This implies that the intervals $I_m$ given by

$$I_m:\ m\delta \leqslant |x_i - x| < (m+1)\delta,$$

where $(m+1)\delta \leqslant \frac{1}{2}$, contain at most two points $x_j$, while every $x_j$ falls in some $I_m$. Using the inequality $\left|\sin\frac{\pi}{2}y\right| \geqslant |y|$ for $|y| \leqslant 1$ we get

$$\sum_{\substack{j=1\\ j\neq i}}^{R} \frac{1}{(\sin\pi(x_j - x_i))^2} = \sum_{\substack{m\neq 0\\ x_j\in I_m}} \frac{1}{(\sin\pi(x_j - x_i))^2} \leqslant 2\sum_{m=1}^{\infty} \frac{1}{4m^2\delta^2} = \frac{\pi^2}{12}\delta^{-2}.$$

It follows that

$$\sum_{j=1}^{R} |(\varphi_i, \varphi_j)| \leqslant 2N + L + \frac{1}{L}\,\frac{\pi^2}{12}\delta^{-2},$$

and (5) follows taking $L$ the nearest integer to $\frac{\pi}{\sqrt{12}}\delta^{-1}$. This completes the proof.

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# On inequalities of Large Sieve type

by

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*In memory of my friend and teacher*
*Professor H. Davenport*

**1. Introduction.** The inequality of the Large Sieve of Yu. V. Linnik, in a general form, can be expressed by:

Let $a_1, \ldots, a_N$ be $N$ complex numbers, then

$$\sum_{q\leqslant Q}\ \sideset{}{^*}\sum_{\chi(\mathrm{mod}\,q)} \left|\sum_{n=1}^{N} a_n\chi(n)\right|^2 \leqslant \sum_{q\leqslant Q}\ \sum_{\substack{b=1\\(b,q)=1}}^{q} \left|\sum_{n=1}^{N} a_n \exp\left(2\pi i\frac{b}{q}n\right)\right|^2$$
$$\leqslant \delta(N, Q)\sum_{n=1}^{N}|a_n|^2$$

where $q$ runs through all the rational integers not exceeding $Q$, * denotes summation over all primitive Dirichlet characters, and $\delta(N, Q)$ is a function of $N$ and $Q$ alone.

That such a function $\delta(N, Q)$ exists, which is in some sense 'not too large', was first proved by Linnik [13]. His result was successively refined by Roth [18], Bombieri [1], Davenport and Halberstam [5], and Davenport and Bombieri [3], [4]. We also mention the papers of Montgomery [15], and Wolke [22], the first of which in particular combines the large sieve with a method of Hálasz [10], and the second of which further refines the function $\delta(N, Q)$ under certain conditions. We state this last result presently. We mention in particular the inequalities

$$\delta(N, Q) \leqslant N + c_1Q^2, \qquad \text{Davenport and Bombieri [4],}$$

and

$$\delta(N, Q) \leqslant Q^2 + \pi N, \qquad \text{Gallagher [8],}$$

the final inequality having a very simple proof.

In their paper [5] Davenport and Halberstam prove that one can take $\delta(N, Q) = 2.2\max(N, Q^2)$ and ask whether there exists any simple best possible function $\delta(N,Q)$. Moreover, Erdös and Rényi [7] also enquired whether any improvement could be effected if, instead of all integers, $q$ was restricted to run through, say, the rational primes not exceeding $Q$. The result of Wolke [22] referred to above, states that if $\varepsilon, c$ are arbitrary positive constants, and if $12 \leqslant N \leqslant Q(\log Q)^c$, then we can take

$$\delta(N,Q) \leqslant c_2 Q^2 (\log Q)^{-1+\varepsilon}.$$

We note that it was proved in [6] that if $Q, N \to \infty$ so that $Q^2 N^{-1} \to 0$, then one must have

$$\delta(N,Q) \geqslant \big(1+o(1)\big) N.$$

In fact, using the statistical properties of integers a similar more explicit lower bound was given subject only to the condition that $q$ runs through a sequence of prime moduli $p$ for which $\sum p^{-1}$ diverges.

**2. Statement of results.** In the present paper we make a few simple remarks concerning the inequality of the Large Sieve, and similar inequalities. In particular we point out that the derivation of inequalities of this type is equivalent to the determination of the spectral radii of certain Hermitian operators; and, moreover, that these inequalities always come in pairs.

As a first example we investigate as a function of $N$ and $Q$ the minimal $\delta(N,Q)$. When one of the variables, $N, Q$ becomes large at the expense of the other. Lower bounds for $\delta(N,Q)$ can be obtained quite simply. As a further example we consider a continuous analogue of the Large Sieve.

To be specific, let $P$ be a set of $|P|$ primes not exceeding $Q$. Define $\Delta(N,Q)$ to be (for each pair $N, Q$) the least number for which the inequality

$$\sum_{p\leqslant Q}{}' \sum_{b=1}^{p-1} \left| S\left(\frac{b}{p}\right)\right|^2 \leqslant \Delta(N,Q) \sum_{n=1}^{N} |a_n|^2$$

is satisfied for all complex numbers $a_1, \ldots, a_N$. Here, as for the duration of Theorem 1, for real numbers $\alpha$ we define

$$S(\alpha) = \sum_{n=1}^{N} a_n e(n\alpha), \qquad e(\alpha) = e^{2\pi i\alpha},$$

and $'$ indicates summation over members of the set $P$.

All of the following remarks could be proved for arbitrary sets of moduli $q$ in place of $P$.

THEOREM 1. *We have*

(i) $\Delta(N,Q) \geqslant \max\big(N, \sum_{p\leqslant Q}' (p-1)\big)$,



(ii) $\Delta(N,Q) = \sum_{p\leqslant Q}' p + O\big(N^2(\log N)^{-1}\big)$,

*in fact*

(iii) $$\sum_{p\leqslant Q}{}' \sum_{b=1}^{p-1} \left| S\left(\frac{b}{p}\right)\right|^2 = \Big(\sum_{p\leqslant Q}{}' p + O\big(N|P| + N^2(\log N)^{-1}\big)\Big) \sum_{n=1}^{N} |a_n|^2$$

*with a similar estimate for*

(iv) $\sum_{p\leqslant Q}' \sum_{\chi(\mathrm{mod}\,p)}^{*} \big| \sum_{n\leqslant N} a_n \chi(n)\big|^2$,

(v) $\Delta(N,Q) = N + O(Q^2|P|)$.

CONJECTURE.

$$\Delta(N,Q) \leqslant \big(1+o(1)\big)\big(N + \sum_{p\leqslant Q}{}' p\big) \qquad (\text{as } N, |P| \to \infty).$$

Remarks. (ii) and (iii) show that if $Q$ and $N$ become large then one of the terms $N$ and $\sum_{p\leqslant Q}' p$ dominates, whilst (i) states, essentially, that at least one of these events always occurs.

An estimate of the type (i) can be obtained for analogous inequalities. Thus in the second of the two inequalities stated in the introduction we have

$$\delta(N,Q) \geqslant \max\Big(N, \sum_{q\leqslant Q} \varphi(q)\Big)$$

where $\varphi(q)$ denotes the number of reduced residue classes (mod $q$), (Euler's phi-function). We note that for $Q \geqslant 2$,

$$\sum_{q\leqslant Q} \varphi(q) = \frac{3}{\pi^2} Q^2 + O(Q\log Q).$$

(See for example Hardy and Wright [12].)

As a second example let $N$ and $T$ be respectively a positive integer $\geqslant 2$ and a positive real number. We define $\varrho(N,T)$ to be the least number $\varrho$ with the property that

$$\int_{-T}^{T} \Big| \sum_{n=1}^{N} a_n n^{-it}\Big|^2 dt \leqslant \varrho \sum_{n=1}^{N} |a_n|^2$$

holds for all complex numbers $a_1, \ldots, a_N$.

We remark that it follows from a result of Davenport (see for example Theorem 1 of Montgomery [15] where one multiplies both sides by $\delta$ and lets $\delta \to 0+$), proved using the method of the large sieve, that

$$\varrho \leqslant 2T + O(N\log N).$$

We investigate to what extent this result might be improved.

THEOREM 2. *For each value of* $N \geqslant 2$ *and every value of* $T$ *we have*

(i) $$\varrho(N, T)\begin{cases} = 2T + O(N\log N), \\ \geqslant 2T. \end{cases}$$

(ii) *For any fixed* $\varepsilon > 0$, *and all* $N \geqslant 2T+4$,

$$\varrho(N, T)\begin{cases} \leqslant (2\pi)^{3/2}N + O(N^{1/2+\varepsilon}T), \\ \geqslant \dfrac{N}{2\sqrt{\pi}} + O(N^{1/2+\varepsilon}T). \end{cases}$$

Remark. It follows from these results that at any rate for large values of $N$ comparison with $T$ ($N^{1/2-\varepsilon} \geqslant cT$ will suffice) one has

$$\varrho(N, T) \geqslant \max(N/4, 2T)$$

so that the first estimate in (i) could be unconditionally little improved. An inequality of this last type probably holds for all $N \geqslant 2$, $T \geqslant 2$.

As in the above statements, $c_1, c_2, \ldots$ will denote positive constants, which will be renumbered from time to time when no confusion can arise. These will either be absolute, or depend upon some (small) fixed given positive number $\varepsilon$. In § 6 we use the notation $A \ll B$ of Vinogradov, with the meaning that there exists a certain constant $c$, whose dependence upon the various parameters concerned will be clear, so that the inequality

$$|A| \leqslant cB$$

holds.

**3. Proof of Theorem 1.** Expanding the sum in question, and inverting the order of summation we have

$$\sum_{m=1}^{N}\sum_{n=1}^{N} a_m\bar{a}_n \sum_{p\leqslant Q}{}' \sum_{b=1}^{p-1} e\left(\frac{b}{p}(m-n)\right).$$

This is a Hermitian form in the variables $a_1, \ldots, a_N$ which we can write shortly as

$$\boldsymbol{a}\boldsymbol{B}\bar{\boldsymbol{B}}^T\bar{\boldsymbol{a}}^T$$

where ${}^T$ denotes the transpose of a matrix, $\bar{}$ denotes complex conjugation, and

$$\boldsymbol{a} = (a_1, \ldots, a_N),$$

$$\boldsymbol{B} = \left(e\left(\frac{b}{p}m\right)\right) \quad \begin{array}{l} \text{columns } 1 \leqslant m \leqslant N \\ \text{rows } 1 \leqslant b \leqslant p-1,\ 2 \leqslant p \leqslant Q,\ p \in P. \end{array}$$

The matrix $\boldsymbol{B}$ need not, of course, be square.



We denote the eigenvalues of $\boldsymbol{B}\bar{\boldsymbol{B}}^T$, in decreasing order, by $\lambda_1, \ldots, \lambda_N$. These are real, and since our form is non-negative definite the $\lambda_j$ are also non-negative.

It is well known that any Hermitian form can be diagonalised by a suitable unitary transformation. Thus, there is a matrix $\boldsymbol{U}$, satisfying

$$\boldsymbol{U}\bar{\boldsymbol{U}}^T = \boldsymbol{I},$$

where $\boldsymbol{I}$ is the $N \times N$ identity matrix, so that if $\boldsymbol{a} = \boldsymbol{U}\boldsymbol{y}^T$, then

$$\boldsymbol{a}\boldsymbol{B}\bar{\boldsymbol{B}}^T\bar{\boldsymbol{a}}^T = \sum_{j=1}^{N}\lambda_j|y_j|^2.$$

Hence

$$\boldsymbol{a}\boldsymbol{B}\bar{\boldsymbol{B}}^T\bar{\boldsymbol{a}}^T \leqslant \lambda\|\boldsymbol{y}\|^2 = \lambda\|\boldsymbol{a}\|^2$$

where $\lambda$ is a maximal eigenvalue of $\boldsymbol{B}\bar{\boldsymbol{B}}^T$. In particular, we can choose a value of $\boldsymbol{a} \neq \boldsymbol{0}$ to effect equality.

It follows that $\Delta(N, Q)$ can be taken to be $\lambda$, the greatest eigenvalue (= spectral radius) of the matrix $\boldsymbol{B}\bar{\boldsymbol{B}}^T$, and that when the $a_n$ are to be unrestricted, this choice is best possible.

We note that a typical non-diagonal component of $\boldsymbol{B}\bar{\boldsymbol{B}}^T$ has the form

$$\sum_{p\leqslant Q}{}' \sum_{b=1}^{p-1} e\left(\frac{b}{p}(m-n)\right) = \sum_{\substack{p|(m-n)\\ p\leqslant Q}}{}' p - |P|,$$

and that down the principal diagonal all the elements have the value

$$\sum_{p\leqslant Q}{}'(p-1) = t,$$

say.

Since

$$\sum_{j=1}^{N}\lambda_j = \operatorname{trace}(\boldsymbol{B}\bar{\boldsymbol{B}}^T) = N\sum_{p\leqslant Q}{}'(p-1),$$

we see that

$$\Delta(N, Q) = \lambda \geqslant t,$$

which is the second half of assertion (i) of Theorem 1.

We write

$$\boldsymbol{B}\bar{\boldsymbol{B}}^T = \operatorname{diagonal}(t)_{N\times N} + \Big(\sum_{\substack{p|(m-n)\\ p\leqslant Q}}{}' p\Big)_{N\times N} + (-|P|)_{N\times N},$$

where it is understood that in each of the final two matrices, the elements of the principal diagonal are to have the value zero.

A simple calculation shows that the last matrix on the right hand side of the above equation has eigenvalues

$$|P|,\quad (N-1)\ \text{times};\qquad (1-N)|P|,\quad \text{once}.$$

The existence of the exceptional eigenvalue $(1-N)|P|$ is suggested by the fact that

$$-(-|P|)_{N\times N}$$

is a matrix, all of whose terms are non-negative, so that a form of the Perron–Frobenius theorem (Gantmacher [9], § 2, p. 53) applies.

Each eigenvalue $\beta$ of

$$\Big(\sum_{\substack{p\mid(m-n)\\ p\leqslant Q}}{}' p\Big)_{N\times N}$$

lies in a Gershgorin disc (Wilkinson [21], § 13, pp. 71–72), centred on a diagonal element, which, in our present case, is at the origin. Thus it is bounded by

$$|\beta|\leqslant \max_{1\leqslant n\leqslant N}\sum_{\substack{m=1\\ m\neq n}}^{N}\Big|\sum_{\substack{p\mid(m-n)\\ p\leqslant Q}}{}' p\Big|\leqslant \sum_{w=1}^{N}\sum_{\substack{p\mid w\\ p\leqslant Q}} p$$

$$\leqslant \sum_{p\leqslant \min(N,Q)} p\left[\frac{N}{p}\right]\leqslant c_1N^2(\log N)^{-1}.$$

In particular, the spread

$$\max_{1\leqslant i<j\leqslant N}|\beta_i-\beta_j|$$

of possible $\beta$-values does not exceed $2c_1N^2(\log N)^{-1}$.

It is well known (see for example Wilkinson [21], (44.10), p. 102) that if $A$ and $B$ are real symmetric matrices, then the eigenvalues of $A+B$ and $A$ differ, when in corresponding order, by at most the spread of the eigenvalues of $\boldsymbol{B}$. Applying this to the above situation, and taking into account the translation by $t$, we see that the eigenvalues of $\boldsymbol{B}\bar{\boldsymbol{B}}^T$ are

$$\sum_{p\leqslant Q}{}' p+O\big(N^2(\log N)^{-1}\big),\qquad (N-1)\ \text{times},$$

and

$$\sum_{p\leqslant Q}{}' p-N|P|+O\big(N^2(\log N)^{-1}\big),\qquad \text{once}.$$

From this the assertions (ii) and (iii) (from the reduced form of $\boldsymbol{B}\bar{\boldsymbol{B}}^T$) of Theorem 1 are proved.



We can prove (iv) in a like fashion, save that corresponding to the matrix $\boldsymbol{B}\bar{\boldsymbol{B}}^T$ we now consider the matrix

$$\text{diagonal}(t)_{N\times N}+\Big(\sum_{\substack{p\mid(m-n),\,p\nmid mn\\ p\leqslant Q}}{}'(p-1)+\sum_{\substack{p\mid mn\\ p\leqslant Q}}{}' 1\Big)_{N\times N}+(-|P|)_{N\times N}.$$

The details are similar to the case (v).

**4.** In order to prove a result of the type (ii) in the statement of the theorem we should like to reverse the rôles of $Q$ and $N$. One way in which to do this is to consider the conjugate inequality (but which is concerned with another space than $C^N$, since $\boldsymbol{B}$ is not necessarily square). Thus in place of the inequality

$$(\alpha)\qquad \sum_{p\leqslant Q}{}'\sum_{b=1}^{p-1}\Big|\sum_{n=1}^{N}a_ne\Big(\frac{b}{p}n\Big)\Big|^2\leqslant \Delta(N,Q)\sum_{n=1}^{N}|a_n|^2$$

we consider the inequality

$$(\beta)\qquad \sum_{n=1}^{N}\Big|\sum_{p\leqslant Q}{}'\sum_{b=1}^{p-1}c_{b,p}e\Big(\frac{b}{p}n\Big)\Big|^2\leqslant \varkappa(N,Q)\sum_{p\leqslant Q}{}'\sum_{b=1}^{p-1}|c_{b,p}|^2,$$

which is to be valid for some function $\varkappa(N,Q)$, and all vectors $\boldsymbol{c}$ with components $c_{b,p}$; $1\leqslant b\leqslant p-1,\ 2\leqslant p\leqslant Q,\ p\in P$.

Expanding once again we can express $(\beta)$ in the form

$$\boldsymbol{c}\boldsymbol{B}^T\bar{\boldsymbol{B}}\bar{\boldsymbol{c}}^T$$

where $\boldsymbol{B}$ is the same matrix as in § 3. Here, once again, $\boldsymbol{B}^T\bar{\boldsymbol{B}}$ is a non-negative definite Hermitian matrix, now of order $t\times t$ ($t$ defined as in § 3). Let us denote its eigenvalues in decreasing order by $\mu_1,\ldots,\mu_t$. Then as in § 3 we can choose

$$\varkappa(N,Q)=\mu=\max_{1\leqslant j\leqslant t}\mu_j,$$

and for unrestricted vectors $\boldsymbol{c}$ this choice is best possible.

The situation is now clarified by the fact that the $\lambda_i$ and $\mu_j$ essentially coincide.

LEMMA 1. *We have*

$$\lambda_j=\mu_j\quad \textit{for}\quad 1\leqslant j\leqslant \min(N,t),$$

*and all remaining eigenvalues of* $\boldsymbol{B}\bar{\boldsymbol{B}}^T$ *or* $\boldsymbol{B}^T\bar{\boldsymbol{B}}$ *are zero.*

Remark. Considerations of rank show that $\lambda_j=\mu_j=0$ if $j>\min(N,t)$.

Proof. A proof of this well-known lemma can be given shortly as follows. For any vectors $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s$, $1 \leqslant s \leqslant \min(N, t)$,

$$\operatorname{Max} \|\boldsymbol{aB}\|/\|\boldsymbol{a}\| \geqslant \lambda_{s+1}$$

holds when the maximum is taken over all those vectors $\boldsymbol{a} \neq \boldsymbol{0}$ which satisfy

$$\boldsymbol{p}_\nu \bar{\boldsymbol{a}}^T = 0 \qquad (\nu = 1, \ldots, s).$$

Moreover, a set of vectors $\boldsymbol{p}_\nu$ exists for which equality is attained. This is known as the Courant–Fisher theorem (see for example Courant and Hilbert [2], or Wilkinson [21], § 43, pp. 99–101). Then for any vector $\boldsymbol{c}$,

$$\begin{aligned} |\boldsymbol{aB}\bar{\boldsymbol{c}}^T| &\leqslant \|\boldsymbol{aB}\|\,\|\bar{\boldsymbol{c}}^T\| && \text{(the inequality of Cauchy–Schwarz)} \\ &\leqslant \lambda_{s+1}\|\boldsymbol{a}\|\,\|\bar{\boldsymbol{c}}^T\| && \text{(by hypothesis, with suitable } \boldsymbol{p}_\nu). \end{aligned}$$

Hence we can set $\bar{\boldsymbol{a}}^T = \boldsymbol{B}\bar{\boldsymbol{c}}^T$ to deduce that

$$(*) \qquad \|\bar{\boldsymbol{c}}^T\|^{-1}\|\boldsymbol{B}\bar{\boldsymbol{c}}^T\| \leqslant \lambda_{s+1}$$

holds for all vectors $\boldsymbol{c} \neq \boldsymbol{0}$ which belong to the space

$$\boldsymbol{p}_\nu \boldsymbol{B}\bar{\boldsymbol{c}}^T = 0 \qquad (\nu = 1, \ldots, s).$$

By the Courant–Fisher theorem the maximum of the left hand side of $(*)$ is at least $\mu_{s+1}$, so that $\mu_{s+1} \leqslant \lambda_{s+1}$ holds for $0 \leqslant s \leqslant \min(N, t)$. The converse inequality, and so the lemma is now clear.

In particular, the 'best possible' choices of $\Delta(N, Q)$ and $\varkappa(N, Q)$ coincide. However,

$$\sum_{j=1}^{t} \mu_j = \operatorname{trace}(\boldsymbol{B}^T\bar{\boldsymbol{B}}) = tN,$$

so that $\lambda = \mu \geqslant N$, and the first part of assertion (i) of Theorem 1 is proved.

We now consider the location of the eigenvalues $\mu_j$. These lie in the Gershgorin discs

$$|\mu_j - N| \leqslant \sideset{}{'}\sum_{\substack{p \leqslant Q \\ (b,p) \neq (b',p')}} \sum_{b=1}^{p-1} \left| \sum_{n=1}^{N} e\left(\left(\frac{b}{p} - \frac{b'}{p'}\right)n\right)\right|,$$

where $1 \leqslant b' \leqslant p'-1$, $2 \leqslant p' \leqslant Q$, $p' \epsilon P$. The multiple sum here can be estimated not to exceed

$$\sideset{}{'}\sum_{p \leqslant Q} \sum_{b=1}^{p-1} \frac{1}{2\left\|\dfrac{b}{p} - \dfrac{b'}{p'}\right\|} \leqslant \sideset{}{'}\sum_{p \leqslant Q}\left(Q^2 + c_2 p \sum_{r=1}^{p-1} \frac{1}{r}\right) \leqslant c_3 Q^2 |P|,$$



the $Q^2$ term arising in each sum over $b$, $1 \leqslant b \leqslant p-1$, from the at most one term $b/p$ which falls within a distance of $1/2Q$ of $b'/p'$. For this term, since $p'b \neq pb'$,

$$\left|\frac{b}{p} - \frac{b'}{p'}\right| \geqslant \frac{1}{pp'} \geqslant \frac{1}{Q^2}.$$

Note that in the above double sum we use $\|\alpha\|$ to denote the distance of the real number $\alpha$ from the nearest integer.

In this way we have shown that every eigenvalue $\mu_j$, and thus $\mu$ and $\lambda$ also, satisfies

$$|\mu_j - N| \leqslant c_3 Q^2 |P|.$$

This proves Theorem 1 (ii), and so recaptures the result

$$\Delta(N, Q) \leqslant c_4(N + Q^3)$$

of Rényi.

**5.** In Theorem 2 we are interested in functions $\varrho_1$ and $\varrho_2$ of $N$ and $T$, so that

$$(\gamma) \qquad \int_{-T}^{T} \left|\sum_{n=1}^{N} a_n n^{-it}\right|^2 dt \leqslant \varrho_1 \sum_{n=1}^{N} |a_n|^2,$$

holds for all complex numbers $a_1, \ldots, a_N$, and

$$(\delta) \qquad \sum_{n=1}^{N} \left|\int_{-T}^{T} f(t) n^{-it} dt\right|^2 \leqslant \varrho_2 \int_{-T}^{T} |f(t)|^2 dt$$

holds for all functions of the Lebesgue class $L^2(-T, T)$.

We see that, as in § 4, Lemma 1, we have without loss of generality $\varrho_1 = \varrho_2$. For here we have case $s = 0$ of that lemma, and the proof uses only the inequality of Cauchy–Schwarz, with no appeal to the Courant–Fisher theorem.

In $(\gamma)$ we have a Hermitian form

$$\sum_{m=1}^{N} \sum_{n=1}^{N} a_m \bar{a}_n \int_{-T}^{T} (mn^{-1})^{-it} dt$$

once again, so that we can take for $\varrho_1$ the greatest eigenvalue of the associated matrix.

In $(\delta)$ we have a (linear) operator from the space $L^2(-T, T)$ into itself, given by

$$A\colon f(x) \to \int_{-T}^{T} f(u) g(u, x)\, du$$

with

$$g(u,v) = \sum_{n=1}^{N} n^{-i(u-v)}.$$

With the usual definition of inner product $L^2(-T,T)$ becomes a Hilbert space, and our operator satisfies for functions $f_1, f_2$ of the space

$$(Af_1, f_2) = (f_1, Af_2)$$

and so is Hermitian. As with all self-adjoint operators, the norm of $A$

$$\varrho_2 = \sup_{\|f\|=1} \|Af\| = \sup_{f\neq 0} \|f\|^{-1}\|Af\|$$

belongs to the spectrum of $A$, which in particular is a closed set. Moreover, since $g(u,v)$ is continuous on $[-T,T]\times[-T,T]$, the operator $A$ is completely continuous (transforms bounded sets into compact sets), and its spectrum consists entirely of eigenvalues (Halmos [11]). Hence, in our second case, also, $\varrho_2$ can be taken to be the largest eigenvalue of the operator concerned.

**6. Proof of Theorem 2.** (i) (cf. Titchmarsh [20], Theorem 7.1, pp. 115–116, where an application of the Cauchy–Schwarz inequality yields a result of type (i) with an error term $O(N^{3/2})$).

We have

$$\int_{-T}^{T} \Big|\sum_{n=1}^{N} a_n n^{-it}\Big|^2 dt = \boldsymbol{a}G\bar{\boldsymbol{a}}^T,$$

where a typical component of the $N\times N$ matrix $G$ is

$$\int_{-T}^{T} (mn^{-1})^{-it} dt = \begin{cases} 2T & \text{if} \quad m = n, \\ O\left(\dfrac{1}{\left|\log \dfrac{m}{n}\right|}\right) & \text{if} \quad m \neq n. \end{cases}$$

Exactly as in § 3

$$\varrho N \geqslant \operatorname{trace} G = 2TN$$

so that $\varrho \geqslant 2T$.

Each eigenvalue of $G$ lies in a Gershgorin disc,

$$|\lambda - 2T| \leqslant \sum_{\substack{m=1\\ m\neq n}}^{N} c_1 \left|\log\frac{m}{n}\right|^{-1} \leqslant c_1 N\log N$$

for some value of $n$, $1\leqslant n\leqslant N$. This proves part (i).



Before proceeding to the proof of part (ii) of Theorem 2 we need two results.

The integral operator $A$ has a kernel $g(x-y)$ where

$$g(w) = \sum_{n\leqslant N} n^{-iw}.$$

We give an estimate for this function.

LEMMA 2.

$$g(w) = \frac{N^{1-iw}}{1-iw} + O\Big(N^{\sigma}\big(\max(N, |w|+2)\big)^{\frac{1}{2}(1-\sigma)+\varepsilon}\Big)$$

*holds for each fixed* $\varepsilon > 0$, *uniformly for* $0\leqslant\sigma\leqslant 1$, $N\geqslant 2$ *and all real* $w$.

Thus if $N\geqslant |w|+2$, then $g(w) = N^{1-iw}(1-iw)^{-1} + O(N^{1/2+\varepsilon})$ holds for any fixed $\varepsilon > 0$.

Proof. Let $N_1$ be the nearest rational number of the form $m+\frac{1}{2}$, $m$ integral, to $N$. If we replace $N$ by $N_1$ in the definition of $g(w)$ we change its value by at most an absolutely bounded amount. Moreover, it is clear that

$$\frac{1}{1-iw}(N^{1-iw} - N_1^{1-iw}) = \int_{N_1}^{N} z^{-iw} dz \ll 1.$$

It will therefore suffice to prove the lemma with $N_1$ in place of $N$. For the remainder of the proof of this lemma we therefore assume that $N = N_1$.

By a standard application of a theorem of Perron (see, for example, Titchmarsh [20], pp. 53–55) we have

$$(\mu) \qquad \sum_{n\leqslant N} n^{-iw} = \frac{1}{2\pi i}\int_{c-iD}^{c+iD} \zeta(iw+s)\frac{N^s}{s} ds + O\left(\frac{N^c}{D(c-1)} + \frac{N\log N}{D}\right)$$

uniformly for $c > 1$, $D\geqslant 2$. We choose $c = 1+(\log N)^{-1}$, so that both of the error terms here are

$$\ll D^{-1} N\log N.$$

The integrand has a simple pole at the point $s = 1-iw$. Assuming that $D\geqslant |w|+2$ we move the line-segment $(c+it, |t|\leqslant D)$ to

$$c-iD \xrightarrow{L_1} \sigma - iD \xrightarrow{L_2} \sigma + iD \xrightarrow{L_3} c+iD,$$

where $\sigma$ satisfies $0<\sigma<1$. We then pass over the above pole, and from Cauchy's theorem obtain for the integral in $(\mu)$ the estimate

$$\frac{N^{1-iw}}{1-iw} + \sum_{j\leqslant 3}\frac{1}{2\pi i}\int_{L_j} \zeta(iw+s)\frac{N^s}{s} ds.$$

We now appeal to the inequality

$$\zeta(a+it) \ll (|t|+2)^{\frac{1}{2}(1-a)+\varepsilon}$$

which is valid for any fixed $\varepsilon > 0$, in the region $0 \leqslant a \leqslant 1$, $|a+it-1| \geqslant \frac{1}{2}$ (see for example Titchmarsh [20], Chapter V). By means of this inequality and the fact that $D \geqslant |w|+2$ we see that

$$\frac{1}{2\pi i} \int_{L_1 \cup L_3} \zeta(iw+s) \frac{N^s}{s} ds \ll D^{-1} N^c D^{\frac{1}{2}(1-\sigma)+\varepsilon} \ll ND^{-\frac{1}{2}-\sigma/2+\varepsilon}.$$

Moreover, the corresponding integral over the line-segment $L_2$

$$\ll D^{\frac{1}{2}(1-\sigma)+\varepsilon} N^{\sigma},$$

so that altogether

$$g(w) = \frac{N^{1-iw}}{1-iw} + O\left(\frac{N \log N}{D} + \frac{N}{D^{\frac{1}{2}(1-\sigma)-\varepsilon}} + N^{\sigma} D^{\frac{1}{2}(1-\sigma)+\varepsilon}\right)$$

holds uniformly for all $D \geqslant 2$, and $0 \leqslant \sigma \leqslant 1$. Apart from the factor $D^{\varepsilon} \log N$ the second of these three error terms is larger than the first, and for $D \ll N$ is smaller than the third. Choosing $D = \max(N, |w|+2)$ we obtain the result stated in the lemma.

Our second preliminary result is an analogue of Gershgorin's theorem.

LEMMA 3. *Let $\lambda$ be an eigenvalue of the operator $L^2(-T, T) \to L^2(-T, T)$ which is given by*

$$f(x) \to \int_{-T}^{T} f(u) h(x, u) du,$$

*where $h(x, y)$ is continuous on the square $[-T, T]^2$. Then $\lambda$ satisfies*

$$|\lambda| \leqslant \sup_{|x| \leqslant T} \int_{-T}^{T} |h(x, u)| du.$$

Proof. We can clearly assume that $\lambda \neq 0$. Let $g(x)$ be an eigenfunction corresponding to $\lambda$. Then $g(x)$ is essentially bounded, and we set

$$M = \operatorname*{ess\,sup}_{|x| \leqslant T} |g(x)|.$$

If $M = 0$ then $\lambda g(x)$ is identically zero, and since $g(x)$ is not identically zero, we have $\lambda = 0$, contrary to assumption. Hence $M > 0$, and for each positive number $\varepsilon_1$, $0 < \varepsilon_1 < 1$, we can find a point $z$ so that $|g(z)| > (1-\varepsilon_1) M$. Hence

$$|\lambda|(1-\varepsilon_1) M \leqslant \int_{-T}^{T} |f(u)|\, |h(z, u)| du \leqslant M \sup_{|x| \leqslant T} \int_{-T}^{T} |h(x, u)| du,$$

and since $\varepsilon_1$ can be taken arbitrarily small, the lemma is proved.



Consider now the functions

$$g_1(w) = \frac{N^{1-iw}}{1-iw}, \quad g_2(w) = g(w) - g_1(w),$$

and their corresponding integral operators:

$$A_j: f(x) \to \int_{-T}^{T} g_j(x-y) f(y) dy \quad (j = 1, 2).$$

Let $\varrho$ $(= \varrho_1 = \varrho_2)$, $\lambda_1$ denote the largest eigenvalues of the operators $A$ and $A_1$ respectively. Then we assert that

(v) $$\varrho = \lambda_1 + O(N^{\frac{1}{2}+\varepsilon} T)$$

*holds for any fixed $\varepsilon > 0$, uniformly for all $N \geqslant 2T+4$.* To see this we apply Lemma 4 to the operator $A_2$, to deduce that every eigenvalue $\mu$ of $A_2$ satisfies

$$|\mu| \leqslant \sup_{|x| \leqslant T} \int_{-T}^{T} |g_2(x-y)| dy \leqslant \int_{-2T}^{2T} |g_2(w)| dw.$$

Since $2T+2 \leqslant N$, Lemma 2 shows that for any fixed value of $\sigma$ which satisfies $0 < \sigma < 1$ the integrand in this final integral is uniformly

$$\ll N^{\sigma + \frac{1}{2}(1-\sigma) + \varepsilon/2}.$$

Choosing $\sigma = \varepsilon$ we see that the spectrum of $A_2$ lies entirely with a disc, centered at the origin, of radius $O(N^{\frac{1}{2}+\varepsilon} T)$. Taking $f(x)$ to be in turn a suitable eigenfunction of $A$, and then of $A_1$, we justify the assertion (v).

For large values of $N$ in comparison with $T$, the case in which we are interested, the study of the operator $A$ therefore reduces to the study of the operator

$$A_1: f(x) \to \int_{-T}^{T} \frac{N^{1-i(x-y)}}{1-i(x-y)} f(y) dy.$$

Upper bound for $\lambda_1$. We obtain the upper bound $\lambda_1 \leqslant (2\pi)^{3/2} N$ by interpreting $A_1$ as two successive Fourier transforms. For any function $f(x)$ of the class $L^2(-T, T)$ we can write the action of $A_1$ in the form

$$f(x) \to \int_0^N z^{-ix} \left( \int_{-T}^{T} z^{iy} f(y) dy \right) dz,$$

the inversion in the order of integration being justified by an application of Fubini's theorem. Letting $z = e^w$ we express this double integral in the form

$$\int_{-\infty}^{\log N} e^{w(1-ix)} \left( \int_{-T}^{T} e^{-iwy} f(z) dy \right) dw.$$

Set

$$f_1(y) = \begin{cases} \sqrt{2\pi} f(y) & \text{if } |y| \leqslant T, \\ 0 & \text{otherwise}. \end{cases}$$

Then $f_1(y)$ belongs to both the class $L(-\infty, \infty)$ and the class $L^2(-\infty, \infty)$. Moreover, in the usual notation

$$\frac{1}{\sqrt{2\pi}} \int_{-T}^{T} e^{iwy} f(y)\, dy = \hat{f}_1(w),$$

the $L(-\infty, \infty)$ Fourier transform of the function $f_1(y)$. By an argument of F. Riesz (see Titchmarsh [79], pp. 75–76), for example, one deduces that $\hat{f}_1(z)$ belongs to the class $L^2(-\infty, \infty)$, and that

$$\int_{-\infty}^{\infty} |\hat{f}_1(y)|^2 dy \leqslant \int_{-\infty}^{\infty} |f_1(y)|^2 dy = 2\pi \int_{-T}^{T} |f(y)|^2 dy.$$

We now have

$$f(x) \to \int_{-\infty}^{\log N} e^{w(1-ix)} \sqrt{2\pi} \hat{f}_1(w)\, dw.$$

Further set

$$f_2(w) = \begin{cases} 2\pi e^{w} \overline{\hat{f}_1(w)} & \text{if } w \leqslant \log N, \\ 0 & \text{otherwise}. \end{cases}$$

Then

$$A_1 f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-wix} \overline{\hat{f}_2(w)}\, dw = \hat{f}_2(x)$$

where the function $f_2(w)$ belongs to the classes $L(-\infty, \infty)$ and $L^2(-\infty, \infty)$, so that $\hat{f}_2(w)$ certainly belongs to the class $L^2(-\infty, \infty)$. Taking norms we obtain

$$\begin{aligned} \|A_1 f\| \ (\text{in } L^2(-T, T)) &\leqslant \|\hat{f}_2\| \ (\text{in } L^2(-\infty, \infty)) \\ &\leqslant \|f_2\| \ (\text{Bessel's inequality in } L^2(-\infty, \infty)) \\ &= \Big(\int_{-\infty}^{\log N} 4\pi^2 e^{2y} |\hat{f}_1|^2 dy\Big)^{1/2} \leqslant 2\pi N \Big(\int_{-\infty}^{\infty} |\hat{f}_1|^2 dy\Big)^{1/2} \\ &\leqslant 2\pi N \|\hat{f}_1\| \ (\text{in } L^2(-\infty, \infty)) \\ &\leqslant 2\pi N \|f_1\| \ (\text{Bessel's inequality in} \\ &\qquad\qquad L^2(-\infty, \infty) \text{ once again}) \\ &\leqslant (2\pi)^{3/2} N \|f\| \ (\text{in } L^2(-T, T)). \end{aligned}$$



Since the operator $A_1$ is completely continuous Hermitian we deduce that

$$\lambda_1 \leqslant (2\pi)^{3/2} N.$$

Lower bound for $\lambda_1$. We shall prove that for $T \geqslant 64$, $N \geqslant 2$,

$$\lambda_1 \geqslant \frac{N}{2\sqrt{\pi}}.$$

We consider the action of the operator $A_1$ on the function

$$f_0(x) = \frac{N^{1-ix}}{1-ix},$$

which belongs to the space $L^2(-T, T)$. We have

$$A_1 f_0 = N^{2-ix} \int_{-T}^{T} \frac{dy}{\big(1-i(x-y)\big)(1-iy)}.$$

For values of $x$ which satisfy $32|x| \leqslant T$ we replace the integral in this expression by the contour integral

$$J = \int_{L} \frac{dz}{\big(1-i(x-z)\big)(1-iz)}$$

taken over the line-segment $-T \leqslant \operatorname{Re} z \leqslant T$, $\operatorname{Im} z = 0$. The integrand is regular over the whole plane save at the points $z = -i$ and $z = i+x$, where it has simple poles, with respective residues

$$-\frac{1}{x+2i} \quad \text{and} \quad \frac{1}{x+2i}.$$

We apply Cauchy's theorem to a semicircle on the line-segment $L$ as a diameter (we can choose either side) to deduce that

$$J = \frac{1}{x+2i} + \int_{\Gamma} \frac{dz}{\big(1-i(x-z)\big)(1-iz)},$$

where $\Gamma$ consists of one half of the circle $|z| = T$. On this arc the inequalities

$$\begin{aligned} \big|(1-i(x-z))\big|\, |(1-iz)| &\geqslant \big(|z| - (1+|x|)\big)(|z|-1) \\ &\geqslant T^2\Big(1-\frac{1}{T}\Big)\Big(\frac{31}{32}-\frac{1}{T}\Big) \geqslant \frac{T^2}{8} \end{aligned}$$

are satisfied, so that

$$J = \frac{1}{x+2i} + \frac{8\vartheta}{T} \qquad (|\vartheta| \leqslant 1).$$

On taking norms in the space $L^2(-T, T)$ it follows that

$$\|A_1 f_0\|^2 \geqslant \int_{-T/32}^{T/32} N^4 |J|^2 dx$$

$$\geqslant \frac{N^4}{4} \int_{-T/32}^{T/32} \frac{dx}{|x+2i|^2} = \frac{N^4}{4}\left(\int_{-\infty}^{\infty} \frac{dx}{4+x^2} - 2\int_{T/32}^{\infty} \frac{dx}{4+x^2}\right) \geqslant \frac{N^4}{2}.$$

On the other hand

$$\|f_0\|^2 \leqslant N^2 \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi N^2.$$

Hence

$$\lambda_1 \geqslant \|A_1 f_0\| \, \|f_0\|^{-1} = \frac{N}{2\sqrt{\pi}},$$

as was asserted.

In view of our estimate (v) the proof of part (ii) of Theorem 2 is now complete.

**7. Concluding remarks.** One can reformulate the problems of the type considered in Theorem 2 in the following manner. One could ask for the best value of $\mu$ so that the inequality

$$\int_{-T}^{T} \Big|\sum_{n=1}^{\infty} a_n e^{-n/N} n^{-it}\Big|^2 dt \leqslant \mu \sum_{n=1}^{\infty} |a_n|^2$$

holds for all vectors $\boldsymbol{a} = (a_1, a_2, \ldots)$ with $\|\boldsymbol{a}\|^2 = \sum |a_n|^2 < \infty$. Thus the vectors $\boldsymbol{a}$ belong to a Hilbert space. The presence of the exponential factors on the left hand side lays emphasis on the early coordinates $a_n$ with $n \ll N$ of $\boldsymbol{a}$, whilst being technically convenient to use. The transferred problem then becomes that of the integral operator $B\colon L^2(-T, T) \to L^2(-T, T)$ given by

$$f(x) \to \int_{-T}^{T} k(x-y) f(y)\, dy,$$

where

$$k(w) = \sum_{n=1}^{\infty} e^{-n/N} n^{-iw}.$$

This kernel can be computed in a manner analogous to that used in Lemma 2, using the fact that

$$k(w) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \zeta(s)\Gamma(s-iw) N^{2-iw} ds.$$



Here $\Gamma(s)$ denotes the usual analytic continuation of the gamma function of Euler. The presence of this function enables one to move the line of integration $\mathrm{Re}\, s = 2$ over into the left half-plane $\mathrm{Re}\, s < 0$ to deduce that

$$k(w) = \Gamma(1-iw) N^{1-iw} + \zeta(iw) + O\big(N^{-1+\varepsilon}(|w|+2)^{3/2}\big)$$

holds for any fixed $\varepsilon > 0$, uniformly for all real $w$. In this representation the dependence of $k(w)$ upon $N$ is clear. An upper bound for $\mu$ is easily obtained using the analogue of Gershgorin's theorem. To obtain a lower bound one can consider the action of the operator upon the function

$$f_0(x) = \Gamma(1+ix) N^{1+ix}$$

and proceed as for the estimation of $\lambda_1$, using the fact that

$$\int_{-\infty}^{\infty} \Gamma(1+it) N^{1+it} n^{-it} e^{-n/N} dt = 2\pi n \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Gamma(s) \left(\frac{n}{N}\right)^{-s} e^{-n/N} ds = 2\pi n e^{-2n/N}.$$

However, for large values of $T$, but still small in comparison with $N$, one might expect that the operator $B$ would behave somewhat like the operator $C\colon L^2(-\infty, \infty) \to L^2(-\infty, \infty)$ given by

$$f(x) \to \int_{-\infty}^{\infty} \Gamma\big(1-i(u-x)\big) f(u)\, du.$$

This operator is bounded and Hermitian. It is amenable to iteration, since (see for example Titchmarsh [19], 7.7.9, p. 192)

$$\frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s)\Gamma(a-s) x^{-s} ds = \frac{\Gamma(a)}{(1+x)^a}, \quad 0 < k < R(a)$$

that is to say $(1+x)^{-a}$ and $\Gamma(s)\Gamma(a-s)\big(\Gamma(a)^{-1}\big)$ are Mellin transforms if $0 < \mathrm{Re}\, s < \mathrm{Re}\, a$. One can therefore expect to approach $\mu$ closely by means of the Euclidean norm of the iterations of the operator $C$.

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# Ovali ed altre curve nei piani di Galois di caratteristica due

di

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*Il presente lavoro viene dedicato con profonda ammirazione alla memoria degli eminenti matematici H. Davenport e W. Sierpiński*

**Prefazione.** L'introduzione e lo studio dei $k$-archi di un piano $S_{2,q}$ di Galois e delle loro estensioni agli spazi superiori devesi essenzialmente a B. Segre ed a suoi discepoli. Ne sono derivate le cosiddette *geometrie di Galois*, vari aspetti salienti delle quali trovansi esposti nella nota [24](1) e nella monografia [27] (in quest'ultima, accanto a nuovi risultati), le quali contengono altresì un'ampia bibliografia sull'argomento, a cui rinviamo con l'aggiunta dei lavori elencati alla fine della presente Memoria.

Un *k-arco* è un insieme di punti di $S_{2,q}$, a tre a tre non allineati; esso denominasi un'*ovale* quando $k$ sia tale che in $S_{2,q}$ non esista nessun $(k+1)$-arco. Riguardo a queste ultime, occorre distinguere due casi a seconda che $q = p^h$ è dispari o pari, ossia a seconda che il numero primo $p$ è maggiore od eguale a 2.

Mentre nel primo caso risulta $k = q+1$ ed ogni $(q+1)$-arco, come insieme di punti, è quello dei punti di una conica non singolare di $S_{2,q}$, e viceversa ([21]; [25], nn. 173–174), nel secondo caso — e cioè se $q = 2^h$ — per un'ovale si ha $k = q+2$, la struttura algebrica dei $(q+2)$-archi (e delle loro estensioni agli spazi superiori) essendo però in generale assai più complessa che nel caso dispari e ben lungi dall'essere pienamente nota.

Più precisamente, qualunque sia $q = 2^h$, si ottiene intanto un'ovale di $S_{2,q}$ coll'aggregare ai punti di una conica non singolare di $S_{2,q}$ il nucleo di questa. Tuttavia ([25], n. 178), mentre per $h = 1, 2, 3$ non vi sono altre ovali all'infuori degli insiemi così definiti, nell'ipotesi che sia $h > 3$ — ad esclusione al più soltanto dei casi $h = 4$, $h = 6$, il primo dei quali è poi stato trattato direttamente con l'uso di un calcolatore elettronico, cfr. [18] — si hanno fra l'altro le *ovali* ottenibili in $S_{2,q}$ con l'aggregare

(1) I numeri tra [ ] rimandano alla bibliografia posta in fine del lavoro.