Repeating the reasoning in Section 10 we can derive from (11.6)

$$(11.7)\qquad \sum_{m\geqslant M_1} Q_m(N) \leqslant \frac{2}{r_1} Q(N)\exp\Big(O(N^{\alpha/(\alpha+1)}\log^{-(\beta+1)/(\alpha+1)}N\log\log N)+ + r_1\Big\{-M_1+(1+r_1)C_1N^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)}N\Big(1+O\Big(\frac{\log\log N}{\log N}\Big)\Big)\Big\}\Big).$$

Now choosing

$$(11.8)\qquad \begin{aligned} M_1 &= C_1N^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)}N(1+2\log^{-1/(4\alpha+4)}N),\\ r_1 &= \log^{-1/(4\alpha+4)}N \end{aligned}$$

(11.7) gives

$$\sum_{m\geqslant M_1} Q_m(N) \leqslant Q(N)\exp(-cN^{\alpha/(\alpha+1)}\log^{-(\beta+\frac14)/(\alpha+1)}N)$$

with an unspecified positive c. This completes the proof.

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On the order function of a transcendental number

by

K. Mahler (Columbus, Ohio)

To the memory of Harold Davenport

Some forty years ago, I introduced the classification of all (real or complex) transcendental numbers into three disjoint classes S, T, and U (see the detailed treatment of this classification and of an equivalent one by J. F. Koksma in Th. Schneider [5], Kapitel III). This classification possessed the *Invariance Property*; i.e., two numbers which are algebraically dependent over the rational field $\boldsymbol{Q}$ always belong to the same class.

In the present paper, a new classification will be introduced. I associate with each transcendental number ξ a positive valued non-decreasing function $O(u\,|\,\xi)$ of an integral variable $u\geqslant 1$, called the *order function* of ξ. For such order functions, both a partial ordering and an equivalence relation will be defined, and it will be proved that if any two transcendental numbers ξ and η are algebraically dependent over $\boldsymbol{Q}$, then $O(u\,|\,\xi)$ and $O(u\,|\,\eta)$ are equivalent. We may now put any transcendental numbers into one and the same class whenever their order functions are equivalent. In this way we evidently obtain a classification of the transcendental numbers into infinitely many disjoint classes.

The order function $O(u\,|\,\xi)$ is defined in terms of the approximation properties of ξ. Unfortunately, the actual determination of $O(u\,|\,\xi)$ for a given ξ is a difficult problem, and more work on such order functions is called for.

1. The following notation will be used. We denote by V the set of all polynomials

$$p(x) = p_0+p_1x+\ldots+p_mx^m \quad \text{where} \quad p_m\neq 0,$$

by W the set of such polynomials with integral coefficients. The exact degree of a polynomial in V is denoted by

$$\partial_x(p) = \partial(p) = m,$$

and we further put

$$L_x(p) = L(p) = |p_0|+|p_1|+\ldots+|p_m|,\qquad A_x(p) = A(p) = 2^{\partial(p)}L(p).$$

When the variable is y, we write instead ∂_y, L_y, and Λ_y. The function $L(p)$ has the two properties

$$L(p+q) \leqslant L(p)+L(q) \quad \text{and} \quad L(pq) \leqslant L(p)L(q),$$

and analogous inequalities hold for $\Lambda(p)$. In addition, $\Lambda(p)$ has the basic property that there are for every integer $u \geqslant 1$ only *finitely many* polynomials $p(x)$ in W for which

$$\Lambda(p) \leqslant u.$$

The set of these polynomials is denoted by $W(u)$. It contains the constant polynomial 1, and when $u < u'$, then $W(u)$ is a subset of $W(u')$.

For any algebraic number ξ, denote by $P(x|\xi)$ the primitive irreducible polynomial with integral coefficients and positive highest coefficient for which

$$P(\xi|\xi) = 0.$$

We then put

$$\partial^0(\xi) = \partial(P), \quad L^0(\xi) = L(P), \quad \Lambda^0(\xi) = \Lambda(P).$$

In particular, $\partial^0(\xi)$ is the degree of ξ.

Next let $a(u)$ and $b(u)$ be any two positive valued non-increasing functions of u. If there exist two positive integers c and u_0 and a positive number γ such that

$$a(u^c) \geqslant \gamma b(u) \quad \text{for} \quad u \geqslant u_0,$$

then we write

$$a(u) >> b(u) \quad \text{or} \quad b(u) << a(u).$$

This relation $>>$ evidently defines a partial ordering. If, simultaneously,

$$a(u) >> b(u) \quad \text{and} \quad a(u) << b(u),$$

then we write

$$a(u) >< b(u).$$

It is clear that this sign $><$ defines an equivalence relation. With respect to this relation, the functions $a(u)$ can be distributed into disjoint classes, and then the sign $>>$ defines a partial ordering of these classes.

2. Let ξ be any real or complex number; put

$$\sigma(\xi) = \begin{cases} 1 & \text{if } \xi \text{ is real,} \\ 2 & \text{if } \xi \text{ is not real.} \end{cases}$$

For every positive integer u denote by $\Omega(u)$ the set of all polynomials $p(x)$ in $W(u)$ for which

$$p(\xi) \neq 0.$$

Thus, for all u, $\Omega(u)$ is a finite set which contains the polynomial 1, and $\Omega(u)$ is a subset of $\Omega(u')$ when $u < u'$. Therefore the minimum

$$o(u|\xi) = \inf_{p(x)\in\Omega(u)} |p(\xi)|$$

exists for all u, satisfies the inequality

$$0 < o(u|\xi) \leqslant 1,$$

and is a non-increasing function of u. In the special case when ξ is a rational integer, or an integer in an imaginary quadratic field, always

$$o(u|\xi) = 1.$$

On the other hand, as is easily proved, for all other ξ

$$0 < o(u|\xi) < 1$$

as soon as u is sufficiently large.

We also introduce the derived function

(1) $$O(u|\xi) = \log\{1/o(u|\xi)\} = \sup_{p(x)\in\Omega(u)} \log|1/p(\xi)|$$

which we call the *order function* of ξ. This function is non-negative and non-decreasing for all u; it vanishes identically if ξ is a rational integer or an integer in an imaginary quadratic field, and otherwise is positive as soon as u is sufficiently large.

We shall use the notations

$$\xi >> \eta \quad \text{if} \quad O(u|\xi) >> O(u|\eta),$$
$$\xi >< \eta \quad \text{if} \quad O(u|\xi) >< O(u|\eta).$$

Evidently $\xi >> \eta$ defines a partial ordering, and $\xi >< \eta$ an equivalence relation, on the set of all real and complex numbers.

3. A result due to R. Güting [3] allows to formulate an upper estimate for the order function when ξ is algebraic.

Let ξ be an algebraic number, and let $p(x)$ be a polynomial in W. Then either

$$p(\xi) = 0,$$

or

$$|p(\xi)| \geqslant \frac{\max(1, |\xi|)^{\partial(p)}}{L^0(\xi)^{\partial(p)/\sigma(\xi)} L(p)^{(\partial^0(\xi)/\sigma(\xi))-1}}.$$

Assume here, in particular, that $p(x)$ lies in $\Omega(u)$. Then the first case is excluded, and $\Lambda(p) = 2^{\partial(p)}L(p)$ does not exceed u. Hence there exist two positive numbers c_1 and c_2 independent of u and $p(x)$ such that

$$|p(\xi)| \geqslant c_1 u^{-c_2} \quad \text{if} \quad p(x) \in \Omega(u).$$

We can express this result in the following form.

THEOREM 1. *If ξ is an algebraic number, then*

$$O(u\,|\,\xi) << \log u.$$

4. Consider next the case when m is a given positive integer, and either is transcendental, or it is algebraic but of a degree greater than m. We shall construct polynomials $p(x)$ in W, with degrees not greater than m, for which $|p(\xi)|$ is small and $\Lambda(p)$ does not exceed a given value u.

The easiest method of finding such polynomials uses an inequality from the theory of positive definite quadratic forms

$$F(x_1, \ldots, x_n) = \sum_{h=1}^{n}\sum_{k=1}^{n} F_{hk}x_h x_k \qquad (F_{hk} = F_{kh}).$$

Denote by

$$D_F = \begin{vmatrix} F_{11} & \ldots & F_{1n} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ F_{n1} & \ldots & F_{nn} \end{vmatrix} > 0$$

the discriminant of F. On writing the form as the sum of the squares of n linear forms and applying Minkowski's theorem on linear forms it can easily be proved that there exist to F integers $x_1^0, \ldots, x_n^0$ not all zero such that

(2) $$F(x_1^0, \ldots, x_n^0) \leqslant nD_F^{1/n}.$$

Depending on whether ξ is real or not, two different cases of this estimate will be applied.

5. Firstly, let ξ be real. Put $n = m+1$, and denote by s and t two parameters such that

$$s \geqslant \max(1, |\xi|)^{-m/(m+1)}, \qquad t = (m+1)(m+2)^{1/\{2(m+1)\}}\max(1, |\xi|)^{m/(m+1)}s$$

and hence

$$t \geqslant (m+1)(m+2)^{1/\{2(m+1)\}}.$$

Take for F the positive definite quadratic form

$$F(x_0, x_1, \ldots, x_m) = s^{2(m+1)}(x_0 + x_1\xi + \ldots + x_m\xi^m)^2 + x_0^2 + x_1^2 + \ldots + x_m^2$$

which is easily seen to have the discriminant

$$D_F = 1 + s^{2(m+1)}(1 + \xi^2 + \ldots + \xi^{2m})$$
$$\leqslant 1 + s^{2(m+1)}(m+1)\max(1, |\xi|)^{2m} \leqslant s^{2(m+1)}(m+2)\max(1, |\xi|)^{2m}.$$

By the property (2), there exists then a polynomial

$$p(x) = p_0 + p_1x + \ldots + p_mx^m$$

with integral coefficients not all zero such that

$$s^{2(m+1)}p(\xi)^2 + p_0^2 + p_1^2 + \ldots + p_m^2$$
$$\leqslant (m+1)s^2(m+2)^{1/(m+1)}\max(1, |\xi|)^{2m/(m+1)} = t^2/(m+1).$$

Since $p(x) \not\equiv 0$, and since ξ is not algebraic at most of degree m, this implies that

$$0 < |p(\xi)| < (m+1)^{1/2}(m+2)^{1/\{2(m+1)\}}\max(1, |\xi|)^{m/(m+1)}s^{-m}$$
$$\leqslant (m+1)^{(2m+1)/2}(m+2)^{1/2}\max(1, |\xi|)^m t^{-m}$$

and therefore

(3) $$0 < |p(\xi)| < (m+2)^{m+1}\max(1, |\xi|)^m t^{-m}.$$

It further follows that also

$$0 < p_0^2 + p_1^2 + \ldots + p_m^2 < t^2/(m+1),$$

whence, by Cauchy's inequality,

(4) $$0 < L(p) < t.$$

Secondly, let ξ be a non-real complex number, and assume now that the parameters s and t are such that

$$s \geqslant \max(1, |\xi|)^{-2m/(m+1)}, \qquad t = (m+1)(m+2)^{1/(m+1)}\max(1, |\xi|)^{2m/(m+1)}s,$$

hence that

$$t \geqslant (m+1)(m+2)^{1/(m+1)}.$$

The case $m = 1$ is now trivial and will be excluded.

We split the powers

$$\xi^k, = \lambda_k + i\mu_k \quad \text{say} \quad (k = 0, 1, \ldots, m),$$

into their real and imaginary parts. The positive definite quadratic form

$$F(x_0, x_1, \ldots, x_m) = s^{m+1}|x_0 + x_1\xi + \ldots + x_m\xi^m|^2 + x_0^2 + x_1^2 + \ldots + x_m^2$$

in $x_0, x_1, \ldots, x_m$ can easily be shown to have the discriminant

$$D_F = 1 + s^{m+1}\sum_{k=0}^{m}(\lambda_k^2 + \mu_k^2) + s^{2(m+1)}\sum_{0\leqslant k_1<k_2\leqslant m}(\lambda_{k_1}\mu_{k_2} - \lambda_{k_2}\mu_{k_1})^2$$

where evidently

$$D_F \leqslant s^{2(m+1)}(m+2)^2\max(1, |\xi|)^{4m}.$$

We find thus just as in the real case that there exists a polynomial

$$p(x) = p_0 + p_1x + \ldots + p_mx^m$$

with integral coefficients not all zero such that

$$s^{m+1}|p(\xi)|^2 + p_0^2 + p_1^2 + \ldots + p_m^2 \leqslant (m+1)s^2(m+2)^{2/(m+1)}\max(1, |\xi|)^{4m/(m+1)}.$$

As in the real case, this inequality implies that simultaneously

(5) $$0 < |p(\xi)| < (m+1)^{m/2}(m+2)^{1/2}\max(1, |\xi|)^m t^{-(m-1)/2}$$

and

(6) $$0 < L(p) < t.$$

On combining the two results (3), (4) and (5), (6), we have thus proved:

Let $m \geqslant \sigma(\xi)$, and also $m < \partial^0(\xi)$ if ξ is algebraic; let further

(7) $$t \geqslant (m+2)^{(m+2)/(m+1)}.$$

Then there exists a polynomial $p(x)$ with integral coefficients satisfying

(8) $$\partial(p) \leqslant m, \quad 0 < L(p) < t, \quad \textit{hence also} \quad A(p) < 2^m t,$$

and

(9) $$0 < |p(\xi)| < (m+2)^{(m+1)/\sigma(\xi)}\max(1, |\xi|)^m t^{-\{(m+1)/\sigma(\xi)\}+1}.$$

6. Assume now, firstly, that ξ is algebraic but is neither rational nor lies in an imaginary quadratic field. Choose $m = \sigma(\xi)$, and allow t to tend to infinity. We obtain then infinitely many distinct polynomials $p(x)$ with integral coefficients for which

$$0 < |p(\xi)| < \begin{cases} 3^2\max(1, |\xi|)t^{-1} < 2\cdot 3^2\max(1, |\xi|)A(p)^{-1} & \text{if } \xi \text{ is real,} \\ 4^{3/2}\max(1, |\xi|)^2 t^{-1/2} < 2^4\max(1, |\xi|)^2 A(p)^{-1/2} & \text{if } \xi \text{ is not real.} \end{cases}$$

Thus, in either case, for all sufficiently large u,

$$O(u\,|\,\xi) \geqslant c_3 \log u$$

where $c_3 > 0$ depends only on ξ. Hence, by Theorem 1, we find as a first result.

THEOREM 2. *If ξ is algebraic, but is neither a rational number nor lies in an imaginary quadratic field, then*

$$O(u\,|\,\xi) > < \log u.$$

This result remains valid in the excluded case provided ξ is not an algebraic *integer*.

Secondly, let ξ be transcendental. We now choose

$$t = 2^m.$$

Then, for sufficiently large m, the condition (7) is satisfied, and

$$A(p) < 4^m.$$

Further

$$0 < |p(\xi)| < (m+2)^{(m+1)/\sigma(\xi)}\max(1, |\xi|)^m t^{-\{(m+1)/\sigma(\xi)\}+1} < 2^{-m^2/3}$$

as soon as m is sufficiently large because $\sigma(\xi) \leqslant 2$.

This means that for every sufficiently large positive integer there exists a polynomial $p(x) \not\equiv 0$ with integral coefficients for which both

$$0 < |p(\xi)| < e^{-c_4(\log u)^2} \quad \text{and} \quad A(p) < u.$$

Here $c_4 > 0$ is a certain absolute constant. From this result, the following theorem follows at once.

THEOREM 3. *If ξ is transcendental, then*

$$O(u\,|\,\xi) >> (\log u)^2.$$

7. We proceed now to the study of the order functions of two transcendental numbers ξ and η which are algebraically dependent over the rational field $\boldsymbol{Q}$.

By this hypothesis, there exists a primitive irreducible polynomial

$$A(x, y) = \sum_{h=0}^{M}\sum_{k=0}^{N} A_{hk}x^h y^k \not\equiv 0$$

with rational integral coefficients and, say, of the exact degrees $M \geqslant 1$ in x and $N \geqslant 1$ in y, such that

$$A(\xi, \eta) = 0.$$

From this we shall deduce that $\xi > < \eta$.

Put

$$A_h(y) = \sum_{k=0}^{N} A_{hk}y^k \quad (h = 0, 1, \ldots, M),$$

so that

$$A(x, y) = \sum_{h=0}^{M} A_h(y)x^h.$$

By the hypothesis,

$$A_M(y) \not\equiv 0,$$

and

(10) $$\max_{0\leqslant h\leqslant M} \partial_y(A_h) = N.$$

We shall use the notation

$$C = \max_{0\leqslant h\leqslant M} L_y(A_h).$$

8. The equation $A(\xi, \eta) = 0$ can be written in the form

$$A_M(\eta)\xi^M = -\{A_0(\eta) + A_1(\eta)\xi + \ldots + A_{M-1}(\eta)\xi^{M-1}\}.$$

We multiply this formula repeatedly by ξ and each time eliminate the term in ξ^M on the right-hand side by means of the formula. We so obtain an infinite sequence of equations

(11) $$A_M(\eta)^k \xi^k = \sum_{h=1}^{M-1} a_{hk}(\eta)\xi^h \quad (k = 0, 1, 2, \ldots).$$

Here the $a_{hk}(y)$ denote certain polynomials in y with integral coefficients which are defined by the initial values

$$(12)\qquad a_{hk}(y) = \begin{cases} A_M(y)^k & \text{if } h = k \\ 0 & \text{if } h \neq k \end{cases} \quad \text{and} \quad k = 0, 1, \ldots, M-1,$$

and, for $k = M, M+1, M+2, \ldots$, by the recursive formulae

$$(13)\qquad a_{h,k+1}(y) = \begin{cases} -A_0(y)\, a_{M-1,k}(y) & \text{if } h = 0, \\ -A_h(y)\, a_{M-1,k}(y) + A_M(y) a_{h-1,k}(y) & \text{if } h = 1, 2, \ldots, M-1. \end{cases}$$

From these formulae and from (10),

$$(14)\qquad \partial_y(a_{hk}) \leqslant kN \quad \text{for all } h \text{ and } k.$$

Further, for all h, by (12),

$$L_y(a_{hk}) \leqslant C^k \quad \text{if} \quad k = 0, 1, \ldots, M-1,$$

and by (13),

$$L_y(a_{h,k+1}) \leqslant 2C \max_{0 \leqslant h \leqslant M-1} L_y(a_{hk}) \quad \text{if} \quad k \geqslant M-1.$$

It follows therefore by induction for k that

$$(15)\qquad L_y(a_{hk}) \leqslant (2C)^k \quad \text{for all } h \text{ and } k.$$

It is convenient to replace the last formulae by slightly different ones. Denote by m any positive integer not less than $M-1$. The formulae (11) imply that also

$$(16)\qquad A_M(\eta)^m \xi^k = \sum_{h=0}^{M-1} B_{hk}(\eta) \xi^h \quad (k = 0, 1, \ldots, m)$$

where the $B_{hk}(y)$ denote new polynomials in y with integral coefficients defined by

$$(17)\qquad B_{hk}(y) = A_M(y)^{m-k} a_{hk}(y).$$

Therefore, by (14) and (15),

$$(18)\qquad \partial_y(B_{hk}) \leqslant mN \quad \text{and} \quad L_y(B_{hk}) \leqslant (2C)^m \quad \text{for all } h \text{ and } k.$$

9. Let

$$p(x) = p_0 + p_1 x + \ldots + p_m x^m, \quad \text{where} \quad p_m \neq 0,$$

be any polynomial in x with integral coefficients, of the exact degree

$$\partial_x(p) = m.$$

Here it is assumed that

$$m \geqslant M-1.$$

Therefore, by (16),

$$A_M(\eta)^m p(\xi) = \sum_{h=0}^{M-1} \sum_{k=0}^{m} p_k B_{hk}(\eta) \xi^h,$$

say

$$(19)\qquad A_M(\eta)^m p(\xi) = \sum_{h=0}^{M-1} b_h(\eta) \xi^h.$$

Here we have put

$$(20)\qquad b_h(y) = \sum_{k=0}^{m} p_k B_{hk}(y) \quad (h = 0, 1, \ldots, M-1),$$

so that also the $b_h(y)$ are polynomials in y with integral coefficients. From the estimates (18), it follows immediately that

$$(21)\quad \partial_y(b_h) \leqslant mN \quad \text{and} \quad L_y(b_h) \leqslant (2C)^m L_x(p) \quad (h = 0, 1, \ldots, M-1).$$

Denote now by $q(y)$ the resultant relative to x of the two polynomials

$$A(x, y) = A_0(y) + A_1(y) x + \ldots + A_M(y) x^M$$

and

$$A^*(x, y) = b_0(y) + b_1(y) x + \ldots + b_{M-1}(y) x^{M-1}.$$

This resultant is given explicitly by the determinant

$$(22)\qquad q(y) = \begin{vmatrix} A_0(y) & A_1(y) & \ldots & A_M(y) & \ldots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \ldots & A_0(y) & A_1(y) & \ldots & A_M(y) \\ b_0(y) & b_1(y) & \ldots & b_{M-1}(y) & \ldots & 0 \\ \vdots & \ddots & & & \ddots & \vdots \\ 0 & \ldots & b_0(y) & b_1(y) & \ldots & b_{M-1}(y) \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} M-1 \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} M \text{ rows} \end{matrix}$$

Hence $q(y)$ is a polynomial with integral coefficients. By (10) and (21),

$$\partial_y(q) \leqslant (M-1) \cdot mN + M \cdot mN$$

and therefore

$$(23)\qquad \partial_y(q) \leqslant m(2M-1)N.$$

It follows further from the trivial estimate for a determinant and from (21) that

$$L_y(q) \leqslant (2M-1)!\,(2C)^{m(M-1)} \{(2C)^m L_x(p)\}^M$$

and hence

$$(24)\qquad L_y(q) \leqslant (2M-1)!\,(2C)^{m(2M-1)} L_x(p)^M.$$

10. Next multiply the 2nd, 3rd, ..., $(2M-1)$st columns of the determinant for $q(y)$ by the factors

$$x, x^2, \ldots, x^{2M-2},$$

respectively, and add to the first column. The new first column becomes then

$$A(x,y), A(x,y)x, \ldots, A(x,y)x^{M-2}, A^*(x,y), A^*(x,y)x, \ldots, A^*(x,y)x^{M-1}.$$

Here put

$$x = \xi \quad \text{and} \quad y = \eta.$$

Then

$$A(\xi,\eta) = 0 \quad \text{and} \quad A^*(\xi,\eta) = A_M(\eta)^m p(\xi),$$

whence

$$q(\eta) = A_M(\eta)^m p(\xi)\cdot q^*(\xi,\eta), \tag{25}$$

where $q^*(\xi,\eta)$ denotes the determinant obtained from that defining $q(y)$ by replacing its first column by the new column

$$0, 0, \ldots, 0, 1, \xi, \xi^2, \ldots, \xi^{M-1}$$

and substituting η for y. Thus $q^*(\xi,\eta)$ can be written as a polynomial in ξ of the form

$$q^*(\xi,\eta) = q_0^*(\eta) + q_1^*(\eta)\xi + \ldots + q_{M-1}^*(\eta)\xi^{M-1}. \tag{26}$$

Here, for $h = 0, 1, \ldots, M-1$, the $q_h^*(y)$ denote the cofactors of the last M elements of the first column of the determinant for $q(y)$. They are thus polynomials in y with integral coefficients. Just as for (23) and (24), we find the estimates

$$\partial_y(q_h^*) \leqslant 2m(M-1)N \quad \text{and} \quad L_y(q_h^*) \leqslant (2M-2)!(2C)^{2m(M-1)}L_x(p)^{M-1} \tag{27}$$
$$(h = 0, 1, \ldots, M-1).$$

11. The resultant $q(y)$ does not vanish identically because $A(x,y)$ is irreducible and has the exact degree M in x, while $A^*(x,y)$ has at most the degree $M-1$ in this variable. The transcendency of η implies then that

$$q(\eta) \neq 0.$$

By (23) and (24),

$$\Lambda_y(q) \leqslant 2^{m(2M-1)N}(2M-1)!(2C)^{m(2M-1)}L_x(p)$$

and also

$$\Lambda_x(p) = 2^m L_x(p).$$

Hence there exist two positive integers C_1 and Γ_1 depending only on C, M, and N, and so only on the polynomial $A(x,y)$, such that

$$\Lambda_y(q) \leqslant \Lambda_x(p)^{C_1} \quad \text{if} \quad \Lambda_x(p) \geqslant \Gamma_1. \tag{28}$$

Next put

$$|A_M(\eta)| = c_5, \quad \max(1, |\xi|) = c_6, \quad \text{and} \quad \max(1, |\eta|) = c_7.$$

By (26) and (27),

$$|q^*(\xi,\eta)| \leqslant Mc_6^{M-1}(2M-2)!(2C)^{2m(M-1)}L_x(p)^{M-1}c_7^{2m(M-1)N},$$

so that, by (25),

$$\left|\frac{q(\eta)}{p(\xi)}\right| \leqslant c_5 M c_6^{M-1}(2M-2)!(2C)^{2m(M-1)}L_x(p)^{M-1}c_7^{2m(M-1)N}.$$

By this inequality, there exist two further positive integers C_2 and Γ_2 which depend only on the polynomial $A(x,y)$ and on the two numbers ξ and η such that

$$|q(\eta)| \leqslant \Lambda_x(p)^{C_2}|p(\xi)| \quad \text{if} \quad \Lambda_x(p) \geqslant \Gamma_2. \tag{29}$$

12. Assume now that the parameter u is not less than

$$\Gamma = \max(\Gamma_1, \Gamma_2).$$

Further choose in $\Omega(u)$ a polynomial $p(x)$ satisfying the equation

$$\log|1/p(\xi)| = O(u\,|\,\xi).$$

By this choice,

$$\Lambda_x(p) \leqslant u.$$

Further, by (28),

$$\Lambda_y(q) \leqslant u^{C_1}, \tag{30}$$

and by (29),

$$|q(\eta)| \leqslant |p(\xi)|u^{C_2}. \tag{31}$$

We found already, in the proof of Theorem 3, that

$$\log|1/p(\xi)| > c_4(\log u)^2,$$

where $c_4 > 0$ was a certain absolute constant. Hence, if Γ_0 is a sufficiently large positive integer, then, by (31),

$$\log|1/q(\eta)| \geqslant \tfrac{1}{2}\log|1/p(\xi)| = O(u\,|\,\xi)/2 \quad \text{if} \quad u \geqslant \Gamma_0. \tag{32}$$

On the other hand, $q(\eta) \neq 0$, and so, by (30), $q(y)$ belongs to the set $\Omega(u^{C_1})$. But then, necessarily,

$$O(u^{C_1}\,|\,\eta) \geqslant \log|1/q(\eta)|,$$

so that, by (32), we arrive finally at the estimate

$$O(u^{C_1}\,|\,\eta) \geqslant \tfrac{1}{2}O(u\,|\,\xi) \quad \text{if} \quad u \geqslant \Gamma_0.$$

Naturally, on interchanging ξ and η, we also obtain an analogous estimate

$$O(u^{C_1^*}\,|\,\xi) \geqslant \tfrac{1}{2}O(u\,|\,\eta) \quad \text{if} \quad u \geqslant \Gamma_1^*,$$

where C_1^* and Γ_1^* are two further positive integers.

We have thus established the following *Invariance Property*.

THEOREM 4. *Let ξ and η be two transcendental numbers which are algebraically dependent over the rational field $\boldsymbol{Q}$. Then*

$$O(u\,|\,\xi) >< O(u\,|\,\eta) \quad \textit{and therefore} \quad \xi >< \eta.$$

13. Denote by $\mathcal{T}$ the set of all transcendental numbers. Let us then subdivide $\mathcal{T}$ into disjoint subsets or *classes* $\Xi, H, Z, \ldots$ by putting numbers ξ and η into the same class if and only if $\xi >< \eta$. Thus, by what has just been proved, *numbers which are algebraically dependent over* $\boldsymbol{Q}$ *belong always to the same class.*

There are evidently non-countably many positive valued non-decreasing functions $a(u), b(u), \ldots$ of the integer $u \geqslant 1$ no two of which stand in the relation

$$a(u) >< b(u),$$

but it is not evident which of these functions are order functions of transcendental numbers. It is further clear that there exist transcendental numbers ξ (e.g. Liouville numbers) for which $O(u\,|\,\xi)$ tends arbitrarily rapidly to infinity; but it does not seem to be easy to find the exact size of these order functions. Thus the following two problems remain open.

PROBLEM 1. *Do there exist non-countably many distinct classes $\Xi, H, Z, \ldots$?* (¹)

PROBLEM 2. *Let $a(u)$ be any positive valued non-decreasing function of the integer $u \geqslant 1$. To establish necessary and sufficient conditions for the existence of a number $\xi \in \mathcal{T}$ such that*

$$a(u) >< O(u\,|\,\xi).$$

In addition to the equivalence relation $><$ we had also defined an order relation $>>$ for both functions and numbers, and it is easily seen that it can be extended to classes. With respect to this order relation, the following two questions arise.

PROBLEM 3. *Does there exist a pair of numbers ξ and η in $\mathcal{T}$ such that neither $\xi >> \eta$ nor $\xi << \eta$?*

PROBLEM 4. *Does there exist a number $\zeta \in \mathcal{T}$ such that*

$$\xi >> \zeta \quad \textit{for all} \quad \xi \in \mathcal{T}\,?$$

(¹) Note added on January 12, 1971. S. Świerczkowski has recently proved that the answer is affirmative.

The following metrical question also has some interest.

PROBLEM 5. *To decide whether there exist, and if so, to determine, two positive valued non-decreasing functions $a(u)$ and $b(u)$ of the integer $u \geqslant 1$ such that*

(i) $\quad O(u\,|\,\xi) << a(u)$ *for almost all real numbers* $\xi \in \mathcal{T}$,

and

(ii) $\quad O(u\,|\,\xi) << b(u)$ *for almost all complex numbers* $\xi \in \mathcal{T}$,

and that, in addition, $a(u)$ and $b(u)$ increase as slowly as possible.

I conjecture that this problem has the solution

$$a(u) >< (\log u)^2, \quad b(u) >< (\log u)^2.$$

The actual determination of $O(u\,|\,\xi)$ for any given $\xi \in \mathcal{T}$ presents a difficult problem which has as yet not even be solved for the two classical transcendental numbers e and π. For the order functions of these two numbers the best *lower* bounds known seem to be those given in Theorem 3.

The best *upper* bounds known at present are those due to N. I. Feldman ([1] and [2]) which state that

$$O(u\,|\,e) << (\log u)^3(\log\log u)^3,$$

$$O(u\,|\,\pi) << (\log u)^2(\log\log u)^3.$$

We had defined the order function $O(u\,|\,\xi)$ in terms of the functional

$$\Lambda(p) = 2^{\partial(p)}L(p).$$

No essentially different results are obtained if 2 is here replaced by any other constant greater than 1. It may, however, be useful to consider other functionals.

Just as in Koksma's approach ([4]) to my old classification, one can replace the order function $O(u\,|\,\xi)$ by a new function

$$O^*(u\,|\,\xi) = \sup_{\alpha \in \Omega^*(u)} \log\{1/|\xi - \alpha|\}$$

where $\Omega^*(u)$ denotes the set of all algebraic numbers α for which

$$\alpha \neq \xi \quad \text{and} \quad \Lambda^0(\alpha) \leqslant u.$$

However, both Koksma's work and a recent paper by Wirsing ([6]) suggest that the results will be completely analogous to those for $O(u\,|\,\xi)$.

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OHIO STATE UNIVERSITY
Columbus, Ohio

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A larger sieve

by

P. X. GALLAGHER (New York, N. Y.)

1. Linnik's 'large sieve' gives an upper bound for the number of integers which remain in an interval of length N after $f(p)$ different residue classes (mod p) have been removed, for each prime p. In its refined form, due to Bombieri and Davenport [1], [2], and Montgomery [4], the upper bound is

$$\frac{N+CQ^2}{S(Q)}, \quad \text{where } S(Q) = \sum_{q\leqslant Q} \mu^2(q) \prod_{p|q} \frac{f(p)}{p-f(p)}, \tag{1}$$

and C is a positive constant. In the applications, Q is chosen a little less than $N^{1/2}$ to minimise the bound.

In some cases, the bound obtained is nearly best possible. For example, if the quadratic nonresidues (mod p) are removed for each prime p, the perfect squares remain. Here $f(p) = \frac{1}{2}(p-1)$ for odd p, so $S(Q) \gg Q$. Thus the upper bound is $\ll N^{1/2}$ for $Q = N^{1/2}$.

In this note we give a simple sieve method which gives a comparable bound in this example and is more effective than the large sieve when $f(p)$ is close to p. We put $g(p) = p-f(p)$ and consider also prime power moduli.

THEOREM 1. *If all but $g(q)$ residue classes* (mod q) *are removed for each prime power q in a finite set $\mathcal{S}$, then the number of integers which remain in any interval of length N is at most*

$$\Big(\sum_{q\in\mathcal{S}} \Lambda(q) - \log N\Big) \Big/ \Big(\sum_{q\in\mathcal{S}} \frac{\Lambda(q)}{g(q)} - \log N\Big) \tag{2}$$

provided the denominator is positive. Here $\Lambda(q) = \log p$ for $q = p^a$.

Proof. Assume Z integers n remain in a given interval of length N, and of these $Z(h, q)$ satisfy $n \equiv h$ (mod q). Then

$$Z^2 = \Big(\sum_{h=1}^{q} Z(h, q)\Big)^2 \leqslant g(q) \sum_{h=1}^{q} \big(Z(h, q)\big)^2$$