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Johnson Reprint Corporation, 111 Fifth Ave., New York, N. Y.

PRINTED IN POLAND

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ACTA ARITHMETICA
XIX (1971)

On the error term in the linear sieve

by

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§ 1. Introduction. The aim of this paper is to improve certain results of Jurkat and Richert [4] concerning the linear sieve and obtained by Selberg's method. In the paper a variant of Brun's method is used, which was considered earlier in an unpublished manuscript of B. Rosser (see A. Selberg [6]). In order to formulate the theorem we introduce the following notation:

$M = \{a_1, \ldots, a_n\}$ is a fixed set of integers,

$A_k(M; z)$ the number of elements of M not divisible by any prime $p < z$, $p \nmid k$,

M_d the set of all elements of M divisible by d,

$|M_d|$ the number of the elements of M_d,

$$R_k(z) = \prod_{\substack{p<z\\ p\nmid k}} \left(1 - \frac{1}{p}\right), \quad R_1(z) = R(z).$$

Theorem 1. *Let* $y > 1$, $s < \dfrac{\log y}{(\log\log 3y)^{11/5}}$ *and assume*

$$(*) \qquad \left| |M_d| - \frac{y}{d} \right| \leqslant 1 \quad \textit{for} \quad (d, k) = 1.$$

There exists an absolute constant c_0 *such that*

$$(1.1) \qquad A_k(M; y^{1/s}) < yR_k(y^{1/s}) \left\{ 1 + \frac{F(s)}{s} + c_0 \left(1 + \frac{s^2 \log^5 s}{\log^2 y}\right)^{5s} \frac{\mathfrak{M}(s) \log\log 3k}{\log y} \right\} \quad \textit{for } s \geqslant 3,$$

$$(1.2) \qquad A_k(M; y^{1/s}) > yR_k(y^{1/s}) \left\{ 1 - \frac{f(s)}{s} - c_0 \left(1 + \frac{s^2 \log^5 s}{\log^2 y}\right)^{5s} \frac{\mathfrak{M}(s) \log\log 3k}{\log y} \right\} \quad \textit{for } s \geqslant 2,$$

where $f(s)$, $F(s)$, $\mathfrak{M}(s)$ *are defined in* § 3 *and satisfy the conditions*

$$f(s) = s-2e^{\gamma}\log(s-1) \quad \text{for} \quad 2 \leqslant s \leqslant 4,$$

$$F(s) = s-6+2e^{\gamma}\left(1-\int\limits_{1}^{s-2}\frac{\log x}{x+1}\,dx\right) \quad \text{for} \quad 3 \leqslant s \leqslant 5,$$

$$\mathfrak{M}(s) = \exp\left\{-s\left(\log s+\log\log s-1+\frac{\log\log s}{\log s}\right)+O\left(\frac{s}{\log s}\right)\right\}$$

(γ is Euler's constant).

COROLLARY. *Let* $y > 1$ *and assume* $(*)$. *Then*

$$A_k(M; z) < yR_k(z)\left\{1+\frac{F\left(\frac{\log y}{\log z}\right)}{\frac{\log y}{\log z}}+c_1\frac{\log\log 3k}{\log y}\right\}, \tag{1.3}$$

$$A_k(M; z) > yR_k(z)\left\{1-\frac{f\left(\frac{\log y}{\log z}\right)}{\frac{\log y}{\log z}}-c_1\frac{\log\log 3k}{\log y}\right\}. \tag{1.4}$$

The main terms of the bounds for $A_k(M; z)$ given above coincide with those of Jurkat and Richert. On the other hand the error term occurring in their main Theorem 5 is not $O\left(\frac{\log\log 3k}{\log y}\right)$ but $O\left(\frac{\log\log 3k}{(\log y)^{1/14}}\right)$. This difference has some significance in applications. It is a well known old problem to estimate for a given r the maximal length $C_0(r)$ of the sequence of consecutive integers each divisible by one of the first r primes. Legendre gave for $C_0(r)$ an upper bound which turned out to be false (cf. [1], p. 415). Then many authors gave lower bounds, the best at present due to Rankin [5] being

$$C_0(r) \geqslant e^{\gamma-\varepsilon}\frac{r\log^2 r\log\log\log r}{(\log\log r)^2}.$$

Jacobsthal (cf. [2]) has modified the problem considering the maximal length $C(r)$ of a sequence of consecutive integers each divisible by one of r suitably chosen primes. He asked whether $C(r) = C_0(r)$ and whether $C(r) \ll r^2$.

Corollary implies easily

THEOREM 2.

$$C_0(r) \ll r^2\log^2 r.$$

The results of [4] imply only $C_0(r) < r^2\exp(\log r)^{13/14}$.

From Theorem 1 one can easily deduce the following

THEOREM 3. *Let* $y > 1$ *and assume* $(*)$. *Then*

$$A_k(M; y^{1/s}) = yR_k(y^{1/s})\left\{1+O\left(\frac{e}{s\log s}\right)^s\right\} \quad \text{for} \quad s < \frac{\log y}{(\log\log 3y)^6}, \tag{1.5}$$

$$A_k(M; z) = yR_k(z)\left\{1+O\left(\frac{1}{\log y}\right)\right\} \quad \text{for} \quad \log z < \frac{\log y\,\log\log\log 16y}{2\log\log 3y}. \tag{1.6}$$

This is an improvement of Theorem 3 of [4].

In the sequel $c_0, c_1, \ldots$ denote absolute constants.

§ 2. The general principles of sieve.

LEMMA 1. *Let* $f_i(x_1, \ldots, x_i)$ *be a real function of* i *real variables. For every* $r \geqslant 1$ *and every* $z \geqslant 2$ *we have*

$$\begin{aligned} A_k(M; z) &\leqslant |M|-\sum_{p_1<z}|M_{p_1}|+\sum_{p_2<p_1<z}|M_{p_1p_2}|-\sum_{\substack{p_3<p_2<p_1<z\\ p_3<f_2(p_1,p_2)}}|M_{p_1p_2p_3}|+\ldots+ \\ &\quad +\sum_{\substack{p_{2r}<\ldots<p_1<z\\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i})\\ i=1,\ldots,r-1}}|M_{p_1\ldots p_{2r}}|, \end{aligned} \tag{2.1}$$

$$\begin{aligned} A_k(M; z) &\geqslant |M|-\sum_{p_1<z}|M_{p_1}|+\sum_{\substack{p_2<p_1<z\\ p_2<f_1(p_1)}}|M_{p_1p_2}|-\sum_{\substack{p_3<p_2<p_1<z\\ p_2<f_1(p_1)}}|M_{p_1p_2p_3}|+\ldots- \\ &\quad -\sum_{\substack{p_{2r-1}<\ldots<p_1<z\\ p_{2i}<f_{2i-1}(p_1,\ldots,p_{2i-1})\\ i=1,\ldots,r-1}}|M_{p_1\ldots p_{2r-1}}|, \end{aligned} \tag{2.2}$$

where $p_i \nmid k$, $1 \leqslant i \leqslant 2r$.

Proof. We use the obvious identity

$$A_k(M; z) = |M|-\sum_{p<z,\,p\nmid k}A_k(M_p; p). \tag{2.3}$$

Applying (2.3) to $A_k(M_{p_1}; p_1)$ and then to $A_k(M_{p_1p_2}; p_2)$ we obtain

$$\begin{aligned} A_k(M; z) &= |M|-\sum_{p_1<z}|M_{p_1}|+\sum_{p_2<p_1<z}A_k(M_{p_1p_2}; p_2) \\ &= |M|-\sum_{p_1<z}|M_{p_1}|+\sum_{p_2<p_1<z}|M_{p_1p_2}|-\sum_{p_3<p_2<p_1<z}A_k(M_{p_1p_2p_3}; p_3), \end{aligned} \tag{2.4}$$

where $p_i \nmid k$, $i = 1, 2, 3$. Since $A_k(M; z) \geqslant 0$, (2.4) implies (2.1) and (2.2) for $r = 1$. For $z_1 \leqslant z_2$ $A_k(M; z_2) \leqslant A_k(M; z_1)$, thus

$$A_k(M_{p_1p_2}; p_2) \leqslant A_k\big(M_{p_1p_2}; \min(p_2, f_2(p_1, p_2))\big).$$

Applying to $A_k\big(M_{p_1p_2}; \min(p_2, f_2(p_1, p_2))\big)$ the inequality (2.1) with $f_{2i}(p_3, \ldots, p_{2i+2})$ replaced by $f_{2i+2}(p_1, \ldots, p_{2i+2})$ and then applying the obtained estimation for $A_k(M_{p_1p_2}; p_2)$ to the middle term of (2.4) we get (2.1) for $r+1$. Analogously $A_k(M_{p_1}; p_1) \leqslant A_k\big(M_{p_1}; \min(p_1, f_1(p_1))\big)$. Applying to $A_k\big(M_{p_1}; \min(p_1, f_1(p_1))\big)$ the inequality (2.1) with $f_{2i}(p_2, \ldots, p_{2i+1})$ replaced by $f_{2i+1}(p_1, \ldots, p_{2i+1})$ and then applying the obtained estimation for $A_k(M_{p_1}; p_1)$ to (2.3) we get (2.2) for $r+1$.

Corollary. *For $r \geqslant 1$ we have*

$$(2.5)\qquad A_k(M; z) \leqslant y\Bigg(1 - \sum_{p_1<z}\frac{1}{p_1} + \sum_{p_2<p_1<z}\frac{1}{p_1p_2} - \sum_{\substack{p_3<p_2<p_1<z\\ p_3<f_2(p_1,p_2)}}\frac{1}{p_1p_2p_3} + \ldots + \sum_{\substack{p_{2r}<\ldots<p_1<z\\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i})\\ i=1,\ldots,r-1}}\frac{1}{p_1\ldots p_{2r}}\Bigg) + \sum_{\substack{d=p_1\ldots p_l,\, l\leqslant 2r\\ p_l<\ldots<p_1>z\\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i})\\ i=1,\ldots,[l/2-1]}} 1,$$

$$(2.6)\qquad A_k(M; z) \geqslant y\Bigg(1 - \sum_{p_1<z}\frac{1}{p_1} + \sum_{\substack{p_2<p_1<z\\ p_2<f_1(p_1)}}\frac{1}{p_1p_2} - \sum_{\substack{p_3<p_2<p_1<z\\ p_2<f_1(p_1)}}\frac{1}{p_1p_2p_3} + \ldots - \sum_{\substack{p_{2r-1}<\ldots<p_1<z\\ p_{2i}<f_{2i-1}(p_1,\ldots,p_{2i-1})\\ i=1,\ldots,r-1}}\frac{1}{p_1\ldots p_{2r-1}}\Bigg) - \sum_{\substack{d=p_1\ldots p_l,\, l\leqslant 2r-1\\ p_l<\ldots<p_1<z\\ p_{2i}<f_{2i-1}(p_1,\ldots,p_{2i-1})\\ i=1,\ldots,[l/2]}} 1,$$

where $p_i \nmid k$, $1 \leqslant i \leqslant 2r$.

We write (2.5), (2.6) in the shorter form

$$(2.7)\qquad A_k(M; z) \leqslant yG_{r,k}(z) + \Sigma_1(z),$$

$$(2.8)\qquad A_k(M; z) \geqslant yD_{r,k}(z) - \Sigma_2(z).$$

Lemma 2. *Let for $r \geqslant 1$*

$$(2.9)\qquad S_{r,k}(z) = R_k(z) + \sum_{\substack{p_3<p_2<p_1<z\\ f_2(p_1,p_2)\leqslant p_3}}\frac{R_k(p_3)}{p_1p_2p_3} + \sum_{\substack{p_5<\ldots<p_1<z\\ p_3<f_2(p_1,p_2)\\ f_4(p_1,\ldots,p_4)\leqslant p_5}}\frac{R_k(p_5)}{p_1p_2p_3p_4p_5} + \ldots + \sum_{\substack{p_{2r+1}<\ldots<p_1<z\\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i})\\ i=1,\ldots,r-1\\ f_{2r}(p_1,\ldots,p_{2r})\leqslant p_{2r+}}}\frac{R_k(p_{2r+1})}{p_1\ldots p_{2r+1}} + \sum_{\substack{p_{2r+1}<\ldots<p_1<z\\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i})\\ i=1,\ldots,r}}\frac{R_k(p_{2r+1})}{p_1\ldots p_{2r+1}},$$

$$(2.10)\qquad T_{r,k}(z) = R_k(z) - \sum_{\substack{p_2<p_1<z\\ f_1(p_1)\leqslant p_2}}\frac{R_k(p_2)}{p_1p_2} - \sum_{\substack{p_4<p_3<p_2<p_1<z\\ p_2<f_1(p_1)\\ f_3(p_1,p_2,p_3)\leqslant p_4}}\frac{R_kp_4)}{p_1p_2p_3p_4} - \ldots - \sum_{\substack{p_{2r}<\ldots<p_1<z\\ p_{2i}<f_{2i-1}(p_1,\ldots,p_{2i-1})\\ i=1,\ldots,r-1\\ f_{2r-1}(p_1,\ldots,p_{2r-1})\leqslant p_{2r}}}\frac{R_k(p_{2r})}{p_1\ldots p_{2r}} \sum_{\substack{p_{2r}<\ldots<p_1<z\\ p_{2i}<f_{2i-1}(p_1,\ldots,p_{2i-1})\\ i=1,\ldots,r}}\frac{R_k(p_{2r})}{p_1\ldots p_{2r}}$$

where $p_i \nmid k$, $1 \leqslant i \leqslant 2r+1$.

Then

$$G_{r,k}(z) = S_{r,k}(z), \qquad D_{r,k}(z) = T_{r,k}(z).$$

Proof. We use the obvious identity

$$(2.11)\qquad R_k(z) = 1 - \sum_{p<z,\, p\nmid k}\frac{R_k(p)}{p}.$$

Applying (2.11) to $R_k(p_1)$ and then to $R_k(p_2)$ we get

$$R_k(z) = 1 - \sum_{p_1<z}\frac{1}{p_1} + \sum_{p_2<p_1<z}\frac{R_k(p_2)}{p_1p_2} = 1 - \sum_{p_1<z}\frac{1}{p_1} + \sum_{p_2<p_1<z}\frac{1}{p_1p_2} - \sum_{p_3<p_2<p_1<z}\frac{R_k(p_3)}{p_1p_2p_3},$$

which imples Lemma 2 for $r = 1$.

Assuming $G_{r,k}(z) = S_{r,k}(z)$ we get

$$S_{r+1,k}(z) = S_{r,k}(z) + \sum_{\substack{p_{2r+3}<\ldots<p_1<z\\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i})\\ i=1,\ldots,r\\ f_{2r+2}(p_1,\ldots,p_{2r+2})\leqslant p_{2r+3}}}\frac{R_k(p_{2r+1})}{p_1\ldots p_{2r+1}} + \sum_{\substack{p_{2r+3}<\ldots<p_1<z\\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i})\\ i=1,\ldots,r+1}}\frac{R_k(p_{2r+3})}{p_1\ldots p_{2_{r+3}}} \sum_{\substack{p_{2r+1}<\ldots<p_1<z\\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i})\\ i=1,\ldots,r}}\frac{R_k(p_{2r+1})}{p_1\ldots p_{2r+1}}$$

$$= G_{r,k}(z) + \sum_{\substack{p_{2r+3}<\ldots<p_1<z\\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i})\\ i=1,\ldots,r}}\frac{R_k(p_{2r+3})}{p_1\ldots p_{2r+3}} - \sum_{\substack{p_{2r+1}<\ldots<p_1<z\\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i})\\ i=1,\ldots,r}}\frac{1 - \sum_{p_{2r+2}<p_{2r+1}}\frac{1}{p_{2r+2}} + \sum_{p_{2r+3}<p_{2r+2}<p_{2r+1}}\frac{R_k(p_{2r+3})}{p_{2r+2}p_{2r+3}}}{p_1p_2p_3\ldots p_{2r+1}}$$

$$= G_{r,k}(z) - \sum_{\substack{p_{2r+1}<\ldots<p_1<z \\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i}) \\ i=1,\ldots,r}} \frac{1}{p_1\cdots p_{2r+1}} + \sum_{\substack{p_{2r+2}<\ldots<p_1<z \\ p_{2i+1}<f_{2i}(p_1,\ldots,p_{2i}) \\ i=1,\ldots,r}} \frac{1}{p_1\cdots p_{2r+2}}$$

$$= G_{r+1,k}(z),$$

where $p_i \nmid k$, $1 \leqslant i \leqslant 2r+3$ and the identity $G_{r,k}(z) = S_{r,k}(z)$ follows for all r by induction. The proof that $T_{r,k}(z) = D_{r,k}(z)$ is analogous.

For $a \leqslant b$ we have

$$R_k(a) = \frac{R_k(b)}{R(b)} R(a) \prod_{\substack{a\leqslant p<b \\ p\nmid k}} \left(1-\frac{1}{p}\right) \leqslant \frac{R_k(b)}{R(b)} R(a),$$

thus

$$S_{r,k}(z) \leqslant \frac{R_k(z)}{R(z)} S_{r,1}(z), \tag{2.12}$$

$$T_{r,k}(z) \geqslant \frac{R_k(z)}{R(z)} T_{r,1}(z). \tag{2.13}$$

Let

$$f_l(x_1, \ldots, x_l) = \left(\frac{y}{x_1\ldots x_l}\right)^{1/3}, \quad x_i > 0,\ 1 \leqslant i \leqslant l.$$

If the meaning of p_i is clear from the context we shall write

$$y_i = \frac{y}{p_1\cdots p_i}, \quad \text{whence} \quad y_{i+1} = \frac{y_i}{p_{i+1}}, \quad f_i(p_1, \ldots, p_i) = y_i^{1/3}.$$

We set further

$$d_{2,y}(s) = \sum_{y_1^{1/3}\leqslant p_2<p_1<y^{1/s}} \frac{R(p_2)}{p_1p_2} \quad \text{for} \quad s \geqslant 2,$$

$$d_{2n+2,y}(s) = \sum_{\substack{y_{2n+1}^{1/3}\leqslant p_{2n+2}<\ldots<p_1<y^{1/s} \\ p_{2i}<y_{2i-1}^{1/3},\, i=1,\ldots,n}} \frac{R(p_{2n+2})}{p_1\cdots p_{2n+2}} \quad \text{for} \quad s \geqslant 2,$$

$$d_{2n+1,y}(s) = \begin{cases} \displaystyle\sum_{\substack{y_{2n}^{1/3}\leqslant p_{2n+1}<\ldots<p_1<y^{1/s} \\ p_{2i+1}<y_{2i}^{1/3},\, i=1,\ldots,n-1}} \frac{R(p_{2n+1})}{p_1\cdots p_{2n+1}} & \text{for} \quad s \geqslant 3, \\ d_{2n+1,y}(3) & \text{for} \quad 1 \leqslant s \leqslant 3, \end{cases}$$

$n = 1, \ldots, r$.

The empty sums are 0. It follows

$$S_{r,1}(y^{1/s}) = R(y^{1/s}) + \sum_{i=1}^{r} d_{2i+1,y}(s) + \sum_{\substack{p_{2r+1}<\ldots<p_1<y^{1/s} \\ p_{2i+1}<y_{2i}^{1/3},\, i=1,\ldots,r}} \frac{R(p_{2r+1})}{p_1\cdots p_{2r+1}}, \tag{2.14}$$

$$T_{r,1}(y^{1/s}) = R(y^{1/s}) - \sum_{i=1}^{r} d_{2i,y}(s) - \sum_{\substack{p_{2r}<\ldots<p_1<y^{1/s} \\ p_{2i}<y_{2i-1}^{1/3},\, i=1,\ldots,r}} \frac{R(p_{2r})}{p_1\cdots p_{2r}}. \tag{2.15}$$

§ 3. Functions $f(s)$, $F(s)$, $\mathfrak{f}(s)$, $\mathfrak{F}(s)$. We set

$$g_2(s) = \begin{cases} 3\log\dfrac{3}{s-1} + s - 4 & \text{if} \quad 2 \leqslant s \leqslant 4, \\ 0 & \text{if} \quad s \geqslant 4, \end{cases} \tag{3.1}$$

$$g_{2n+1}(s) = \begin{cases} \displaystyle\int_s^{3+2n} \frac{g_{2n}(t-1)}{t-1}\,dt & \text{if} \quad 3 \leqslant s, \\ g_{2n+1}(3) & \text{if} \quad 1 \leqslant s \leqslant 3, \end{cases} \tag{3.2}$$

$$g_{2n+2}(s) = \int_s^{4+2n} \frac{g_{2n+1}(t-1)}{t-1}\,dt \quad \text{if} \quad s \geqslant 2. \tag{3.3}$$

Remark. The upper limits of integration can be replaced by ∞ since $g_i(s) = 0$ for $s \geqslant 2+i$. The function $g_2(s)$ is of class C^1, since $\lim\limits_{s\to 4+} g_2'(s) = 0$,

$$\lim_{s\to 4-} g_2'(s) = \lim_{s\to 4-}\left(1 - \frac{3}{s-1}\right) = 0.$$

The function $g_{2n+1}(s)$ for $s \geqslant 3$ and $g_{2n+2}(s)$ for $s \geqslant 2$ are clearly of class C^1. The functions $g_i(s)$ are continuous, nonnegative, decreasing.

The following two lemmata serve to estimate $g_i(s)$ from above.

Lemma 3. *Let* $\psi(s) = s^{-1}e^{s-3}\left(1+3\log\dfrac{3}{s-1}\right)$, $2 \leqslant s \leqslant 4$. *Then*

$$\psi(s) \leqslant \psi(2) = 2e^{-1}(1+3\log 3).$$

Proof. Let $\varphi(s) = \dfrac{3s}{(s-1)^2} - \left(1+3\log\dfrac{3}{s-1}\right)$, $2 \leqslant s \leqslant 4$. Then $\varphi'(s) = 3s(s-3)/(s-1)^3$, thus $\varphi(s)$ has minimum for $s = 3$. It follows that $\varphi(s) \geqslant \varphi(3) = 9/4 - (1+3\log(3/2)) > 0$ and $e^s\psi'(s) = -\varphi(s)e^s(s-1)s^{-2} < 0$, hence $\psi(s)$ is decreasing.

Lemma 4. *Let* $a = \sup\limits_{2\leqslant s\leqslant 4} g_2(s)s^{-1}e^s$, $A = \dfrac{e}{3}\psi(2) = \dfrac{1+3\log 3}{6} < 0.72$.

Then

$$g_{2n+1}(s) \leqslant a\frac{e}{3}A^{n-1}se^{-s} \quad \text{for} \quad s \geqslant 1,\ n \geqslant 1, \tag{3.4}$$

$$g_{2n+2}(s) \leqslant aA^{n}se^{-s} \quad \text{for} \quad s \geqslant 2,\ n \geqslant 0. \tag{3.5}$$

Proof. For $n = 0$, (3.5) follows from the definition of a. Assume (3.5) for $n = m-1$. If $s \geqslant 3$, then

$$g_{2m+1}(s) = \int_s^\infty \frac{g_{2m}(t-1)}{t-1}\,dt \leqslant aA^{m-1}\int_s^\infty e^{-(t-1)}dt$$
$$= aA^{m-1}e^{-(s-1)} \leqslant a\frac{e}{3}A^{m-1}se^{-s},$$

if $1 \leqslant s \leqslant 3$, then

$$g_{2m+1}(s) = g_{2m+1}(3) \leqslant a\frac{e}{3}A^{m-1}se^{-s},$$

i.e. (3.4) holds for $n = m$.

If $s \geqslant 4$, then

$$g_{2m+2}(s) = \int_s^\infty \frac{g_{2m+1}(t-1)}{t-1}\,dt \leqslant a\frac{e}{3}A^{m-1}\int_s^\infty e^{-(t-1)}dt$$
$$\leqslant aA^{m-1}\frac{e^2}{12}se^{-s} \leqslant aA^m se^{-s},$$

if $2 \leqslant s \leqslant 4$, then

$$g_{2m+2}(s) = g_{2m+2}(4) + g_{2m+1}(3)\int_s^4 \frac{dt}{t-1} \leqslant a\frac{e}{3}A^{m-1}\left(e^{-3}+3e^{-3}\log\frac{3}{s-1}\right)$$
$$= a\frac{e}{3}A^{m-1}\psi(s)se^{-s} \leqslant aA^m se^{-s},$$

i.e. (3.5) holds for $n = m$. It follows by induction that (3.4) holds for all $n \geqslant 1$ and $s \geqslant 1$, (3.5) holds for all $n \geqslant 0$ and $s \geqslant 2$.

Corollary. *The series*

$$f(s) = \sum_{i=1}^{\infty} g_{2i}(s),\quad s \geqslant 2, \qquad F(s) = \sum_{i=1}^{\infty} g_{2i+1}(s),\quad s \geqslant 1$$

are convergent and

$$0 < f(s) = O(se^{-s}) = F(s) > 0, \tag{**}$$

$f(s)$ is of class C^1, $F(s)$ is of class C^1 for $s \geqslant 3$. It follows from the definition of $f(s)$ and $F(s)$ that

$$f(s) = \int_s^\infty \frac{F(t-1)}{t-1}\,dt + g_2(s) \quad \text{for} \quad s \geqslant 2, \tag{3.6}$$

$$F(s) = \begin{cases} \displaystyle\int_s^\infty \frac{f(t-1)}{t-1}\,dt & \text{for} \quad s \geqslant 3, \\ F(3) & \text{for} \quad 1 \leqslant s \leqslant 3. \end{cases} \tag{3.7}$$

The computation of $f(s)$ and $F(s)$ in the intervals $\langle 2, 4\rangle$, $\langle 1, 3\rangle$ is thus reduced to the determination of $f(2)$, $F(3)$. We shall show later that the equations (3.6), (3.7) and the condition (**) determine $f(s)$ and $F(s)$ uniquely for all s. The difficulty of the problem lies in the fact that $f(s)$ and $F(s)$ are subject to linear differential equations with shifted argument and with a boundary condition in infinity (and not in an interval of suitable length).

Lemma 5.

$$f(s) = s - 2e^\gamma \log(s-1) \quad \text{for} \quad 2 \leqslant s \leqslant 4,$$

$$F(s) = s - 6 + 2e^\gamma\left(1 - \int_1^{s-2} \frac{\log x}{x+1}\,dx\right) \quad \text{for} \quad 3 \leqslant s \leqslant 5.$$

Proof. Set $M(s) = f(s) + F(s)$, $W(s) = f(s) - F(s)$ for $s \geqslant 3$. $M(s)$, $W(s)$ are clearly of class C^1. It follows from (3.6), (3.7) that

$$M'(s) = -\frac{M(s-1)}{s-1}, \qquad W'(s) = \frac{W(s-1)}{s-1} \quad \text{for} \quad s \geqslant 4, \tag{3.8}$$

$$f(s) = f(4) + \big(F(3)+3\big)\log\frac{3}{s-1} + s - 4 \quad \text{for} \quad 2 \leqslant s \leqslant 4. \tag{3.9}$$

Hence

$$M(s) = f(4) + \big(F(3)+3\big)\log\frac{3}{s-1} + s - 4 + \int_s^\infty \frac{f(t-1)}{t-1}\,dt$$
$$\text{for} \quad 3 \leqslant s \leqslant 4,$$

$$W(s) = f(4) + \big(F(3)+3\big)\log\frac{3}{s-1} + s - 4 - \int_s^\infty \frac{f(t-1)}{t-1}\,dt$$
$$\text{for} \quad 3 \leqslant s \leqslant 4.$$

The equations (3.8) give a continuation of $M(s)$, $W(s)$ on the half-line $s \geqslant 2$. It is enough to put

$$M(s) = -sM'(s+1) = 3 + F(3) - s + f(s) \quad \text{for} \quad 2 \leqslant s \leqslant 3,$$
$$W(s) = sW'(s+1) = -3 - F(3) + s + f(s) \quad \text{for} \quad 2 \leqslant s \leqslant 3.$$

The extended functions $M(s)$, $W(s)$ are continuous. The equations (3.8) are now satisfied for $s>3$. We have for $s>3$

$$\big((s-1)M(s)\big)' = M(s)-M(s-1),$$

thus there exists a constant C such that

$$(s-1)M(s) = \int_{s-1}^{s} M(x)\,dx + C \quad \text{for} \quad s \geqslant 3.$$

Since $M(s) = O(se^{-s})$ (see the condition (**)), it follows $C=0$.

We have thus in particular

$$2M(3) = \int_2^3 M(x)\,dx = 2M(3)-M(2)-\int_2^3 (x-1)M'(x)\,dx$$

$$= 2M(3)-M(2)-\int_2^3 (x-1)\big(f'(x)-1\big)\,dx$$

$$= 2M(3)-M(2)+\int_2^3 \big(F(3)+3\big)\,dx$$

$$= 2M(3)-3-F(3)+2-f(2)+3+F(3),$$

hence

(3.10) $$f(2) = 2.$$

The function $W(s)$ is of class C^1 for $s \geqslant 2$. Indeed,

$$\lim_{s\to 3-} W'(s) = f'(3)+1 = -\frac{F(3)+3}{2}+1+1,$$

$$\lim_{s\to 3+} W'(s) = \frac{W(2)}{2} = 1+\frac{f(2)}{2}-\frac{F(3)+3}{2}.$$

The formula (3.8) gives a continuation $W(s)$ on the half-line $s \geqslant 1$. Indeed, it is enough to put for $1 \leqslant s < 2$

$$W(s) = sW'(s+1) = s\big(f'(s+1)+1\big) = s\left(2-\frac{F(3)+3}{s}\right) = 2s-\big(F(3)+3\big).$$

The extended function $W(s)$ is continuous since $W(2-) = 4-\big(F(3)+3\big) = W(2)$ and it satisfies the equation

$$W'(s) = W(s-1)/(s-1) \quad \text{for} \quad s \geqslant 2.$$

Set $w(s) = \dfrac{2s-W(s)}{s\big(3+F(3)\big)}$. It follows that $w(s)$ is continuous and

$$\begin{cases} w(s) = 1/s & \text{if} \quad 1 \leqslant s \leqslant 2, \\ \big(sw(s)\big)' = w(s-1) & \text{if} \quad s \geqslant 2, \end{cases}$$

hence $w(\infty) = e^{-\gamma}$ (see [4], p. 225). Since $W(s) = O(se^{-s})$ we have $W(\infty) = 0$ and $F(3) = 2e^{\gamma}-3$. (3.9) and (3.10) imply

$$f(s) = f(s)-f(2)+2 = s-\big(3+F(3)\big)\log(s-1) = s-2e^{\gamma}\log(s-1) \quad \text{for} \quad 2 \leqslant s \leqslant 4,$$

$$F(s) = s-6+2e^{\gamma}\left(1-\int_3^{s-2}\frac{\log x}{x+1}\,dx\right) \quad \text{for} \quad 3 \leqslant s \leqslant 5.$$

Corollary. *$M(s) = 2e^{\gamma}\big(1-\log(s-1)\big)$ for $2 \leqslant s \leqslant 3$.*

Definition:

$$\begin{cases} \mathfrak{W}(s) = \dfrac{1}{s-1}-\log(s-1), & 2 \leqslant s \leqslant 3, \\ \mathfrak{W}'(s) = \dfrac{s}{(s-1)^2}\mathfrak{W}(s-1), & s > 3, \end{cases}$$

$$\begin{cases} \mathfrak{M}(s) = 2+\dfrac{1}{s-1}-\log(s-1), & 2 \leqslant s \leqslant 3, \\ \mathfrak{M}'(s) = -\dfrac{s}{(s-1)^2}\mathfrak{M}(s-1), & s > 3. \end{cases}$$

Functions $\mathfrak{W}(s)$ and $\mathfrak{M}(s)$ are continuous.

We have for $s>3$

$$\left(\frac{\mathfrak{W}(s)}{s}\right)' = \frac{\mathfrak{W}(s-1)}{(s-1)^2}-\frac{\mathfrak{W}(s)}{s^2},$$

thus there exists a constant C such that

$$\frac{\mathfrak{W}(s)}{s} = -\int_{s-1}^{s}\frac{\mathfrak{W}(x)}{x^2}\,dx + C \quad \text{for} \quad s \geqslant 3.$$

Since

$$\frac{\mathfrak{W}(3)}{3} = \frac{\frac{1}{2}-\log 2}{3} = -\int_2^3 \frac{1/(x-1)-\log(x-1)}{x^2}\,dx,$$

it follows $C=0$ and for $s \geqslant 3$ we get

(3.11) $$\mathfrak{W}(s) = -s\int_{s-1}^{s}\frac{\mathfrak{W}(x)}{x^2}\,dx,$$

hence $\mathfrak{W}(s) = O(e^{-s})$.

Let $r(s) = (s-1)^2-\frac{1}{2}$, thus $r(s)/s = r(s+1)s^{-2}$ and

$$\left(\frac{r(s)\,\mathfrak{M}(s)}{s}\right)' = \frac{r(s)}{s}\mathfrak{M}'(s)+\left(\frac{r(s)}{s}\right)'\mathfrak{M}(s)$$
$$= -\frac{r(s)}{(s-1)^2}\mathfrak{M}(s-1)+\frac{r(s+1)}{s^2}\mathfrak{M}(s)\quad \text{for}\quad s>3.$$

Therefore, there exists a constant C such that

$$\frac{r(s)\,\mathfrak{M}(s)}{s} = \int\limits_{s-1}^{s}\frac{r(x+1)\,\mathfrak{M}(x)}{x^2}\,dx+C\quad \text{for}\quad s\geqslant 3.$$

Since

$$\frac{r(3)\,\mathfrak{M}(3)}{3} = \frac{3.5(2.5-\log 2)}{3} = \int\limits_{2}^{3}\frac{x^2-\frac{1}{2}}{x^2}\left(2+\frac{1}{x-1}-\log(x-1)\right)dx,$$

it follows $C=0$, thus for $s\geqslant 3$ we get

$$\frac{(s-1)^2-\frac{1}{2}}{s}\mathfrak{M}(s) = \int\limits_{s-1}^{s}\left(1-\frac{1}{2x^2}\right)\mathfrak{M}(x)\,dx. \tag{3.12}$$

It follows hence that $0<\mathfrak{M}(s) = O(e^{-s})$, thus $\mathfrak{M}(s)$ is decreasing. Set

$$\mathfrak{f}(s) = \frac{\mathfrak{M}(s)+\mathfrak{W}(s)}{2},\qquad \mathfrak{F}(s) = \frac{\mathfrak{M}(s)-\mathfrak{W}(s)}{2},$$

$\mathfrak{F}(s) = 1$ for $1\leqslant s\leqslant 2$. It follows from the definition of $\mathfrak{M}(s)$ and $\mathfrak{W}(s)$ that

$$\mathfrak{f}(s) = 1+\frac{1}{s-1}-\log(s-1)\ \text{ for }\ 2\leqslant s\leqslant 3,\qquad \mathfrak{F}(s) = 1\ \text{ for }\ 1\leqslant s\leqslant 3.$$

For $s>3$ we have

$$\mathfrak{F}'(s) = \frac{\mathfrak{M}'(s)-\mathfrak{W}'(s)}{2} = -\frac{s\mathfrak{f}(s-1)}{(s-1)^2},$$

thus in view of $\mathfrak{F}(s) = O(e^{-s})$

$$\mathfrak{F}(s) = \int\limits_{s}^{\infty}\frac{\mathfrak{f}(t-1)t}{(t-1)^2}\,dt.$$

Analogously

$$\mathfrak{f}(s) = \int\limits_{s}^{\infty}\frac{\mathfrak{F}(t-1)t}{(t-1)^2}\,dt\quad \text{for}\quad s\geqslant 3.$$

For $2\leqslant s\leqslant 3$ we have

$$\int\limits_{s}^{\infty}\frac{\mathfrak{F}(t-1)t}{(t-1)^2}\,dt = \mathfrak{f}(3)+\mathfrak{F}(3)\int\limits_{s}^{3}\frac{t}{(t-1)^2}\,dt = 1+\frac{1}{s-1}-\log(s-1) = \mathfrak{f}(s).$$

Eventually

$$\mathfrak{F}(s) = \begin{cases}\displaystyle\int\limits_{s}^{\infty}\frac{\mathfrak{f}(t-1)t}{(t-1)^2}\,dt, & s\geqslant 3,\\ 1, & 1\leqslant s\leqslant 3,\end{cases} \tag{3.13}$$

$$\mathfrak{f}(s) = \int\limits_{s}^{\infty}\frac{\mathfrak{F}(t-1)t}{(t-1)^2}\,dt,\qquad s\geqslant 2. \tag{3.14}$$

LEMMA 6.

$$|\mathfrak{W}(s)|\leqslant \tfrac{1}{3}\mathfrak{M}(s)\quad \textit{for}\quad s\geqslant 2.$$

Proof. It can easily be verified that

$$\left|\frac{\mathfrak{W}(s)}{\mathfrak{M}(s)}\right|\leqslant\left|\frac{\mathfrak{W}(2)}{\mathfrak{M}(2)}\right| = \frac{1}{3}\quad \text{for}\quad 2\leqslant s\leqslant 3.$$

Suppose that the set $\{s\geqslant 2,\ |\mathfrak{W}(s)|>\frac{1}{3}\mathfrak{M}(s)\}$ is not empty and denote by a its infimum. It follows that $a\geqslant 3$ and $|\mathfrak{W}(a)| = \frac{1}{3}\mathfrak{M}(a)$. Since $x^2-\frac{1}{2}>(a-1)^2-\frac{1}{2}$ for $x>a-1$ and $\mathfrak{M}(x)>0$, we get

$$\left|\frac{\mathfrak{W}(a)}{a}\right|\leqslant\int\limits_{a-1}^{a}\frac{|\mathfrak{W}(x)|}{x^2}\,dx\leqslant\frac{1}{3}\int\limits_{a-1}^{a}\frac{\mathfrak{M}(x)}{x^2}\,dx$$
$$<\frac{1}{3\left((a-1)^2-\frac{1}{2}\right)}\int\limits_{a-1}^{a}\frac{\mathfrak{M}(x)\,(x^2-\frac{1}{2})}{x^2}\,dx = \frac{1}{3}\,\frac{\mathfrak{M}(a)}{a}.$$

The contradiction obtained proves the lemma.

COROLLARY.

$$\tfrac{1}{3}\mathfrak{M}(s)\leqslant\mathfrak{f}(s)\leqslant\tfrac{2}{3}\mathfrak{M}(s)\quad \textit{for}\quad s\geqslant 2,$$
$$\tfrac{1}{3}\mathfrak{M}(s)\leqslant\mathfrak{F}(s)\leqslant\tfrac{2}{3}\mathfrak{M}(s)\quad \textit{for}\quad s\geqslant 2.$$

Remark. The functions $\mathfrak{F}(s)$, $\mathfrak{f}(s)$ are continuous, positive, decreasing.

Let

$$f_r(s) = \begin{cases}f(s) & \text{for}\quad r\ \text{even},\\ F(s) & \text{for}\quad r\ \text{odd},\end{cases}$$

$$G_r(s) = \begin{cases}\mathfrak{f}(s) & \text{for}\quad r\ \text{even},\\ \mathfrak{F}(s) & \text{for}\quad r\ \text{odd}.\end{cases}$$

Lemma 7.
$$f_r(s) < M(s) \leqslant 4\mathfrak{M}(s) \quad \textit{for} \quad s \geqslant 2.$$

Proof.
$$f_r(s) < \sum_{k=2}^{\infty} g_k(s) = \begin{cases} M(s) & \text{for} \quad s \geqslant 3, \\ f(s)+F(3) \leqslant 3+F(3)-s+f(s) = M(s) & \text{for} \quad 2 \leqslant s \leqslant 3. \end{cases}$$

It can easily by verified that for $2 \leqslant s \leqslant 3$
$$\frac{M(s)}{\mathfrak{M}(s)} = \frac{2e^{\gamma}(1-\log(s-1))}{2+\frac{1}{s-1}-\log(s-1)} \leqslant 4.$$

Suppose that the set $\{s;\, M(s) > 4\mathfrak{M}(s)\}$ is not empty and denote by a its infimum. It follows that $a \geqslant 3$ and $M(a) = 4\mathfrak{M}(a)$. Since $1-\frac{1}{2x^2} > 1-\frac{1}{2(a-1)^2}$, for $x > a-1$ and $\mathfrak{M}(x) > 0$ we get
$$(a-1)\,M(a) = \int_{a-1}^{a} M(x)\,dx \leqslant 4\int_{a-1}^{a}\mathfrak{M}(x)\,dx$$
$$< \frac{4}{1-\frac{1}{2(a-1)^2}} \int_{a-1}^{a} \mathfrak{M}(x)\left(1-\frac{1}{2x^2}\right)dx = 4(a-1)\,\mathfrak{M}(a).$$

The contradiction obtained proves the lemma.

Lemma 8.
$$\frac{\mathfrak{M}(s-1)}{\mathfrak{M}(s)} < 4s^2 \quad \textit{for} \quad s \geqslant 3.$$

Proof. For $s \geqslant 3$
$$\mathfrak{M}(s) = \frac{s}{(s-1)^2-\frac{1}{2}} \int_{s-1}^{s} \mathfrak{M}(x)\left(1-\frac{1}{2x^2}\right)dx$$
$$> \frac{s}{(s-1)^2-\frac{1}{2}} \int_{s-1}^{s-\frac{1}{2}} \mathfrak{M}(x)\left(1-\frac{1}{2x^2}\right)dx \geqslant \frac{s}{2(s-1)^2}\,\mathfrak{M}(s-\tfrac{1}{2}),$$

thus for $s \geqslant 3.5$ we get
$$\mathfrak{M}(s) \geqslant \frac{s}{2(s-1)^2}\,\frac{s-\frac{1}{2}}{2(s-\frac{3}{2})^2}\,\mathfrak{M}(s-1) > \frac{\mathfrak{M}(s-1)}{4s^2}.$$

On the other hand, for $3 \leqslant s \leqslant 3.5$ we have
$$\frac{\mathfrak{M}(s-1)}{4s^2\mathfrak{M}(s)} \leqslant \frac{\mathfrak{M}(2)}{36\mathfrak{M}(3.5)} < 1.$$

Lemma 9. *We have*
$$G_r(s)\exp(s\log s + s\log\log s) \leqslant \exp(c_2 s).$$

Proof. By Corollary to Lemma 6
$$\tfrac{1}{3}\mathfrak{M}(s) \leqslant G_r(s) \leqslant \tfrac{2}{3}\mathfrak{M}(s).$$

For $s \geqslant 3$
$$\mathfrak{M}(s) = \frac{s}{(s-1)^2-\frac{1}{2}} \int_{s-1}^{s}\left(1-\frac{1}{2x^2}\right)\mathfrak{M}(x)\,dx < \frac{1}{s-2}\int_{s-1}^{s}\mathfrak{M}(x)\,dx,$$
$$\mathfrak{M}(s+2) < \frac{1}{s}\int_{s-1}^{s}\mathfrak{M}(x+2)\,dx.$$

Hence
$$\mathfrak{M}(s+2) \leqslant \exp\left\{-s\left(\log s+\log\log s-1+\frac{\log\log s}{\log s}+O\left(\frac{1}{\log s}\right)\right)\right\}$$

(see [3]), which gives the upper bound for $G_r(s)$.

Let
$$\begin{cases} \varrho(s) = 1 & \text{if} \quad 0 < s \leqslant 1, \\ s\varrho'(s) = -\varrho(s-1) & \text{if} \quad s > 1. \end{cases}$$

It follows
$$\varrho(s) = \frac{1}{s}\int_{s-1}^{s}\varrho(x)\,dx \quad \text{for} \quad s > 1.$$

Since $\mathfrak{M}(s) > \frac{1}{s}\int_{s-1}^{s}\mathfrak{M}(x)\,dx$ for $s \geqslant 3$, we have $\varrho(s) = O(\mathfrak{M}(s))$. Since
$$\varrho(s) = \exp\left\{-s\left(\log s+\log\log s-1+\frac{\log\log s}{\log s}+O\left(\frac{s}{\log s}\right)\right)\right\}$$

(see [3]), the proof is complete.

Lemma 10. *We have*
$$\frac{\mathfrak{M}(t)\,t\log t}{\mathfrak{M}(t-1)} = O(1).$$

Proof. Let $\eta(t) = \frac{\mathfrak{M}(t-1)}{\mathfrak{M}(t)}$. We have for $s > 4$
$$\log\eta(t) = \log\frac{\mathfrak{M}(t-1)}{\mathfrak{M}(t)} = \int_{t-1}^{t} -(\log\mathfrak{M}(x))'\,dx = \int_{t-1}^{t} -\frac{\mathfrak{M}'(x)}{\mathfrak{M}(x)}\,dx$$
$$= \int_{t-1}^{t}\frac{\mathfrak{M}(x-1)\,x}{\mathfrak{M}(x)\,(x-1)^2}\,dx = \int_{t-1}^{t}\eta(x)\,\frac{x}{(x-1)^2}\,dx < \frac{\eta(t)}{t-3},$$

thus

$$\frac{\eta(t)}{\log\eta(t)} > t-3, \qquad \frac{t\log t}{\eta(t)} = O(1).$$

Corollary. *We have for* $t > (5+(-1)^r)/2$

$$\frac{G_{r+1}(t)\,(t-1)^2\log t}{G_r(t-1)t} < c_3.$$

Proof follows from Corollary to Lemma 6.

Lemma 11. *Let* $\xi = \dfrac{\log y}{(\log\log 3y)^{11/5}}$. *If* $e^{(120c_3)^{5/2}} = s_0 \leqslant s \leqslant \xi$, *then*

$$\int_s^{\xi}\left(1+\frac{t^2\log^5 t}{\log^2 y}\right)^{5(t-1)}\frac{G_r(t-1)t}{(t-1)^2}\,dt < \left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s} G_{r+1}(s). \tag{3.15}$$

Proof. Let $\tau(t) = \left(1+\dfrac{t^2\log^5 t}{\log^2 y}\right)^{5t} G_{r+1}(t)$. In the interval $\dfrac{5+(-1)^r}{2} < t < \xi$, $\tau(t)$ is differentiable and $G'_{r+1}(t) = -\dfrac{G_r(t-1)t}{(t-1)^2}$.

$$-\tau'(t) = \left(1+\frac{t^2\log^5 t}{\log^2 y}\right)^{5t}\frac{G_r(t-1)t}{(t-1)^2}\times$$

$$\times\left\{1+\frac{G_{r+1}(t)\,(t-1)^2}{G_r(t-1)t}\left[5\log\left(1+\frac{t^2\log^5 t}{\log^2 y}\right)+\frac{5t(2t\log^5 t+5t\log^4 t)}{t^2\log^5 t+\log^2 y}\right]\right\}$$

$$>\left(1-\frac{t^2\log^5 t}{\log^2 y}\right)^{5t}\frac{G_r(t-1)t}{(t-1)^2}\left[1-100\,\frac{G_{r+1}(t)\,(t-1)^2t^2\log^5 t}{G_r(t-1)t\log^2 y}\right]$$

$$>\left(1+\frac{t^2\log^5 t}{\log^2 y}\right)^{5t}\frac{G_r(t-1)t}{(t-1)^2}\left(1-\frac{100c_3t^2\log^4 t}{\log^2 y}\right).$$

Since

$$100c_3\frac{t^2\log^4 t}{\log^2 y} \leqslant 100c_3\frac{\xi^2\log^4\xi}{\log^2 y} < 100c_3\,\xi^{-2/5} < 1,$$

it follows

$$-\tau'(t) > \left(1+\frac{t^2\log^5 t}{\log^2 y}\right)^{5(t-1)}\frac{G_r(t-1)t}{(t-1)^2} \tag{3.16}$$

because

$$\left(1+\frac{5t^2\log^5 t}{\log^2 y}\right)\left(1-\frac{100c_3t^2\log^4 t}{\log^2 y}\right)$$

$$= 1+\frac{(5\log t-100c_3)t^2\log^4 t}{\log^2 y}-\frac{500c_3t^4\log^9 t}{\log^4 y}$$

$$\geqslant 1+\frac{(5\log^{2/5} t-600c_3)t^2\log^{23/5} t}{\log^2 y} \geqslant 1.$$

On integrating the inequality (3.16) in the interval $\langle s, \xi\rangle$ we get inequality (3.15).

Lemma 12. *Let* $\xi = \dfrac{\log y}{(\log\log 3y)^{11/5}}$. *The function*

$$\mu(s) = \left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5(s-1)} G_{r+1}(s)$$

is decreasing in the interval $\langle s_0, \xi\rangle$.

Proof. It follows from (3.16) that $\tau(s)$ is decreasing in the interval in question. Since

$$\mu(s) = \tau(s)\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{-5},$$

$\mu(s)$ is also decreasing.

§ 4. Estimation of $Q_{r,y}(s)$, $A_{r,y}(s)$.

Lemma 13. *Let* $A \geqslant B \geqslant 2$, $b(x)$ *be a nonnegative increasing function in the interval* $B \leqslant x \leqslant A$. *We have*

$$\left|\sum_{B\leqslant p\leqslant A}\frac{b(p)}{p}-\int_B^A\frac{b(x)}{x\log x}\,dx\right| \leqslant c_4 b(A)\exp\left(-\sqrt{\log B}\right).$$

Proof. We can assume that A and B are integers since $b(A)B^{-1} \ll b(A)\exp\left(-\sqrt{\log B}\right)$.

$$\left(1+O\left(\frac{1}{n}\right)\right)\int_n^{n+1}\frac{b(x)}{x\log x}\,dx \geqslant \frac{b(n)}{n\log n} \geqslant \left(1+O\left(\frac{1}{n}\right)\right)\int_{n-1}^{n}\frac{b(x)}{x\log x}\,dx,$$

$$\log\log n-\log\log(n-1) = \frac{1}{n\log n}+O\left(\frac{1}{n^2}\right),$$

$$\exp\left(-\sqrt{\log n}\right)-\exp\left(-\sqrt{\log(n-1)}\right) = O\left(\frac{\exp\left(-\sqrt{\log n}\right)}{n\sqrt{\log n}}\right).$$

We have by the well known formula

$$S(x) = \sum_{p\leqslant x}\frac{1}{p} = \log\log x+c_5+O\left(\exp\left(-\sqrt{\log x}\right)\right).$$

It follows

$$\sum_{B\leqslant p\leqslant A}\frac{b(p)}{p} = \sum_{B\leqslant n\leqslant A}\big(S(n)-S(n-1)\big)b(n)$$

$$= \sum_{B\leqslant n\leqslant A-1} S(n)\big(b(n)-b(n-1)\big)+S(A)b(A)-S(B-1)b(B)$$

$$= \sum_{B\leqslant n\leqslant A-1} (\log\log n + c_5)\big(b(n)-b(n+1)\big) +$$
$$+ (\log\log A + c_5)\, b(A) - \big(\log\log(B-1)+c_5\big)\, b(B) +$$
$$+ O\Big(\sum_{B\leqslant n\leqslant A-1} \exp\big(-\sqrt{\log n}\big)\big(b(n)-b(n+1)\big)\Big) + O\Big(b(A)\exp\big(-\sqrt{\log B}\big)\Big)$$
$$= \sum_{B\leqslant n\leqslant A} b(n)\big(\log\log n - \log\log(n-1)\big) +$$
$$+ O\Big(\exp\big(-\sqrt{\log n}\big) - \exp\big(-\sqrt{\log(n-1)}\big)\Big) + O\Big(b(A)\exp\big(-\sqrt{\log B}\big)\Big)$$
$$= \sum_{B\leqslant n\leqslant A} \frac{b(n)}{n\log n}\Big(1+O\big(\exp(-\sqrt{\log n})\sqrt{\log n}\big)\Big) + O\Big(b(A)\exp\big(-\sqrt{\log B}\big)\Big)$$
$$= \int\limits_B^A \frac{b(x)}{x\log x}\,dx\Big(1+O\big(\exp(-\sqrt{\log x})\sqrt{\log x}\big)\Big) + O\Big(b(A)\exp\big(-\sqrt{\log B}\big)\Big)$$
$$= \int\limits_B^A \frac{b(x)}{x\log x}\,dx + O\Big(b(A)\exp\big(-\sqrt{\log B}\big)\Big).$$

COROLLARY 1. *Let* $\frac{5+(-1)^r}{2} \leqslant \alpha \leqslant \beta \leqslant \frac{1}{2}\log y$. *Then*

$$\sum_{y^{1/\beta}\leqslant p\leqslant y^{1/\alpha}} \frac{f_r\Big(\dfrac{\log y}{\log p}-1\Big)}{p\log\dfrac{y}{p}} \leqslant \int\limits_\beta^\alpha \frac{f_r(t-1)}{t-1}\,dt\log^{-1}y + c_4\frac{f_r(\alpha-1)}{\exp\sqrt{\log y^{1/\beta}}}.$$

Proof. Let

$$b(x) = \frac{f_r\Big(\dfrac{\log y}{\log x}-1\Big)}{\log\dfrac{y}{x}}.$$

Hence

$$\int\limits_{y^{1/\alpha}}^{y^{1/\beta}} \frac{b(x)}{x\log x}\,dx = \int\limits_\alpha^\beta \frac{f_r(t-1)}{t-1}\,dt\log^{-1}y.$$

Clearly $b(x)$ is increasing and for $A = y^{1/\alpha}$, $B = y^{1/\beta}$ we get

$$b(A)\exp\big(-\sqrt{\log B}\big) \leqslant \frac{f_r(\alpha-1)}{\exp\sqrt{\log y^{1/\beta}}}.$$

COROLLARY 2. *Let* $2 \leqslant \alpha \leqslant \beta \leqslant \frac{1}{2}\log y$. *Then*

$$\Bigg|\sum_{y^{1/\beta}\leqslant p\leqslant y^{1/\alpha}} \frac{1}{p\log\dfrac{y}{p}} - \log\frac{\beta-1}{\alpha-1}\log^{-1}y\Bigg| \leqslant c_4\exp-\big(\sqrt{\log y^{1/\beta}}\big).$$

Proof. It is enough to take in Lemma 13 $b(x) = \dfrac{1}{\log\dfrac{y}{x}}$ and to notice that

$$\int\limits_{y^{1/\beta}}^{y^{1/\alpha}} \frac{dx}{x\log x\log\dfrac{y}{x}} = \log\frac{\beta-1}{\alpha-1}\log^{-1}y.$$

COROLLARY 3. *Let* $\xi = \log y/(\log\log 3y)^{11/5} > e^{(120c_3)^{5/2}} = s_0$. *If* $(3+(-1)^r)/2 \leqslant s \leqslant \xi$, $s_1 = \max(s_0, s)$, *then*

$$\sum_{y^{1/\beta}\leqslant p\leqslant y^{1/s}} \left(1+\frac{\log^5\Big(\dfrac{\log y}{\log p}\Big)}{\log^2 p}\right)^{5\left(\frac{\log y}{\log p}-1\right)} \frac{G_r\Big(\dfrac{\log y}{\log p}-1\Big)}{p\log^2\dfrac{y}{p}}$$
$$\leqslant \left(1+\frac{s_1^2\log^5 s_1}{\log^2 y}\right)^{5s_1} \frac{G_{r+1}(s)}{\log^2 y}\left(1+\frac{100c_4\xi^2}{\exp\sqrt{\log y^{1/\xi}}}\right).$$

Proof. Let $s_0 \leqslant s \leqslant \xi$,

$$b(x) = \left(1+\frac{\log^5\Big(\dfrac{\log y}{\log x}\Big)}{\log^2 x}\right)^{5\left(\frac{\log y}{\log x}-1\right)} \frac{G_r\Big(\dfrac{\log y}{\log x}-1\Big)}{\log^2\dfrac{y}{x}}.$$

By Lemma 12, $b(x)$ is decreasing in the interval $y^{1/\xi} \leqslant x \leqslant y^{1/s}$. It follows from Lemmata 11 and 13

$$\sum_{y^{1/\xi}\leqslant p\leqslant y^{1/s}} \left(1+\frac{\log^5\Big(\dfrac{\log y}{\log p}\Big)}{\log^2 p}\right)^{5\left(\frac{\log y}{\log p}-1\right)} \frac{G_r\Big(\dfrac{\log y}{\log p}-1\Big)}{p\log^2\dfrac{y}{p}}$$

$$< \int_{y^{1/\xi}}^{y^{1/s}} \frac{b(x)}{x\log x}\,dx + c_4 \frac{b(y^{1/s})}{\exp\sqrt{\log y^{1/\xi}}}$$

$$= \int_s^\xi \left(1+\frac{t^2\log^5 t}{\log^2 y}\right)^{5(t-1)} \frac{G_r(t-1)t}{(t-1)^2\log^2 y}\,dt +$$

$$+ c_4\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5(s-1)} \frac{G_r(s-1)s}{(s-1)\log^2 y}\exp\left(-\sqrt{\log y^{1/\xi}}\right)$$

$$< \left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\frac{G_{r+1}(s)}{\log^2 y}\left(1+\frac{20c_4\xi^2}{\exp\sqrt{\log y^{1/\xi}}}\right).$$

Let $(3+(-1)^r)/2 \leqslant s\leqslant s_0$

$$\left|\sum_{y^{1/s_0}\leqslant p\leqslant y^{1/s}} \frac{G_r\left(\frac{\log y}{\log p}-1\right)}{p\log^2\frac{y}{p}} - \int_s^{s_0}\frac{G_r(t-1)t}{(t-1)^2}\,dt\log^{-2}y\right|$$

$$\leqslant c_4\frac{sG_r(s-1)}{(s-1)\log^2 y}\exp\left(-\sqrt{\log y^{1/s_0}}\right).$$

$$\sum_{y^{1/\xi}\leqslant p\leqslant y^{1/s}}\left(1+\frac{\log^5\left(\frac{\log y}{\log p}\right)}{\log^2 p}\right)^{5\left(\frac{\log y}{\log p}-1\right)} \frac{G_r\left(\frac{\log y}{\log p}-1\right)}{p\log^2\frac{y}{p}} = \sum_{y^{1/s_0}\leqslant p\leqslant y^{1/s}} + \sum_{y^{1/\xi}\leqslant p< y^{1/s_0}}$$

$$< \left(1+\frac{s_0^2\log^5 s_0}{\log^2 y}\right)^{5s_0}\sum_{y^{1/s_0}\leqslant p\leqslant y^{1/s}}\frac{G_r\left(\frac{\log y}{\log p}-1\right)}{p\log^2\frac{y}{p}} +$$

$$+\left(1+\frac{s_0^2\log^5 s_0}{\log^2 y}\right)^{5s_0}\frac{G_{r+1}(s_0)}{\log^2 y}\left(1+\frac{20c_4\xi^2}{\exp\sqrt{\log y^{1/\xi}}}\right)$$

$$\leqslant \left(1+\frac{s_0^2\log^5 s_0}{\log^2 y}\right)^{5s_0}\frac{1}{\log^2 y}\Bigg[\left(1+\frac{20c_4\xi^2}{\exp\sqrt{\log y^{1/\xi}}}\right)G_{r+1}(s_0)+$$

$$+G_{r+1}(s)-G_{r+1}(s_0)+\frac{s}{s-1}G_r(s-1)c_4\exp\left(-\sqrt{\log y^{1/s_0}}\right)\Bigg]$$

$$\leqslant \left(1+\frac{s_0^2\log^5 s_0}{\log^2 y}\right)^{5s_0}\frac{G_{r+1}(s)}{\log^2 y}\left(1+\frac{100c_4\xi^2}{\exp\sqrt{\log y^{1/\xi}}}\right).$$

LEMMA 14. *Let* $a\geqslant b\geqslant 2$. *Then*

$$\sum_{b\leqslant p\leqslant a}\frac{1}{p\log p} = \frac{1}{\log b}-\frac{1}{\log a}+O\left(\exp\left(-\sqrt{\log b}\right)\right).$$

Proof is analogous to the proof of Lemma 13.

LEMMA 15. *We have*

$$d_{2n+1,y}(s) = \sum_{y^{1/(2n+3)}\leqslant p_1<y^{1/s}} \frac{d_{2n,y_1}\left(\frac{\log y_1}{\log p_1}\right)}{p_1}\quad \text{for}\quad s\geqslant 3,$$

$$d_{2n+2,y}(s) = \sum_{y^{1/(2n+4)}\leqslant p_1\leqslant y^{1/s}}\frac{d_{2n+1,y_1}\left(\frac{\log y_1}{\log p_1}\right)}{p_1}\quad \text{for}\quad s\geqslant 2.$$

Proof. We have for $s\geqslant 3$

$$d_{2n+1,y}(s) = \sum_{\substack{y^{1/(2n+3)}\leqslant p_1\leqslant y^{1/s}\\ y_{2n}^{1/3}\leqslant p_{2n+1}<\dots<p_1=y_1^{(\log p_1/\log y_1)}\\ p_{2i+1}<y_{2i}^{1/3},\, i=1,\dots,n-1}} \frac{R(p_{2n+1})}{p_1\cdots p_{2n+1}}$$

$$= \sum_{y^{1/(2n+3)}\leqslant p_1<y^{1/s}}\frac{d_{2n,y_1}\left(\frac{\log y_1}{\log p_1}\right)}{p_1}.$$

Similarly, for $s\geqslant 2$ we have

$$d_{2n+2,y}(s) = \sum_{\substack{y^{1/(2n+4)}\leqslant p_1<y^{1/s}\\ y_{2n+1}^{1/3}\leqslant p_{2n+2}<\dots<p_2<\min(p_1,\, y_1^{1/3})\\ p_{2i}<y_{2i-1}^{1/3},\, i=1,\dots,n}}\frac{R(p_{2n+2})}{p_1\cdots p_{2n+2}}$$

$$= \sum_{y^{1/(2n+4)}\leqslant p_1<y^{1/s}}\frac{d_{2n+1,y_1}\left(\frac{\log y_1}{\log\min(p_1,\, y_1^{1/3})}\right)}{p_1} = \sum_{y^{1/(2n+4)}\leqslant p_1<y^{1/s}}\frac{d_{2n+1,y_1}\left(\frac{\log y_1}{\log p_1}\right)}{p_1}$$

since

$$d_{2n+1,y_1}\left(\frac{\log y_1}{\log\min(p_1,\, y_1^{1/3})}\right) = d_{2n+1,y_1}\left(\frac{\log y_1}{\log p_1}\right).$$

COROLLARY. *Let*

$$Q_{2n+1,y}(s)=\sum_{k=1}^{n} d_{2k+1,y}(s)\quad \textit{for}\quad s\geqslant 1,$$

$$Q_{2n,y}(s)=\sum_{k=1}^{n} d_{2k,y}(s)\quad \textit{for}\quad s\geqslant 2.$$

Then

$$Q_{2n+1,y}(s)=\sum_{y^{1/(2n+1)}\leqslant p<y^{1/s}}\frac{Q_{2n,y/p}\left(\frac{\log y}{\log p}-1\right)}{p}\quad \textit{for}\quad s\geqslant 3,$$

$$Q_{2n+2,y}(s)=\sum_{y^{1/(2n+4)}\leqslant p<y^{1/s}}\frac{Q_{2n+1,y/p}\left(\frac{\log y}{\log p}-1\right)}{p}+d_{2,y}(s)\quad \textit{for}\quad s\geqslant 2.$$

LEMMA 16. *If* $y>1$, *we have*

$$e^{\gamma}d_{2,y}(s)<g_2(s)\log^{-1}y+c_6\exp\left(-\sqrt{\frac{\log y}{6}}\right).$$

Proof. Using the well known estimate

$$R(z)=e^{-\gamma}\log^{-1}z+O\big(\exp(-\sqrt{\log z})\big),$$

Lemma 14 and Corollary 2 to Lemma 13, we obtain for $y>1$, $2\leqslant s\leqslant 4$

$$\begin{aligned}
d_{2,y}(s) &= \sum_{y_1^{1/3}\leqslant p_2<p_1<y^{1/s}}\frac{R(p_2)}{p_1p_2}\\
&= \sum_{y_1^{1/3}\leqslant p_2<p_1<y^{1/s}}\frac{1}{p_1}\left(\frac{e^{-\gamma}}{p_2\log p_2}+O\big(\exp(-\sqrt{\log p_2})\big)\right)\\
&= e^{-\gamma}\sum_{y^{1/4}\leqslant p_1<y^{1/s}}\left\{\frac{1}{p_1}\left(\frac{3}{\log y_1}-\frac{1}{\log p_1}\right)+O\left(\frac{1}{p_1}\exp\left(-\sqrt{\frac{\log y}{6}}\right)\right)\right\}\\
&= e^{-\gamma}\left(3\log\frac{3}{s-1}+s-4\right)\log^{-1}y+O\left(\exp\left(-\sqrt{\frac{\log y}{6}}\right)\right).
\end{aligned}$$

THEOREM 4. *If* $y>1$, $\xi=\dfrac{\log y}{(\log\log 3y)^{11/5}}\geqslant s\geqslant\dfrac{3+(-1)^r}{2}$, *we have*

$$e^{\gamma}Q_{r,y}(s)<\frac{f_r(s)}{\log y}+c_7\frac{r}{r+1}\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\frac{G_r(s)}{\log^2 y}. \tag{4.1}$$

Proof. We begin by defining the constant c_7. There exists a constant c_8 such that for $y>c_8$

$$200\xi^2\exp\left(-\sqrt{\log y^{1/\xi}}\right)<\log^{-2}y, \tag{4.2}$$

$$c_6\exp\left(-\sqrt{\frac{\log y}{6}}\right)<c_4\frac{G_r(4)}{\log^4 y}, \tag{4.3}$$

$$\exp\left(-\frac{\xi}{2}\log\xi-\xi\log\log\xi-2c_2\xi\right)>\exp(-\xi\log\xi+\xi\log\log\xi+2\xi), \tag{4.4}$$

$$\exp(-\xi\log\xi+2\xi\log\log\xi-3c_2\xi)>\exp(-\xi\log\xi+\xi\log\log\xi+2\xi), \tag{4.5}$$

$$\left(1-\frac{\log\log^2 y}{\log^2 y}\right)\left(1+\frac{4c_4}{\log^2 y}\right)\left(1+\frac{s_0^2\log^5 s_0}{\log^2 y}\right)^{5s_0}<1, \tag{4.6}$$

$$\frac{\log^2 y}{\sqrt{y}}<\exp(-\xi\log\xi-\xi\log\log\xi-2c_2\xi). \tag{4.7}$$

We set

$$c_7=\frac{(1+\log c_8)^3}{\mathfrak{M}(2\log c_8)}+10c_6\sup_{y\geqslant 1}\left\{\log^2 y\exp\left(-\sqrt{\frac{\log y}{6}}\right)\right\}+c_4.$$

If $y<c_8$ and $Q_{r,y}(s)>0$, then $2\leqslant y^{1/s}$, $s\leqslant 2\log y<2\log c_8$ and

$$\begin{aligned}
Q_{r,y}(s) &< \sum_{d\leqslant y}\frac{1}{d}\leqslant\sum_{d\leqslant c_8}\frac{1}{d}<1+\int_1^{c_8}\frac{du}{u}=1+\log c_8\\
&< \frac{f_r(s)}{\log y}+c_7\frac{r}{r+1}\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\frac{G_r(s)}{\log^2 y}.
\end{aligned} \tag{4.8}$$

In virtue of Lemma 16 and (4.3) we get (4.1) for $r=2$, and all $y>1$, and for $r=2,3,\ldots$, and each $y<c_8$. The further proof will proceed by induction with respect to r. For $r=2$, (4.1) is true. Assume it is true for $r=m$. For $y<c_8$, $r=m+1$, (4.1) has just been proved. Let $y\geqslant c_8$. If $m+1>\dfrac{2\log y}{\log\log y}$, then $p_1\ldots p_{m-1}>y$ for $y\geqslant c_8$ and any distinct primes $p_1,\ldots,p_{m-1}$, thus $d_{m+1,y}(s)=d_{m-1,y}(s)=0$ and $Q_{m+1,y}(s)=Q_{m-1,y}(s)$. In this case, (4.1) for $r=m+1$ follows from the inductive assumption.

It remains to consider the case $y \geqslant c_8$, $m+1 \leqslant \dfrac{2\log y}{\log\log y}$.

In virtue of Lemmata 13, 15, 8, 16, Corollary 3 to Lemma 13 and (4.3) we have for $s \geqslant (5+(-1)^m)/2$

$$
\begin{aligned}
& e^{\gamma} Q_{m+1,y}(s) \\
&= e^{\gamma} Q_{m+1,y}(\xi-1) + \sum_{y^{1/(\xi-1)} \leqslant p < y^{1/s}} \frac{e^{\gamma} Q_{m,y/p}\left(\dfrac{\log y}{\log p}-1\right)}{p} + \frac{1+(-1)^{m+1}}{2} e^{\gamma} d_{2,y}(s) \\
&< e^{\gamma} Q_{m+1,y}(\xi-1) + \sum_{y^{1/(\xi-1)} \leqslant p < y^{1/s}} \frac{f_m\left(\dfrac{\log y}{\log p}-1\right)}{p\log\dfrac{y}{p}} + \\
&\quad + c_7 \frac{m}{m+1} \sum_{y^{1/(\xi-1)} \leqslant p < y^{1/s}} \left(1+\frac{\log^5\left(\dfrac{\log y}{\log p}-1\right)}{\log^2 y}\right)^{5\left(\frac{\log y}{\log p}-1\right)} \frac{G_m\left(\dfrac{\log y}{\log p}-1\right)}{p\log^2\dfrac{y}{p}} + \\
&\quad + \frac{1+(-1)^{m+1}}{2} e^{\gamma} d_{2,y}(s) \\
&= e^{\gamma} Q_{m+1,y}(\xi-1) + \int_s^{\xi} \frac{f_m(t-1)}{t-1}\, dt \log^{-1} y + \\
&\quad + c_7 \frac{m}{m+1}\left(1+\frac{s_1^2\log^5 s_1}{\log^2 y}\right)^{5s_1}\left(1+100c_4\xi^2\exp\left(-\sqrt{\log y^{1/\xi}}\right)\right)\frac{G_{m+1}(s)}{\log^2 y} + \\
&\quad + c_4 \frac{f_m(s-1)}{\exp\sqrt{\log y^{1/\xi}}} + \frac{1+(-1)^{m+1}}{2} e^{\gamma} d_{2,y}(s) \\
&< \frac{f_{m+1}(s)}{\log y} + \frac{1-\operatorname{sgn}(s-4)}{2} c_6 \exp\left(-\sqrt{\frac{\log y}{6}}\right) + \\
&\quad + c_7 \frac{m}{m+1}\frac{G_{m+1}(s)}{\log^2 y}\left(1+\frac{s_1^2\log^5 s_1}{\log^2 y}\right)^{5s_1}\left(1+200c_4\xi^2\exp\left(-\sqrt{\log y^{1/\xi}}\right)\right) + \\
&\quad + e^{\gamma} Q_{m+1,y}(\xi-1).
\end{aligned}
$$

$$
\begin{aligned}
Q_{m+1,y}(\xi-1) &\leqslant \sum_{2 \leqslant k \leqslant m+1} d_{k,y}(\xi-1) = \sum_{\xi-3 \leqslant k \leqslant m+1} d_{k,y}(\xi-1) < \sum_{k > \xi-3} \frac{\left(\sum_{p<y} 1/p\right)^k}{k!} \\
&< \left(\frac{5\log\log y}{\xi}\right)^{\xi} < \exp\{-\xi\log\xi + \xi\log\log\xi + 2\xi\}.
\end{aligned}
$$

Therefore, if $s < \xi/2$, we get from (4.4) and Lemma 9 that

$$
\begin{aligned}
(4.9)\qquad \frac{G_{m+1}(s)}{\log^4 y} &> \frac{\exp\{-s\log s - s\log\log s - c_2 s\}}{\log^4 y} \\
&> \exp\{-\tfrac{1}{2}\xi\log\xi - \xi\log\log\xi - 2c_2\xi\} \\
&> \exp\{-\xi\log\xi + \xi\log\log\xi + 2\xi\} > Q_{m+1,y}(\xi-1),
\end{aligned}
$$

if, on the other hand, $s \geqslant \xi/2$, we obtain from (4.5) and Lemma 9

$$
\begin{aligned}
(4.10)\qquad \left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s} \frac{G_{m+1}(s)}{\log^4 y} &> (\log\log y)^{3s}\exp\{-s\log s - s\log\log s - 2c_2 s\} \\
&> \exp\{-s\log s + 2s\log\log s - 3c_2 s\} \\
&> \exp\{-\xi\log\xi + 2\xi\log\log\xi - 3c_2\xi\} > Q_{m+1,y}(\xi-1).
\end{aligned}
$$

It follows now from (4.2), (4.9), (4.10) that for $s \geqslant (5+(-1)^r)/2$

$$
e^{\gamma} Q_{m+1,y}(s) < \frac{f_{m+1}(s)}{\log y} + c_7\frac{m}{m+1}\left(1+\frac{s_1^2\log^5 s_1}{\log^2 y}\right)^{5s_1}\left(1+\frac{3c_4}{\log^2 y}\right)\frac{G_{m+1}(s)}{\log^2 y}.
$$

If m is even and $1 \leqslant s \leqslant 3$, the last inequality is also true since its both sides are constant in the interval $1 \leqslant s \leqslant 3$. Since by (4.6)

$$
\begin{aligned}
& \frac{m}{m+1}\left(1+\frac{s_1^2\log^5 s_1}{\log^2 y}\right)^{5s_1}\left(1+\frac{3c_4}{\log^2 y}\right) \\
&< \frac{m+1}{m+2}\left(1-\frac{1}{(m+1)^2}\right)\left(1+\frac{s_0^2\log^5 s_0}{\log^2 y}\right)^{5s_0}\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\left(1+\frac{3c_4}{\log^2 y}\right) \\
&< \frac{m+1}{m+2}\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s},
\end{aligned}
$$

the proof of (4.1) is complete.

Definition.

$$
\begin{aligned}
a_{l,y}(s) &= \sum_{\substack{d=p_1\ldots p_l \\ p_l<\ldots<p_1<y^{1/s},\, p_{2i+1}<v_{2i}^{1/3} \\ i=1,\ldots,[l/2-1]}} 1 \qquad \text{for} \quad s \geqslant 3,\ l \geqslant 0, \\
a_{l,y}(s) &= a_{l,y}(3) \qquad\qquad \text{for} \quad 1 \leqslant s \leqslant 3,\ l \geqslant 0, \\
b_{l,y}(s) &= \sum_{\substack{d=p_1\ldots p_l \\ p_l<\ldots<p_1<y^{1/s},\, p_{2i}<v_{2i-1}^{1/3} \\ i=1,\ldots,[l/2]}} 1 \qquad \text{for} \quad s \geqslant 2,\ l \geqslant 0,
\end{aligned}
$$

where for $l = 0$, $d = p_1 \ldots p_l$ means $d = 1$.

Let

$$A_{2r,y}(s) = \sum_{l=0}^{2r} a_{l,y}(s), \qquad A_{2r+1,y}(s) = \sum_{l=0}^{2r+1} b_{l,y}(s).$$

LEMMA 17. *We have*

$$A_{2r+1,y}(s) = \sum_{p<y^{1/s}} A_{2r,y/p}\left(\frac{\log y}{\log p}-1\right)+1 \quad \textit{for} \quad s \geqslant 2,$$

$$A_{2r+2,y}(s) = \sum_{p<y^{1/s}} A_{2r+1,y/p}\left(\frac{\log y}{\log p}-1\right) \quad \textit{for} \quad s \geqslant 3.$$

Proof is analogous to the proof of Lemma 15.

THEOREM 5. *If* $y>1$, $\frac{3+(-1)^r}{2} \leqslant s \leqslant \xi$, *then*

$$A_{r,y}(s) < c_7 \frac{r}{r+1}\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s} \frac{G_{r+1}(s)}{\log^2 y}\, y. \tag{4.11}$$

Proof. Since $c_7 > \frac{6\log^2 c_8}{\mathfrak{M}(\log c_8)}$, it follows for $y \leqslant c_8$, $\frac{3+(-1)^r}{2} \leqslant s \leqslant \xi$ and $r = 1, 2, \ldots$

$$A_{r,y}(s) < y < \frac{y}{2\log^2 y}\, c_7 G_{r+1}(s).$$

The inequality (4.10) is satisfied in this case. The further proof will proceed by induction with respect to r.

For $r=1$ we have by Lemma 9 and (4.7)

$$A_{1,y}(s) \leqslant A_{1,y}(2) = 1+\sum_{p\leqslant y^{1/2}} 1 < y^{1/2}$$

$$\leqslant \frac{2}{3}\frac{y}{\log^2 y}\exp\{-\xi\log\xi - \xi\log\log\xi - 2c_2\xi\}$$

$$< \frac{2}{3}\frac{y}{\log^2 y} G_2(\xi) \leqslant \frac{2y}{3\log^2 y} G_2(s),$$

thus (4.10) holds.

Assume that (4.11) holds for $r = m$. If $y \leqslant c_8$, (4.11) is satisfied for $r = m+1$. Let $y \geqslant c_8$. If $m+1 > \frac{2\log y}{\log\log y}$ and $p_1, \ldots, p_{m-1}$ are any distinct primes, then $p_1 \ldots p_{m-1} > y$, $A_{m+1,y}(s) = A_{m-1,y}(s)$, thus (4.11) holds for $r = m+1$ by inductive assumption.

It remains to consider as in the proof of Theorem 4 the really important case

$$\frac{3+(-1)^r}{2} \leqslant s \leqslant \xi, \qquad m+1 \leqslant \frac{2\log y}{\log\log y}.$$

We show in the same way as in the proof of Theorem 4 that

$$A_{m+1,y}(s) = A_{m+1,y}(\xi-1) + \sum_{y^{1/(\xi-1)}\leqslant p<y^{1/s}} A_{m,\frac{y}{p}}\left(\frac{\log y}{\log p}-1\right)$$

$$< A_{m+1,y}(\xi-1)+c_7\frac{m}{m+1}\sum_{y^{1/(\xi-1)}\leqslant p<y^{1/s}}\left(1+\frac{\log^5\left(\frac{\log y}{\log p}-1\right)}{\log^2 p}\right)^{5\left(\frac{\log y}{\log p}-1\right)} \times$$

$$\times \frac{G_{m+1}\left(\frac{\log y}{\log p}-1\right)}{p\log^2\frac{y}{p}}$$

$$< A_{m+1,y}(\xi-1)+c_7\frac{m}{m+1}\frac{y}{\log^2 y}\; G_{m+2}(s)(1+3c_4\log^{-2}y)\left(1+\frac{s_1^2\log^5 s_1}{\log^2 y}\right)^{5s_1}.$$

Let $\xi_1 = \xi-\xi/\log\xi$. We have

$$A_{m+1,y}(\xi-1) \leqslant \sum_{1\leqslant k\leqslant m+1}\{a_{k,y}(\xi-1)+b_{k,y}(\xi-1)\} = \sum_{k<\xi_1}+\sum_{\xi_1\leqslant k\leqslant m+1},$$

$$\sum_{k<\xi_1} < 2\sum_{k<\xi_1}\frac{\left(\pi(y^{1/(\xi-1)})\right)^k}{k!} < 6y^{\xi_1/(\xi-1)} < 6y^{1-1/\log 2\xi} < y\exp(-\xi\log^{6/5}\xi+\xi),$$

$$\sum_{\xi_1\leqslant k\leqslant m+1} < 2y\sum_{\substack{r(d)\geqslant\xi_1\\ p|d\Rightarrow p<y}} d^{-1} < 6y\left(\frac{3\log\log y}{\xi_1}\right)^{\xi_1} < y\exp\{-\xi\log\xi+\xi\log\log\xi\}.$$

Hence by (4.8) and (4.9) we get

$$A_{m+1,y}(\xi-1) < y\exp\{-\xi\log\xi+\xi\log\log\xi+2\xi\}$$

$$< \left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\frac{y}{\log^2 y}\, G_{m+2}(s).$$

It follows

$$A_{m+1,y}(s) < c_7\frac{m}{m+1}\frac{y}{\log^2 y}\, G_{m+2}(s)\left(1+\frac{s_1^2\log^5 s_1}{\log^2 y}\right)^{5s_1}\left(1+\frac{4c_4}{\log^2 y}\right)$$

$$< c_7\frac{m+1}{m+2}\frac{y}{\log^2 y}\, G_{m+2}(s)\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}.$$

and the proof is complete.

I am unable to decide whether in Theorem 5 the right hand side can be replaced by $o(y/\log^2 y)$.

§ 5. Proof of Theorems 1, 2, 3.

Proof of Theorem 1. It follows from Theorem 5 that

$$\sum\nolimits_1 (y^{1/s}) = A_{2r,y}(s) \ll y\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\frac{\mathfrak{F}(s)}{\log^2 y},$$

$$\sum\nolimits_2 (y^{1/s}) = A_{2r-1,y}(s) \ll y\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\frac{\mathfrak{f}(s)}{\log^2 y}$$

for $s < \dfrac{\log y}{(\log\log 3y)^{11/5}} = \xi$ and any integer $r > 1$.

In virtue of Mertens formula

$$\frac{1}{R(y^{1/s})} = \frac{e^{\gamma}\log y}{s} + O(1),$$

$$\frac{1}{R_k(y^{1/s})} < \frac{k}{\varphi(k)}\frac{1}{R(y^{1/s})} \ll \frac{\log\log 3k\log y}{s}.$$

Using Corollary to Lemma 6 we get

$$\frac{\sum_i (y^{1/s})}{R_k(y^{1/s})} \ll \left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\mathfrak{M}(s)\frac{\log\log 3k}{\log y} \quad \text{for} \quad i = 1, 2,\ s < \xi.$$

Using Theorem 4, (2.12) and (2.13) we obtain for $r = [\log y]$

$$\frac{S_{r,k}(y^{1/s})}{R_k(y^{1/s})} \leqslant \frac{S_{r,1}(y^{1/s})}{R(y^{1/s})} = 1+\frac{Q_{2r+1,y}(s)}{R(y^{1/s})}$$

$$< 1+\frac{e^{\gamma}Q_{2r+1,y}(s)\log y}{s} + O\big(Q_{2r+1,y}(s)\big)$$

$$< 1+\frac{F(s)}{s}+O\left(\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\frac{F(s)}{\log y}\right)$$

$$= 1+\frac{F(s)}{s}+O\left(\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\frac{\mathfrak{M}(s)}{\log y}\right),$$

since by Lemma 7 and Corollary to Lemma 6 $F(s) \ll \mathfrak{M}(s)$.

Analogously

$$\frac{T_{r,k}(y^{1/s})}{R_k(y^{1/s})} \geqslant \frac{T_{r,1}(y^{1/s})}{R(y^{1/s})} = 1-\frac{Q_{2r,y}(s)}{R(y^{1/s})}$$

$$> 1-\frac{f(s)}{s}+O\left(\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\frac{\mathfrak{M}(s)}{\log y}\right).$$

The above estimates together with (2.7), (2.8) imply Theorem 1.

Proof of Theorem 2. The set $M = \{n+1, \ldots, n+y\}$ satisfies the assumption of Corollary to Theorem 1 since

$$\left||M_d|-\frac{y}{d}\right| = \left|\left[\frac{n+y}{d}\right]-\left[\frac{n}{d}\right]-\frac{y}{d}\right| \leqslant 1.$$

For $k = 1$, $2 \leqslant s \leqslant 4$, the inequality (1.4) implies

$$A_1(M; y^{1/s}) \geqslant yR(y^{1/s})\left(\frac{2e^{\gamma}\log(s-1)}{s}-\frac{c_1}{\log y}\right).$$

Let P_r be the rth prime, $s = 2+\dfrac{c_1}{\log P_r}$, $y = P_r^s = e^{c_1}P_r^2$. We get for $r \geqslant r_0$

$$A_1(M; P_r) > \frac{yR(P_r)}{s}\left(2e^{\gamma}\log\left(1+\frac{c_1}{\log P_r}\right)-\frac{3c_1}{\log P_r}\right) > 0,$$

thus $C_0(r) \ll r^2\log^2 r$ which completes the proof.

Proof of Theorem 3. Let $k = k_1 k_2$, where $k_1 | P(z) = \prod_{p \leqslant z} p$, $(k_2, P(z)) = 1$. Then

$$A_k(M; z) = A_{k_1}(M; z), \qquad R_k(z) = R_{k_1}(z),$$

$$k_1 < [z+1]!, \qquad \log\log 3k_1 \ll \log z,$$

hence

$$A_k(M; y^{1/s}) = yR_k(y^{1/s})\left(1+O\left(\left(\frac{s^2\log^5 s}{\log^2 y}+1\right)^{5s}\mathfrak{M}(s)\right)\right)$$

for $s < \dfrac{\log y}{(\log\log 3y)^{11/5}}$. We have, however

$$\left(1+\frac{s^2\log^5 s}{\log^2 y}\right)^{5s}\mathfrak{M}(s) < \left(1+\frac{1}{\log s}\right)^{5s}\mathfrak{M}(s) < \exp\left\{\frac{5s}{\log s}\right\}\mathfrak{M}(s)$$

$$< \exp\left\{-s\log s-\log\log s+s-\frac{s\log\log s}{\log s}+O(s)\right\} = O\left(\frac{e}{s\log s}\right)^s$$

for $s < \dfrac{\log y}{(\log\log 3y)^6}$, which implies the first part of Theorem 3.

If $\log z < \dfrac{\log y\log\log\log 16y}{\log\log 3y}$, $z = y^{1/s}$, then $s > \dfrac{2\log\log 3y}{\log\log\log 16y}$, thus

$$\left(\frac{e}{s\log s}\right)^s < O\left(\frac{1}{\log y}\right)$$

and the proof is complete.

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Received on 5. 1. 1970 (21)

ACTA ARITHMETICA
XIX (1971)

A Kuzmin theorem for a class of number theoretic endomorphisms

by

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Recently several papers ([3], [4], [5], [6], [7]) have been concerned with generalizations of a 1928 theorem of Kuzmin. His result gives a rate of $e^{-\lambda\sqrt{n}}$ for the convergence of the iteration of an arbitrary function to the invariant measure for the continued fraction. The present paper gives a generalized Kuzmin theorem for a class of multi-dimensional F-expansions which includes the n-dimensional continued fraction. An earlier paper ([6]) presented such a theorem with a rate of $\left(e^{-\lambda\sqrt{\nu}}+\sigma(\sqrt{\nu})\right)$. Our present theorem improves the rate to $\sigma(\nu)$.

Our F-expansions were first considered in [6], and we include a short summary of notation and assumptions here. Let A be a fixed convex subset of R^n. Suppose F is a one-to-one continuous map of A onto $(0,1)^n$. We assume $J_F(\cdot)$, the Jacobian of F, exists, the components of F have continuous first order partial derivatives, and $J_F(x)\neq 0$ for almost all $x\in A$. Let $D=F^{-1}$, $T(x)=D(x)-[D(x)]$, and $a_\nu(x)=[D(T^{\nu-1}x)]$ (where $[z]=\left([z_1],[z_2],\ldots,[z_n]\right)$). We call $a_\nu(x)$ the *ν-th coordinate of the F-expansion of x*. Letting

$$(0,1)^n_F=\{x\in(0,1)^n\colon\ T^\nu(x)\in(0,1)^n\ \text{for all}\ \nu\geqslant 1\},$$

we impose the assumption $m(0,1)^n_F=1$, where m denotes n-dimensional Lebesgue measure. We will write $F\in\mathscr{F}$ to indicate the satisfaction of these assumptions.

We define the cylinder of order ν generated by a realizable set of coordinates $k_1,k_2,\ldots,k_\nu$ as

$$B_\nu=B_\nu(k_1,k_2,\ldots,k_\nu)=\{x\in(0,1)^n_F\colon\ a_i(x)=k_i,\ i=1,\ldots,\nu\},$$

and the cylinder of order ν generated by $x\in(0,1)^n_F$ as

$$B_\nu=B_\nu(x)=\{y\in(0,1)^n_F\colon\ a_i(y)=a_i(x),\ i=1,\ldots,\nu\}.$$