 1, 2, \dots, K_1).$$

Here c_3 and c_4 are positive constants depending only on c and c_2.

Consequently for such n_i we have firstly (supposing $k \geqslant k_0$)

$$d(n_i) \leqslant l^{(\log\log k)^{c_3+c_4}}\left(1+\frac{\log(u+k)}{\log 2}\right)^{c_2^{-1}(1+c)}$$

and secondly, the maximum l-th power free factor of n_i does not exceed

$$\left(\operatorname{Exp}\operatorname{Exp}\{\log\log k - (\log\log\log k)^2 + c_3\log\log\log k\}\right) k^{2(\log\log\log k)^{c_4}l}$$

$$\leqslant k^{(\log\log k)^{1/2}}.$$

One may also note that if $F_k(n)$ denotes the number of prime factors of n which exceed k, then $F_k(n_1)+\ldots+F_k(n_{K_1}) \leqslant (1+c+o(1))k$.

Proof. In Lemma 5, choose $j = [A_1 \log\log\log\log k]$ where A_1 is a large constant and $A = [A_2 j^{-1} \log\log\log k]$ where A_2 is a large constant. The minimum term of the sum in Lemma 5 does not exceed

$$\left(\frac{k}{K}\right)^{\frac{1}{2}+\frac{1}{4j}(1+\frac{1}{A})} \left((1+2\varepsilon)\log\log k\right)^{\frac{1}{4Aj}} \left((1+2\varepsilon)(\log\log\log k)^2\right)^{\frac{1}{4j}} (1+c+\varepsilon^{(l)}+2\varepsilon)^{\frac{1}{2}}.$$

It follows that $\varepsilon^{1/2j}$ times $(f_k^{(l)}(n)+\varepsilon)^{1/2}$ does not exceed the above expression and this gives our assertion regarding $f_k^{(l)}(n_i)$. The other assertions follow similarly. The last statement follows from the fact that the g.c.d. of any two of the numbers n_i cannot exceed k.

We are now in a position to prove Theorem 2. In view of the results of my earlier note [6], we can assume that $u \geqslant k^{\log\log k}$. We start with the fundamental formula (Lemma 1 of [6])

$$(5) \qquad \sum_{a=1}^{j_0-1} \sum_{n \leqslant uk^{-a}} \left\{\vartheta\left(\left(\frac{u+k}{n}\right)^{1/a}\right) - \vartheta\left(\left(\frac{u}{k}\right)^{1/a}\right)\right\} = k\log\frac{u}{k} + O\left(\frac{k\log u}{\log\log k}\right)$$

where $j_0 = \dfrac{\log u}{\log k} - 1$ if u is a power of k and otherwise $j_0 = \left[\dfrac{\log u}{\log k}\right]$. We set $l_2 = \left[g\dfrac{\log u}{\log k}\right]$ where g is a constant satisfying $\frac{1}{2} < g < 1$.

We assume that $P(u,k) \leqslant (1+c)k\log k$ where $0 < c < 1$. The contribution from $a \geqslant l_2$ is easily seen to be

(6)

$$\sum_{a=l_2}^{j_0} \sum_{n \leqslant uk^{-a}} \sum_{u < np^a \leqslant u+k} \log p \leqslant (1-g)\frac{\log u}{\log k} \log k\,(1+o(1)) \sum_{\substack{u < np^{l_2} \leqslant u+k \\ k < p \leqslant (1+c)k\log k}} 1.$$

Let K_2 be the number of integers in $(u, u+k]$ divisible by p^{l_2} for some p in the range $k < p \leqslant (1+c)k\log k$. Then if I is the number of integers common to these and also the $K_1 = k-K+1$ integers of Theorem 3, we have $K_2+(k-K+1)-I \leqslant k$. Hence $I \geqslant K_2-K+1$. Consider these common integers I in number. Then by Theorem 1 (taking l_2 for l_1)

$$I = O_\varepsilon\left(l^{(\log\log k)^{c_3}+c_4}\left(1+\frac{\log(u+k)}{\log 2}\right)^{c_2^{-1}(1+c)} k^{\frac{1}{2g}+\varepsilon}\, 16^{(1+\varepsilon)g\frac{\log u}{\log k}}\right).$$

Let us confine to $u \leqslant k^{c_5 \log k}$ so that

$$I = O_\varepsilon\left(\mathrm{Exp}\left((\log\log k)^{c_6}\right) k^{\frac{1}{2g}+\varepsilon+gc_5\log(16)(1+\varepsilon)}\right)$$
$$= O_\varepsilon\left(\mathrm{Exp}\left((\log\log k)^{c_6}\right) k^{\frac{1}{2g}+4(\log 2)gc_5+2\varepsilon}\right)$$

if we choose $4(\log 2)gc_5 \leqslant 1$. In fact we define c_5 by $\dfrac{1}{2g}+4(\log 2)gc_5 = 1-3\varepsilon$. Choosing c_5 thus we have

$$I = O_\varepsilon(k^{1-\frac{\varepsilon}{2}}).$$

This gives $K_2 \leqslant K+I-1 \leqslant c_2k(1+o(1))$ and thus the contribution from $a \geqslant l_2$ is

$$\leqslant \left((1-g)c_2+o(1)\right)k\log u.$$

But, since the intervals $\left(\left(\dfrac{u}{n}\right)^{1/a}, \left(\dfrac{u+k}{n}\right)^{1/a}\right]$ for $n \leqslant uk^{-a}$ are disjoint,

$$\sum_{a=1}^{l_2-1} \sum_{u((1+c_7)k\log k)^{-a} < n \leqslant uk^{-a}} \left\{\vartheta\left(\left(\frac{u+k}{n}\right)^{1/a}\right) - \vartheta\left(\left(\frac{u}{n}\right)^{1/a}\right)\right\}$$
$$\leqslant \sum_{a=1}^{l_2-1} (1+c_7+o(1))k\log k \leqslant \left(g(1+c_7)+o(1)\right)k\log u.$$

If we secure that $g(1+c_7)+(1-g)c_2 < 1$ it follows now from (5) that with $c_8 = \min(c, c_7)$, $P(u,k) > (1+c_8)k\log k$. Obviously $c_7 < 1$ and we could take $c = c_7 = c_8$. Now any $c_5 < \left(1-(2g)^{-1}\right)\left(4(\log 2)g\right)^{-1}$ will do. Here the R. H. S. increases from $-\infty$ to $\frac{1}{8}$ in $[0,1]$ and so we could choose c_5 to be any constant $< \dfrac{1}{8\log 2}$. Let us fix $c_5 = \dfrac{1}{8\log 2(1+c_1)}$. Then g will be determined by $\dfrac{1}{g}+\dfrac{g}{1+c_1} = 2-6\varepsilon$, i.e. by $\left(\dfrac{1}{g}\right)^2 - \dfrac{2-6\varepsilon}{g} + \dfrac{1}{1+c_1} = 0$. Now $1+c = 1+c_7$ can be chosen to be any constant less than $1/g$ since c_2 is arbitrary. This proves Theorem 2.

§ 3. Next we shall apply Baker's method (I follow my paper [4] which is slightly more convenient for my purposes) to prove

THEOREM 4. *We have for every $\varepsilon > 0$,*

$$\liminf_{k\to\infty} \left(Q\left(e^{(\log k)^{4+\varepsilon}}\right)(k\log k)^{-1}\right) \geqslant 2.$$

In view of Theorem 2, we may now confine to $u > e^{(\log k)^2(\log\log k)^{-1}}$. For simplicity we shall suppose, in Theorem 3, that $0 < c < 1$. We record a special case of Theorem 3.

LEMMA 6. *Let $u > e^{(\log k)^2(\log\log k)^{-1}}$, $P(u,k) \leqslant (1+c)k\log k$ for some c satisfying $0 < c < 1$ and $k \geqslant k_0(c)$. Then there exist a positive constant*

c_2 ($c_2 < 1$ *depending only on* c), *and* $K_1 = k - [c_2 k] + 1$ *distinct integers* $n_1, \ldots, n_{K_1}$ *in* $(u, u+k]$ *with the properties*

$$n_i = m_i p_i^{l(i)} \quad \text{where} \quad m_i \leqslant k^{(\log\log k)^{1/2}} \text{ and } (p_i, m_i) = 1 \ (i = 1, 2, \ldots, K_1).$$

Remark 1. Let c_8 be a small positive constant and $K_3 = [c_8 k]$. Then by the last statement of Theorem 3, the minimum of $F_k(n_i)$ over K_3 of the integers n_i is $\leqslant \left(\frac{1+c}{c_8} + o(1)\right)$. Hence there exists $K_4 = K_1 - K_3 + 1$ integers n_i with the further property that $F_k(n_i) \leqslant \frac{1+c}{c_8}$. Define $G_k(n)$ to be the number which results from n after replacing in its prime factor decomposition all primes exceeding k by 1. Then by an argument of Erdös (private communication, letter dated 18. 11. 69), the minimum of $G_k(n_i)$ taken over any $K_5 = [c_9 k]$ of the K_4 integers n_i (c_9 small positive constant), does not exceed $(k!)^{2K_5^{-1}}$. So in Lemma 6 we may replace c_2 by a slightly bigger positive constant (but still < 1), and assume $m_i \leqslant k^C$ where C is a positive constant depending only on c.

This unfortunately does not lead to a lowering of $4+\varepsilon$ in Theorem 4.

Remark 2. In view of the fact that m_i are small, it follows that the p_i are all distinct and we can assume that $p_i > k$ for each i. Now let the integers $l^{(i)}$ be all distinct. Then $l^{(i)} \leqslant \frac{\log(2u)}{\log k}$ and since the maximum of the integers $l^{(i)}$ is at least $(c_2 + o(1))k$, u will have to exceed e^k. We have $u < m_1 p_1^{l(1)} < m_2 p_2^{l(2)} \leqslant u+k$ and so

$$0 < \log \frac{m_2 p_2^{l(2)}}{m_1 p_1^{l(1)}} < \log \frac{u+k}{u}, \quad \text{i.e.,} \quad 0 < \left| \log \frac{m_2}{m_1} + l^{(2)} \log p_2 - l^{(1)} \log p_1 \right| < \frac{k}{u}.$$

But it is easy to see that $ku^{-1} < e^{-\frac{1}{2}a}$ where $a = \max(l^{(1)}, l^{(2)})$. By Baker's result [1, IV] it follows that $a < (4^9 \cdot 2 \cdot (\log k)^2)^{49}$ and this contradicts $u > e^k$. It also shows for instance that, whether $l^{(1)} = l^{(2)}$ or not, $a < (\log k)^{100}$.

In view of Remark 2, the proof of Theorem 4 now depends on lower bounds for $L = \left| \log \frac{m_2}{m_1} - a \log \frac{p_1}{p_2} \right|$ so as to contradict $L < \frac{k}{u}$ in the range $u > e^{(\log k)^{4+\varepsilon}}$. To apply the results of my paper [4], it is necessary to have the multiplicative independence of $\frac{m_2}{m_1}$ and $\frac{p_1}{p_2}$. This is trivial to prove. It is perhaps not unreasonable (in view of possible improvements if any on Theorem 4) to prove slightly more, viz.

LEMMA 7. *Suppose that* $u < p_1^a m_1 < p_2^a m_2 < \ldots < p_r^a m_r \leqslant u+k$, *where* $a, m_1, \ldots, m_r, u, k$ *are natural numbers,* $p_1, \ldots, p_r$ *primes, each of the* m_i *are B-free where B is positive integral constant independent of* u, k, a, p_i's, m_i's, p_i *does not divide* m_i, $k < p_i < k^2$ *and finally* $u > k^{\log\log k}$. *Then* $\frac{p_2}{p_1}, \frac{m_2}{m_1}, \frac{p_3}{p_2}, \frac{m_3}{m_2}, \ldots, \frac{p_r}{p_{r-1}}, \frac{m_r}{m_{r-1}}$ *are multiplicatively independent.*

Proof. It suffices to confine our attention to the ratios $\frac{m_{i+1}}{m_i} = \frac{u_{i+1} p_{i+1}^{-a}}{u_i p_i^{-a}}$ (we have written u_i for $p_i^a m_i$). Since $m_i \neq m_j$ trivially, the lemma is true for $r = 2$. Assume now that $\prod_{i=1}^{r-1} \left(\frac{u_i p_i^{-a}}{u_{i+1} p_{i+1}^{-a}}\right)^{b_i} = 1$, where b_i are integers none zero and $(b_1, \ldots, b_{r-1}) = 1$. Now b_i are determined by (finitely many) linear homogeneous equations (uniquely up to a constant multiple; for, uniqueness follows by assuming the truth of the lemma for $r-1$ in place of r) with bounded coefficients, and so bounded. We have

$$a = \frac{b_1 \log \frac{u_1}{u_2} + b_2 \log \frac{u_2}{u_3} + \ldots + b_{r-1} \log \frac{u_{r-1}}{u_r}}{b_1 \log \frac{p_1}{p_2} + \ldots + b_{r-1} \log \frac{p_{r-1}}{p_r}}.$$

Here the denominator has absolute value at least k^{-A_1} (A_1 constant), and so $a = O\left(\frac{k^{A_1}}{u}\right)$ which is a contradiction since $u > k^{\log\log k}$ and $a \geqslant 1$.

Suppose we prove that, under the condition $a < (\log k)^{100}$

$$L > C(\varepsilon) e^{-(\log k)^{4+\varepsilon}}$$

for every fixed $\varepsilon > 0$. Then since $L < k/u$, Theorem 4 follows. We concentrate therefore on proving

THEOREM 5. *Let* a_1 *and* a_2 *be positive multiplicatively independent rational numbers with sizes (i.e. size of a rational number* a/b *is defined to be* $|b| + |a/b|$ *provided* a *and* b *are integers satisfying* $(a, b) = 1$) *not exceeding* S_1 *and* β_0 *a rational number whose size does not exceed* $(\log S_1)^{100} = S$, *say. Then for every fixed* $\varepsilon > 0$,

$$|\beta_0 \log a_1 - \log a_2| > A(\varepsilon) e^{-(\log S_1)^{4+\varepsilon}}$$

where $A = A(\varepsilon)$ *is a constant depending only on* ε *and not on* a_1, a_2, β_0.

Remark. Until we complete the proof of Theorem 5, which shall be along the lines of my paper [4], we shall ignore the other notations of the present paper and follow the notation of [4]. It may also be mentioned that I follow my paper instead of Baker's [1, I] since my paper is more convenient for my purposes.

Proof. In the notation of my paper $n = f = d = 1$. We can take $C_1 = S_1^2$, $C_2 = 2$ (S will be assumed to exceed an absolute constant without loss of generality; note also that all the constants $C_1, C_2, \ldots, D_{11}$, of my paper may be replaced also by bigger constants). However we make a small change. We write $W = |(\log\alpha_1)^{m_1}|$ and we see that it cancels out ultimately. We take instead of our original C_3, $C_3 = C_1^3$ and instead of the estimate (4) on page 3, we write down $|p(\lambda)| \leqslant C_3^{Lh} C_2^{2k} (SL)^{2k}$. Next we take $C_4 = 3\log S_1$ and instead of the estimate $\beta C_6^{Lh_1+k}(SL)^{lk_1+2lk}$ (on page 4 line 9 from the bottom) we write down the estimate

$$\leqslant \beta (L+1)^2 C_3^{Lh} C_2^{2k} (SL)^{2k} C_1^{Lh_1} 7^{Lh_1} (C_2 SL)^{k_1} W C_4^{k_1}$$
$$\leqslant \beta (28 C_1 C_3)^{Lh_1} (C_2 C_4)^{3k} (C_2 SL)^{3k} W.$$

Hence we get

$$|f^{(m)}(r)| \leqslant \beta (28 C_1 C_3)^{Lh_1} (C_2^2 C_4 SL)^{3k} W, \tag{1}$$

where $C_1 = S_1^2$, $C_2 = 2$, $C_3 = C_1^3$ and $C_4 = 3\log S_1$.

Next we have

$$\max_{|z|=4h_2} |f(z)| \leqslant (L+1)^2 C_3^{Lh} C_2^{2k} (SL)^{2k} C_8^{Lh_2} (C_2 SL)^{k_2} W, \quad \text{where } C_8 = C_1^8 \tag{2}$$
$$\leqslant (4 C_3 C_8)^{Lh_2} (C_2 SL)^{3k} W.$$

On page 5, line 9 from the top, we change the estimate to

$$\leqslant \beta (L+1)^2 C_3^{Lh} C_2^{2k} (SL)^{2k} C_1^{Lh_2} 7^{Lh_2} (C_2 SL)^{k_2} \tag{3}$$
$$\leqslant \beta (28 C_1 C_3)^{Lh_2} (C_2 SL)^{3k}.$$

We can take for A some integer $\leqslant C_{10}^{Lh_2} S^{k_2}$ where $C_{10} = C_1$, and it would follow that $|\theta| \geqslant A^{-1}$ and so we have

$$|W^{-1} f(l)| > C_{10}^{-Lh_2} S^{-k} - \beta (28 C_1 C_3)^{Lh_2} (C_2 SL)^{3k}. \tag{4}$$

On page 6, we write down using (1) and (2) the estimate

$$(5)\ |f(l)| < \frac{1}{2\pi} \frac{2\pi 4h_2}{4h_2 - 2h_2} (4 C_3 C_8)^{Lh_2} (C_2 SL)^{3k} \left(\frac{h_2}{4h_2 - 2h_2}\right)^{(k_1 - k_2 + 1)h_1} W +$$
$$+ \frac{1}{2\pi} \sum_{r=0}^{h_1} \sum_{m=0}^{k_1 - k_2} \frac{1}{m!} \beta (28 C_1 C_3)^{Lh_1} (C_2^2 C_4 SL)^{3k} W \cdot 2\pi \cdot \tfrac{1}{2} \cdot \left(\frac{1}{2}\right)^m \cdot 2 \cdot (2h_2)^{h_1(k_1 - k_2 + 1)}$$
$$< (8 C_3 C_8)^{Lh_2} (C_2 SL)^{3k} 2^{-h_1(k_1 - k_2 + 1)} W + \beta W (112 C_1 C_3)^{Lh_1} (C_2^2 C_4 SL)^{3k} \times$$
$$\times (2h_2)^{h_1(k_1 - k_2 + 1)}.$$

We now set $k_2 = [k_1/2]$ and so

$$2^{-h_1(k_1 - k_2 + 1)} < 2^{-\frac{1}{2}h_1 k_1},$$
$$(2h_2)^{h_1(k_1 - k_2 + 1)} < (2h_2)^{h_1(k_1/2 + 2)} \quad \text{and} \quad k_1 - k_2 < k_1/2 + 1.$$

We now come to the final step. We have to see that (4) and (5) contradict, i.e.

$$C_{10}^{-Lh_2} S^{-k} \geqslant (8 C_3 C_8)^{Lh_2} (C_2 SL)^{3k} 2^{-\frac{1}{2}h_1 k_1} + \beta (28 C_1 C_3)^{Lh_2} (C_2 SL)^{3k} + \tag{6}$$
$$+ \beta (112 C_1 C_3)^{Lh_1} (C_2 C_4 SL)^{3k} (2h_2)^{h_1(k_1/2 + 2)}.$$

Here $C_1 = C_{10} = C_3^{1/3} = C_8^{1/8} = S_1^2$, $C_2 = 2$, $C_4 = 3\log S_1$. To satisfy (6) we must have

$$2^{\frac{1}{2}h_1 k_1} \geqslant (8 C_1 C_3 C_8)^{Lh_2} (2SL)^{4k} + \beta (224 C_1 C_3)^{Lh_2} (2 C_4 SL)^{4k} (4h_2)^{h_1(\frac{1}{2}k_1 + 2)},$$

i.e. we must have something like

$$2^{\frac{1}{2}h_1 k_1} \geqslant (1000 C_1^{12})^{Lh_2} (2 C_4 SL)^{4k} (1 + \beta h_2^{h_1 k_1}),$$

i.e. something like

$$2^{\frac{1}{2}h_1 k_1} \geqslant C_1^{13Lh_2 + 4k\log(SL)(\log C_1)^{-1} + 8k\log\log S_1 (\log C_1)^{-1}} (1 + \beta e^{h_1 k_1 \log h_2}),$$

i.e.

$$2^{\frac{1}{2}h_1 k_1} \geqslant C_1^{13Lh_2 + 12k\log(SL)(\log\log S_1)(\log C_1)^{-1}} (1 + \beta e^{h_1 k_1 \log h_2}) \tag{7}$$

where $C_1 = S_1^2$.

Let A_1 be a positive constant (we mean independent of S, S_1), $E > A_1^{-1}$ another positive constant, $\delta > 0$ a small constant,

$$\tilde{r} = [(1 + 2E + \delta)(E - A_1^{-1})^{-1}] + 2, \qquad C_{15} = C_9(\log S + \log C_1)$$

where C_9 is a large constant, $C_{14} = C_{15}^{41} C_{16}$ where C_{16} is a large constant, $h = [C_{14}]$, $L = [h^{1+E}]$, $k = [\frac{1}{9} h^{1+2E}]$. We also define somewhat (but not quite) similar to (with a positive constant b) my paper [4],

$$\begin{cases} h_1 = h,\ k_1 = k;\ h_2 = [C_{16} h_1 \cdot h^b],\ \ k_2 = \left[\dfrac{k_1}{2}\right], \ldots, \\ h_r = [C_{16} h_{r-1} h^b],\ \ k_r = \left[\dfrac{k_{r-1}}{2}\right]; \ldots\ (r = 2, \ldots, \tilde{r}). \end{cases} \tag{8}$$

We choose C_{15} and C_{16} to satisfy

$$2^{\frac{1}{2}h_r k_r} \geqslant e^{13Lh_{r+1}\log C_1 + 13k\log(SL)(\log\log S_1)} (1 + \beta e^{h_r k_r \log h_{r+1}}) \tag{9}$$
$$(r = 1, 2, \ldots, \tilde{r} - 1)$$

as follows.

If S is large we see that all the h_r and k_r are large, h_r is increasing, k_r is decreasing and for $r \leqslant \tilde{r} - 1$,

$$h_r k_r \log h_{r+1} < C_{17} h_{\tilde{r}-1}^{1+\delta_1} h^{1+2E}$$

where $\delta_1 > 0$ is an arbitrarily small constant and the constant C_{17} depends only on δ_1, E, A_1, δ. Suppose now that

$$\beta < e^{-C_{17} h_{\tilde{r}-1}^{1+\delta_1} h^{1+2E}}. \tag{10}$$

Then we have to choose C_{15} and C_{16} such that

$$h_r k_r \geqslant 200\big(Lh_{r+1}\log C_1 + k\log(SL)\log\log S_1\big) \quad (r = 1, 2, \ldots, \tilde{r}-1). \tag{11}$$

It is plain now that it suffices to satisfy

$$k \geqslant C_{18} C_{16} L h^b \log C_1 \tag{12}$$

where C_{18} depends only on A_1, δ, E. It is plain that this inequality can be secured by first fixing C_{16} and then a large C_9 (provided that $bA_1 \leqslant EA_1 - 1$). This gives immediately that (10) is false (see the last sentence of this section). It is plain that $h_{\tilde{r}-1} < (2C_{16})^{\tilde{r}-2} h^{1+(\tilde{r}-2)b}$ and so $C_{17} h_{\tilde{r}-1}^{1+\delta_1} h^{1+2E}$ does not exceed

$$C_{17}(2C_{16})^{(1+\delta_1)(\tilde{r}-2)} h^{2+\delta+2E+(1+\delta_1)(\tilde{r}-2)b}$$

and by the definition of h this does not exceed

$$C_{19}(\log S_1)^{A_1(2+\delta+2E+(1+\delta_1)(\tilde{r}-2)b)}$$

and this by putting $b = E - A_1^{-1}$ and making both A_1 and b small, gives

$$\beta > C_{20} e^{-(\log S_1)^{4+\varepsilon}},$$

where C_{20} depends only on ε.

We have still to check the inequality $(L+1)^2 < h_{\tilde{r}}$. This requires

$$(2h)^{2+2E} < (\tfrac{1}{2}C_{16})^{\tilde{r}-1} h^{1+(\tilde{r}-1)b},$$

i.e. something like

$$2+2E \leqslant 1 + (\tilde{r}-1)b,$$

i.e.

$$2+2E \leqslant 1 + \big([(1+2E+\delta)(E-A_1^{-1})^{-1}]+1\big)b.$$

We are compelled to choose $b = E - A_1^{-1}$.

This completes the proof of Theorem 5, since $h_{\tilde{r}} > (L+1)^2$ contradicts easily the multiplicative independence of α_1 and α_2.

§ 4. Thus we have proved Theorem 4. We now resume the notation of this paper. In view of Theorems 2 and 4 we can now confine to the gap

$$\operatorname{Exp}\big((\log k)^2(\log\log k)^{-1}\big) \leqslant u \leqslant \operatorname{Exp}\big((\log k^{4+\varepsilon})\big)$$

where $\varepsilon > 0$ is an arbitrarily small, but fixed constant. (When u does not lie in this gap we know that by Theorems 2 and 4, $P(u, k)$ exceeds $(2-\varepsilon_1)k\log k$ provided $\varepsilon_1 > 0$ is any constant and $k \geqslant k(\varepsilon_1)$, $u \geqslant k^2$.) In this gap we prove Theorem 6 (below) which is not quite satisfactory. The proof of Theorem 6, which is quite simple, is based on the following

LEMMA 8. *Let $l_3 \geqslant 2$ be a natural number, $X \geqslant 1$ and m, n natural numbers. Then*

$$S = S_{l_3}(X) = \sum_{\substack{x < nm^{l_3} \leqslant 2X \\ m > k}} 1 = O\big(X^{l_3^{-1}} + l_3^{-1} X k^{-l_3+1}\big).$$

Proof. We have easily

$$S = \sum_{k < m \leqslant (2X)^{l_3^{-1}}} \left(\left[\frac{2X}{m^{l_3}}\right] - \left[\frac{X}{m^{l_3}}\right]\right) \leqslant X \sum_{m > k} m^{-l_3} + O\big(X^{l_3^{-1}}\big)$$

and this gives Lemma 8.

We now take $X = u$, write $h = \min\left(k^{l_3}, u^{1-l_3^{-1}}\right)$ and we get $S = o\left(\frac{uk}{h}\right)$.

It follows that there exists a gap of length $[h]$, viz. $x \leqslant nm^{l_3} \leqslant x+h$, with x in $(u, 2u]$, where $S_1 = \sum_{\substack{x \leqslant nm^{l_3} \leqslant x+h \\ m > k}} 1 = o(k)$, provided l_3 exceeds a function of k which tends to infinity with k. As a consequence we have

LEMMA 9. *Let l_3 exceed a function of k which tends to infinity with k and let $h = \min(k^{l_3}, u^{1-l_3^{-1}})$. Then there exists an x satisfying $u \leqslant x < x + h \leqslant 2u$ such that for all y satisfying $x \leqslant y < y+k \leqslant x+h$, we have (uniformly),*

$$\sum_{y < nm^{l_3} \leqslant y+k,\, m > k} 1 = o(k).$$

We can now use Lemma 6 to prove Theorem 6 (below). Suppose $P(u, k) \leqslant (1+c)k\log k$ for some c satisfying $0 < c < 1$. Lemma 6 now gives at least K_1 distinct integers with $l^{(i)}$ satisfying

$$k^{l^{(i)}} \leqslant 2u, \quad \text{but} \quad u \leqslant k^{(\log\log k)^{1/2}}(2k\log k)^{l^{(i)}};$$

i.e.

$$l^{(i)} \leqslant \frac{\log(2u)}{\log k}, \quad \text{but} \quad l^{(i)} \geqslant \frac{\log u}{\log(2k\log k)} - \frac{(\log\log k)^{1/2}\log k}{\log(2k\log k)}.$$

If we take l_3 to be the least integer $l^{(i)}$ which occurs, we then have

$$k^{l_3} \geqslant \operatorname{Exp}\left\{\frac{\log u \log k}{\log(2k\log k)} - \frac{(\log\log k)^{1/2}(\log k)^2}{\log(2k\log k)}\right\}$$
$$= \operatorname{Exp}\left\{(\log u)\left(1 - O\left(\frac{\log\log k}{\log k} + \frac{(\log k)(\log\log k)^{1/2}}{\log u}\right)\right)\right\}$$

and

$$u^{1-l_3^{-1}} \geqslant \operatorname{Exp}\left\{(\log u)\left(1 - O\left(\frac{\log k}{\log u}\right)\right)\right\}.$$

Thus we have (by contradiction to Lemma 9)

THEOREM 6. *Let* $\operatorname{Exp}\left((\log k)^2(\log\log k)^{-1}\right) \leqslant u \leqslant \operatorname{Exp}\left((\log k)^{4+\varepsilon}\right)$, $\varepsilon > 0$ *arbitrarily small but fixed constant and*

$$h = \operatorname{Exp}\left\{(\log u)\left(1 - G\left(\frac{\log\log k}{\log k} + \frac{(\log k)(\log\log k)^{1/2}}{\log u}\right)\right)\right\}$$

with a certain positive constant G.

Then there exists an x *satisfying* $u \leqslant x \leqslant 2u$ *such that for every integer* n *in* $[x, x+h]$ *we have*

$$P(n; k) > (2-\varepsilon_1) k \log k$$

where ε_1 *is an arbitrarily small positive constant (the constant* G *in* h *may depend on* ε_1*, but certainly does not depend on* ε*).*

Remark. We can make slight improvements on this theorem and we do not wish to state them here. We may also remark that in Theorem A of the introduction we can improve the R.H.S. to 2 if we can prove something like (for $k \geqslant 100$)

$$|a \log a_1 - \log a_2| + |a \log a_3 - \log a_4| + |a \log a_5 - \log a_6| > Ce^{-(\log k)^2(\log\log k)^{-1}}$$

where $a_1, a_2, \ldots, a_6$ are multiplicatively independent positive rational numbers with height at most $k^{(\log\log k)^{1/2}}$, a is a positive integer not exceeding $(\log k)^5$ and C is a positive absolute constant.

Added in proof.

A COROLLARY TO THEOREM 2. *Let* $k > k_0$ *and* $n_1, n_2, \ldots$ *the sequence of all natural numbers whose largest prime factors exceed* k. *Then*

$$n_{i+1} - n_i < \frac{k}{\log k} + \frac{7k}{(\log k)^2} \qquad (i = 1, 2, \ldots)$$

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Quotientbasen und (R)-dichte Mengen

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In der Arbeit [3], an welche die vorliegende Arbeit anknüpft, sind die Quotientmengen $R(A)$ für die Mengen A,

$$A \subset \{1, 2, 3, \ldots\} = N$$

so definiert: $R(A)$ bedeutet die Menge aller rationalen Zahlen der Form c/d, wo $c, d \in A$. Diese Definition kann man in folgender natürlicher Weise verallgemeinern.

DEFINITION 1. Wenn $A, B \subset N$, dann bedeutet $R(A, B)$ die Menge aller rationalen Zahlen der Form a/b, $a \in A$, $b \in B$. $R(A, B)$ nennt man die Quotientmenge der Mengen A, B.

Es gilt im allgemeinen $R(A, B) \neq R(B, A)$. Weiter offensichtlich $R(A, A) = R(A)$.

Es sei für die weiteren Bedürfnisse bemerkt, daß für $A \subset N$ das Symbol $\delta_1(A)$ $(\delta_2(A))$ die Zahl $\liminf_{n\to\infty} \frac{A(n)}{n}$ $\left(\limsup_{n\to\infty} \frac{A(n)}{n}\right)$ bezeichnet, wo $A(n) = \sum_{a \leqslant n, a \in A} 1$ ist. Wenn der Grenzwert $\lim_{n\to\infty} \frac{A(n)}{n}$ existiert, dann setzen wir $\delta(A) = \lim_{n\to\infty} \frac{A(n)}{n}$. Die Zahlen $\delta_1(A)$, $\delta_2(A)$ bzw. $\delta(A)$ nennt man die untere, obere asymptotische Dichte von A bzw. die asymptotische Dichte von A.

Es bedeute im weiteren R^+ die Menge aller positiven rationalen Zahlen. Es ergibt sich die Frage, unter welchen Voraussetzungen über die Mengen A, B die Gleichheit $R(A, B) = R^+$ gilt.

SATZ 1. *Die Mengen* $A, B \subset N$ *sollen wenigstens eine der folgenden Bedingungen erfüllen:*

(a) $\delta(A) = 1, \quad \delta_2(B) = 1;$

(b) $\delta_2(A) = 1, \quad \delta(B) = 1.$

Dann existiert zu jedem $r \in R^+$ *eine unendliche Anzahl von Paaren* $(a, b) \in A \times B$, *so daß* $r = a/b$.