The Markoff spectrum*

by

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Dedicated to Harold Davenport

1. Introduction. An indefinite binary quadratic form \( f(x, y) = ax^2 + bxy + cy^2 \) of positive discriminant \( \Delta \) for integers \((x, y)\) not \((0, 0)\) always takes on a minimum less than or equal to \( \sqrt{\Delta} \), and equality is necessary for forms equivalent to \( ax^2 - bxy - cy^2 \), and for all other forms the minimum is at most \( \sqrt{\Delta} \). Markoff [6] showed for \( M(f) \) the lower bound of \( |f(x, y)| \) over integers \((x, y) \neq (0, 0)\), with \( M(f) = m\sqrt{\Delta} \), that only a countable number of values greater than \( 1/3 \) are possible for \( m \), and that in these cases the minimum is attained. He also describes exactly these forms and their minima, which are called the Markoff chain. There are excellent accounts of this by Cassels [1], [2] and by Dickson [4].

The set of values of \( m = M(f)/\sqrt{\Delta} \) is called the Markoff spectrum. In this paper it is shown that if \( M(f) \) is not attained for a form \( f \), there is another form \( f^* \) of the same discriminant with \( M(f^*) = M(f) \) for which \( M(f^*) \) is attained. Hence in studying the spectrum we may consider only those forms which attain their minimum. It is also shown that the spectrum contains every positive number \( m \leq 1/5.1007 \). In addition it is shown that minima \( m \geq 1/\sqrt{10} \) form a set of measure zero. Between \( 1/\sqrt{10} \) and \( 1/\sqrt{21} \) there are gaps in the spectrum. For instance it has long been known that there is a gap between \( 1/\sqrt{12} \) and \( 1/\sqrt{13} \), but there are further gaps between \( 1/\sqrt{13} \) and \( 1/5.1007 \).

2. Let \( f(x, y) = ax^2 + bxy + cy^2 \) be a real indefinite binary quadratic form of positive discriminant \( \Delta = b^2 - 4ac \). We are interested in the minimum of \( f, M(f) \) defined as

\[
M(f) = \inf_{(x,y) \neq (0,0)} |f(x, y)|, \quad x, y \text{ integers.}
\]

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If \( t \neq 0 \) is a real number then \( M(tf) = \|M(f)\| \) and the discriminant of \( tf \) is \( t^2 \alpha \). Hence the quantity \( M(f)/V^2 \alpha \) is the same for forms differing by a constant factor of proportionality, and it is this ratio \( M(f)/V^2 \alpha \) which we consider the minimum of \( f \).

As is customary we say that \( f(x, y) = ax^2 + bxy + cy^2 \) and \( f_1(x_1, y_1) = a_1x_1^2 + b_1x_1y_1 + c_1y_1^2 \) are equivalent if there is a transformation \( T \)

\[
T: \quad x = \tau x_1 + \eta y_1, \quad y = \xi x_1 + \nu y_1,
\]

where \( \tau, \eta, \xi, \nu \) are integers such that \( T \) transforms \( f(x, y) \) into \( f_1(x_1, y_1) \).

Clearly, \( M(f_1) = M(f) \).

Associated with the form \( f \) are its two roots \( \theta_1, \theta_2 \), defined by

\[
f(x, y) = a(x - \theta_1 y)(x - \theta_2 y)
\]

where \( \theta_1 \) and \( \theta_2 \) are real and distinct since \( \alpha > 0 \). If either root is rational, then from (2.3) we can find integers \( (x, y) \neq (0, 0) \) such that \( f(x, y) = 0 \) and so \( M(f) = 0 \). From now on we shall assume that neither root is rational. Following Dickson [3], [4] we say that \( f(x, y) \) is reduced if numbering \( \theta_1, \theta_2 \) appropriately

\[
\theta_1 > 1, \quad -1 < \theta_2 < 0.
\]

Hence \( \theta_1 \) and \( \theta_2 \) can be represented by infinite continued fractions (being irrational) and

\[
\theta_1 = [b_0, b_1, b_2, \ldots],
\]

\[
-\theta_2 = [0, b_0, b_1, b_2, \ldots],
\]

where the \( b_i \) are positive integers.

If we apply the transformation \( T \) to \( f \) where

\[
T: \quad x = b_0x_1 + y_1,
\]

then \( f_1(x_1, y_1) \) is also reduced and for its roots

\[
\varphi_1 = [b_2, b_3, b_4, \ldots],
\]

\[
-\varphi_2 = [0, b_0, b_1, \ldots].
\]

The general theory, due originally to Lagrange may be found in Dickson [3], [4] and asserts that if \( f(x, y) \) and \( g(x, y) \) are equivalent reduced forms, then if the roots of \( f(x, y) \) are \( \theta_1, \theta_2 \) as given by (2.5) and if the roots of \( g(x, y) \) are \( \varphi_1, \varphi_2 \) given by

\[
\varphi_1 = [c_0, c_1, c_2, \ldots],
\]

\[
-\varphi_2 = [0, c_1, c_2, \ldots],
\]

then necessarily there is an integer \( n \) such that

\[
\varphi_i = b_{i+n} \theta_i, \quad \text{for all } i.
\]

Application of transformations \( T \) as in (2.6) shows the converse to be true.

Furthermore from (2.3)

\[
d = a^2(\theta_1 - \theta_2)^2
\]

so that

\[
|a| = V^2 \alpha/|\theta_1 - \theta_2|.
\]

The general theory asserts that every number \( m \) properly represented by \( f \) where \( |m| < V^2 \alpha/2 \) is the leading coefficient of a reduced form equivalent to \( f_1 \) and so \( |m| = |a| \) as represented in (2.11).

The following theorem summarizes what we shall need of the general theory.

**Theorem 2.1.** Let \( f(x, y) = ax^2 + bxy + cy^2 \) be a real binary quadratic form with positive discriminant \( \alpha = b^2 - 4ac \), and suppose also that \( f(x, y) \neq 0 \) for integers \( (x, y) \neq (0, 0) \), and let \( M(f) = \inf|f(x, y)| \), for integers \( (x, y) \neq (0, 0) \). Then there is a doubly infinite sequence \( S \) of positive integers

\[
S: (\ldots, b_{-2}, \ldots, b_{-1}, b_0, b_1, \ldots, b_1, \ldots)
\]

such that if we form the sum \( S_i \) of the two continued fractions

\[
S_i = [b_0, b_{i+1}, \ldots] + [0, b_{i-1}, b_{i-2}, \ldots]
\]

for every \( i \), then \( M(f)/V^2 \alpha = \inf(1/S_i) \). Consecutively a sequence \( S \) defines a class of equivalent forms.

We can now show that in studying the values of \( M(f)/V^2 \alpha \) we may restrict our attention to forms which attain their minimum.

**Theorem 2.2.** If \( m = M(f)/V^2 \alpha \) is the minimum of a form \( f \), there is a form \( f^* \) which attains the minimum \( m \).

**Proof.** If \( m = 0 \), the form \( f^*(x, y) = x^2 - y^2 \) attains this minimum.

If \( m > 0 \) then the \( b_i \)'s in the sequence \( S \) are bounded. Now if \( f(x, y) \) does not attain its minimum for a particular choice of \( i \) in the sum \( S_i \), then there is an infinite sequence \( E_n \) of \( E_n \) in which the central integer \( b_n \) has the same value \( c_i \). Then \( E_n \) has a subsequence \( E_n' \) in which \( b_n = c_0 \) and \( b_{n+1} = c_1 \) for a fixed \( c_1 \). Then

\[
\lim_{n \to \infty} S_n' = \frac{1}{m}.
\]

As the \( b_i \)'s are bounded integers, there is an infinite subsequence \( E_n' \) of \( E_n \) in which the central integer \( b_n \) has the same value \( c_i \). Then \( E_n' \) has a subsequence \( E_n'' \) in which \( b_n = c_0 \) and \( b_{n+1} = c_1 \) for a fixed \( c_1 \). Then
in turn \( E_n \) has an infinite subsequence \( E'_n \) in which \( b_{n-1} = c_{-1} \), \( b_n = c_0 \), and \( b_{n+1} = c_1 \). Continuing there is a doubly infinite sequence

\[
\{ \ldots, c_{-3}, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots \}
\]

with the property that \( b_{n-r} = c_{-r} \), \( \ldots \), \( b_{n-1} = c_{-1} \), \( b_n = c_0 \), \( \ldots \), \( b_{n+r} = c_r \),

for every \( r \), occurs in an infinite subsequence of \( E'_n \). Hence

(3.12)

\[
1/m = \left[ c_0, c_1, c_2, \ldots \right] + [0, c_{-1}, c_{-2}, \ldots]
\]

and for the form \( f(x, y) \) associated with \( S' \) the value \( m = f^*(x, y)\sqrt{d} \) is attained. Furthermore as every finite section of \( S' \) \( c_{-2}, \ldots, c_0 \) is also a section of \( S \) we have \( S' \leq 1/m \) in every case so that \( m \) is in fact the minimum of \( f^*(x, y)\sqrt{d} \).

3. The Markoff spectrum

Definition. The Markoff spectrum is the set of real numbers

\[
m = M(f)/\sqrt{d}
\]

corresponding to all real indefinite binary quadratic forms \( f(x, y) \).

The Markoff chain is a sequence of forms for which \( m \) takes on its largest values, namely all \( m > 1/3 \).

Theorem 3.1 (Markoff [8]). Let \( f(x, y) = ax^2 + bxy + cy^2 \) be an indefinite quadratic form with real coefficients and discriminant \( d = b^2 - 4ac \), and let \( M = M(f) \) be the lower bound of \( |f(x, y)| \) over all integer pairs \( (x, y) \neq (0, 0) \). Then \( M \leq \sqrt{d}/\sqrt{5} \), the sign of equality being necessary for

\[
f = M(a^2 - ax - y^2).
\]

If \( f \) is not equivalent to \( f \), then \( M \leq \sqrt{d}/\sqrt{8} \), the sign of equality being necessary for

\[
f = M(a^2 - 2axy - y^2).
\]

If \( f \) is not equivalent to \( f \) or \( f \), then \( M \leq 5\sqrt{d}/\sqrt{221} \), the sign of equality being necessary for

\[
f = M/5(5a^2 - 31xy - 6y^2).
\]

If \( f \) is not equivalent to \( f \), \( f \), or \( f \), then \( M \leq 13\sqrt{d}/\sqrt{517} \), the sign of equality being necessary for

\[
f = M/13(13a^2 - 29xy - 13y^2),
\]

and so on. The set \( f, f, f, \ldots, f \) continues indefinitely and every \( f \) such that \( M > \sqrt{d}/3 \) is equivalent to some \( f \).

For the proof of this result, the reader is referred to Dickson [3, 4], or Cassels [13, 2]. The forms are exhibited explicitly and every form attains its lower bound.

The Markoff chain describes that part of the Markoff spectrum for which \( m > 1/3 \). It will be proved here that every positive number below 1/5.1007 is in the Markoff spectrum.

Theorem 3.2. If \( 0 < m < 1/3 \), \( s = 4 + \sqrt{21} - 3 \), \( \sqrt{21} - 5 = 5.1006890 \)
then there is a real indefinite form \( f(x, y) \) for which \( M(f)/\sqrt{d} = m \).

Proof. Froeman and Yudin [9] have shown that if \( D \) is the set of continued fractions

\[
[0, a_1, a_2, \ldots]
\]

with \( a_i = 1, 2, 3, 4 \) such that there is no sequence \( a_i, a_{i+1} \) of the form \( 1, 4 \) or \( 2, 4 \), then every number \( z \) with \( 5 - \sqrt{21} \leq z < 5 \) is of the form

\[
z = a_1 + a_2 + a_3 + a_4 + \ldots
\]

From this they were able to show that the Markoff spectrum contained all numbers \( 0 < m < 1/3 \), with \( s = 5.118 \). This theorem is a slight improvement on theirs.

Choose a positive integer \( a_0 \) and define the doubly infinite sequence \( S \) as

\[
S = \{ \ldots, a_1 - 1, a_1, \ldots, a_0, a_1, \ldots \}
\]

where each of \([0, a_1, a_2, \ldots]\) and \([0, a_{i+1}, a_{i+2}, \ldots]\) is a continued fraction of the set \( D \). Then with

\[
S_i = [a_1, a_{i+2}, \ldots] + [0, a_{i+1}, a_{i+2}, \ldots]
\]

we have

\[
m = M(f)/\sqrt{d} = \min(1/S_t).
\]

Choosing \( a_0 = n \), then from (3.1) we may choose \( a_1 \) and \( a_2 \) for \( D \) so that \( S_0 \) is any number in the interval

\[
n + 5 - \sqrt{21} \leq S_0 \leq n + \sqrt{21} - 3
\]

and we note that \( 5 - \sqrt{21} = 0.1472422, \sqrt{21} - 3 = 1.5825757 \) so that this interval is of length greater than 1, so that taking \( n = 5, 6, 7, \ldots \) we obtain every number greater than \( 11 - \sqrt{21} = 5.1472422 \) as a value of \( S_0 \). If \( \delta \neq 0 \) and \( a_3 = 1, 2, 3, \) then \( S_i < 3 + 1 + 1 = 5 \), while if \( a_1 = 4 \) then since there is no sequence \( 1, 4 \) or \( 2, 4 \) in \( D \), then \( a_3 = 1, 2 \) if \( \delta \) is positive while, \( a_{i+1} = 1, 2 \) if \( \delta \) is negative, and so

\[
S_i \leq 4 + [0, 1, \ldots] + [0, 3, \ldots] = 5.3333 < S_0.
\]
Thus in all these cases \( n = 1 / s_0 \) and we have every \( m \) possible \( 0 < m < 1 / s \), \( s = 11 - \sqrt{21} \). Now let \( D_1 \) be the subset of \( D \) for which \( \alpha_1 = 1 \) and \( D_2 \) the subset of \( D \) for which \( \alpha_1 = 2 \). Then for \( \alpha \in D_1 \)

\[
[0, 1, 1, 3, 1] = \frac{\sqrt{21}+1}{10} = .5582576,
\]

and for \( \alpha \in D_2 \)

\[
[0, 1, 3, 1, 3] = \frac{\sqrt{21}-3}{2} = .7912878,
\]

\[
\frac{\sqrt{21}+1}{10} \leq \alpha \leq \frac{\sqrt{21}-3}{2}
\]

and for \( \alpha \in D_3 \)

\[
[0, 2, 1, 3, 1] = \frac{\sqrt{21}-1}{10} = .3582576,
\]

\[
[0, 2, 3, 1, 3] = \frac{9-\sqrt{21}}{10} = .4417424,
\]

\[
\frac{\sqrt{21}-1}{10} \leq \alpha \leq \frac{9-\sqrt{21}}{10}.
\]

We now form the doubly infinite sequence \( S \) by taking \( s_0 = 4 \), and \( s = [0, a_1, a_2, ...] \) from \( D_1 \) and \( s = [0, a_1, a_2, ...] \) in the first instance from \( D_3 \) and in the second instance from \( D_2 \). Alternatively we might take \( a_1 \) from \( D_3 \) and \( a_2 \) from \( D_1 \). In the first instance the values of \( s_0 \) are in the interval from \( (\sqrt{21}+1)/5 = 5.1185151 \) to \( 1+\sqrt{21} = 5.5825737 \). In the second instance the values are in the interval from \( (20+\sqrt{21})/5 = 9.1665151 \) to \( (17+21)/5 = 5.3305036 \). Pfeiffer and Yudin have shown that \( D_1 \) and \( D_2 \) may be obtained by Cantor subdivisions in which the length of the middle interval removed is shorter than either interval remaining. Since the length of \( D_3 \) (0.8348488) is greater than one-third the length of \( D_3 \) (2.330502), it follows from Theorem 2.2 of the author's paper [5] that \( s_0 = 4 + a_1 + a_2 \) takes on every value in the interval in both instances.

To complete the proof of the theorem we need to show that if

\[
S_0 = \frac{\sqrt{21}-3}{3} + \frac{1498-6}{3} = 5.1606890
\]

then for \( \delta > 0 \), \( S_0 < S_0 \). Here if \( a_1 = 1, 2, 3 \) then \( S_1 < 3 + 1 + 1 = 5 < S_0 \) and so we need only consider cases with \( a_1 = 4 \). Again if \( a_1 = 5 \) and \( a_1 + 1 \) then \( S_1 < 4 + 1 = 5 < S_0 \). Thus we need only consider cases in which \( a_1 = 1 \) or \( a_1 = 4 \) is 1. But as 1, 4 does not arise in \( D_1 \) or \( D_2 \), this must have arisen from a sequence \( 4, 1 \) in \( D_3 \) or \( D_4 \). Hence if \( i \) is positive \( a_{i-1} \) may be 1 but \( a_{i-1} = 1 \), while if \( i \) is negative \( a_{i-1} \) may be 1 but \( a_{i+1} \). Let us suppose that \( i \) is positive since replacing \( a \) by \( a_{i-j} \) throughout in \( S_0 \) does not alter the values of \( S_0 \) to be considered. Thus \( a = 4, a_{i-1} \) with \( i > 0 \). Since \( a_1 = 1, 2 \), and \( D_1 \) and \( D_2 \) do not contain a sequence \( 1, 4 \) or \( 2, i \) \( \neq 1 \) and \( i \neq 2 \), so that \( \delta \geq 3 \).

\[
[4, 1, a_{i-1}, \ldots] = [4, 1, \delta] = \frac{21-3}{2} = 4.7912879.
\]

Since \( i \geq 3 \) \( a_{i-1}, 4 \) is a sequence in \( D_3 \) or \( D_4 \) so that \( a_{i-1} \neq 1, 2 \). If \( a_{i-1} = 4 \), then

\[
[0, a_{i-1}, \ldots] = [0, 4, \ldots] < .25
\]

and here \( S_0 < 5.0412789 < S_0 \). If \( a_{i-1} = 3 \), then I assert

\[
[0, 3, a_{i-1}, \ldots] < [0, 3, 4] = \frac{1498-6}{3}.
\]

This is certainly true if \( a_{i-2} = 1, 2, 3 \). If \( a_{i-3} = 4 \), then \( a_{i-2} \neq 3 \), since \( a_1 = 1, 2 \) and here \( a_{i-1} = 3 \). Thus with \( a_{i-2} = 4 \), and \( i-2 \geq 2 \) then \( a_{i-3} = 4 \) is a sequence in \( D_3 \) or \( D_4 \) and so \( a_{i-3} = 4 \). If \( a_{i-3} = 3 \) the inequality (3.6) certainly holds. Hence suppose \( a_{i-3} = 3 \). Continuing suppose

\[
[0, a_{i-3}, a_{i-2}, \ldots, a_{i-n}, a_{i-n-1}, \ldots = [0, 3, 4, \ldots, 3, 4, 4, a_{i-n-1}, \ldots]
\]

with \( a_{i-n-1} \neq 1, n \) odd, or

\[
[0, a_{i-3}, a_{i-2}, \ldots, a_{i-n}, a_{i-n-1}, \ldots = [0, 3, 4, \ldots, 3, 4, a_{i-n-1}, \ldots]
\]

with \( a_{i-n-1} \neq 3, n \) even. In (3.7) with \( a_{i-n-1} = 1, 2 \) or 3 the inequality (3.6) certainly holds. Since \( a_{i-1}, a_3, a_1 \) are \( 1, 4, 1 \), or \( 2, 4, 1 \), or \( 1, 4, 2 \), then in (3.8) \( i-n \) is positive and as \( a_{i-n} = 4 \), and \( D_3 \) and \( D_4 \) do not contain a sequence \( 1, 4 \) or \( 2, 4 \), then \( i-n \geq 3 \) and hence \( a_{i-n-1} = 3 \) or 4. As we assumed \( a_{i-n-1} = 3 \), we must have \( a_{i-n-1} = 4 \) and so the inequality (3.6) holds.

Thus in every case in which \( S_0 \geq 4 + \frac{21-3}{2} + \frac{1498-6}{3} \) we have

\[
S < S_0 \text{ for } i \neq 0 \text{ and we have completed the proof of the theorem.}
\]

Theorem 3.3. The Markoff spectrum is of measure zero for \( m \geq 1 / \sqrt{10} \).

Proof. In a doubly infinite sequence

\[
S = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)
\]
if there is a 3, the smallest possible value \( \max S_i = [b_5, b_{i+1}, \ldots] + [0, b_{i-1}, \ldots] \) is attained by \([3, 3] + [0, 3] = \sqrt{13} \). If the \( b's \) are 1's and 2's the greatest value is \( S = [2, 1, 2] + [0, 1, 2] = \sqrt{12} \). Thus there is a gap in the spectrum between \( 1/\sqrt{12} \) and \( 1/\sqrt{13} \).

Thus for \( n > 1/\sqrt{13} \) we may suppose that \( S \) consists entirely of 1's and 2's. Now suppose that \( S \) has a sequence 1, 2, 1, 2 and we take this first 2 as \( b_0 \). Then

\[
S_0 \geq [2, 1, 2, 2, 1] + [0, 1, 1, 2] = (81 + 14\sqrt{3})/33 = 3.2802639.
\]

Now suppose that \( S \) has a sequence 1, 2, 1, 1, 1. Then

\[
S_0 \geq [2, 1, 1, 1, 1, 2] + [0, 1, 1, 2] = (81 + 14\sqrt{3})/33 = 3.1893548.
\]

Now suppose that \( S \) has a sequence 1, 2, 1, 1, 2, 2. Then

\[
S_0 \geq [2, 1, 1, 2, 2, 1] + [0, 1, 1, 2] = 2 + (82 + 14\sqrt{3})/143 + \sqrt{3}/3 = 3.1628890.
\]

Now we suppose that \( S \) consists entirely of 1's and 2's and does have a sequence 1, 2, 1 but none of the sequences 1, 2, 1, 1 or 1, 2, 1, 1, 1 or 1, 2, 1, 1, 2 or their reverses. Then the only possibility is that \( S \) is the sequence 2, 1, 1 repeated infinitely often. Hence

\[
S_0 = [2, 1, 1, 2] + [0, 1, 1, 2] = \sqrt{10} = 3.1622776.
\]

As \( \sqrt{10} \) is less than the values in (3.10), (3.11) and (3.12) it follows that 
\( 1/\sqrt{10} \) is the largest minimum corresponding to sequences of 1's and 2's containing a subsequence 1, 2, 1. The form corresponding to (3.13) is

\[
f(x, y) = 2x^2 - 4xy - 3y^2
\]

for which \( d = 40 \). It is easily seen that \( \inf f(x, y) \) for \( (x, y) \neq (0, 0) \) and integral is 2 so that \( m = M(f)/\sqrt{d} = 2/\sqrt{40} = 1/\sqrt{10} \). Furthermore from (3.10), (3.11) and (3.12) we see that \( 1/\sqrt{10} \) is an isolated value of \( m \).

The last \( S_0 \) which can be constructed from a sequence of 1's and 2's which does not contain a subsequence 1, 2, 1 is easily found to be

\[
S_0 = [2, 1, 2, 2, 2] + [0, 2, 2, 2, 2]
\]

\[
= (\sqrt{120} + 8)/7 + (\sqrt{120} - 8)/7
\]

\[
= 2\sqrt{120}/7 = 3.129432 < \sqrt{10}.
\]

A corresponding quadratic form is

\[
f(x, y) = 7x^2 - 18xy - 8y^2
\]

for which \( M(f) = 7 \) and \( d = 480 \) and \( m = 7/\sqrt{480} \). Thus there is a gap in the Markoff spectrum between \( 7/\sqrt{480} \) and \( 1/\sqrt{10} \) and all sequences of 1's and 2's not containing a subsequence 1, 2, 1 have \( m > 7/\sqrt{480} \).

Let

\[
u_1 = [b_0, b_1, \ldots, b_r, a_1],
\]

\[
u_2 = [b_0, b_1, \ldots, b_r, a_2].
\]

Then if \( u_{r-1}/v_{r-1} \) and \( u_r/v_r \) are the last two convergents to \([b_0, b_1, \ldots, b_r]\)

\[
u_i = a_{i-1}/a_i + u_{i-1}/v_{i-1}, \quad i = 1, 2
\]

and since, as is shown in Perron [7]

\[
u_r - v_r = [0, b_r, b_{r-1}, \ldots, b_1], \quad u_r/v_r - u_{r-1}/v_{r-1} = (-1)^{-r-1},
\]

we will have from (3.17), (3.18) and (3.19)

\[
u_2 - \nu_1 = (-1)^{-r-1}(a_2 - a_1)/y_r(a_1 + e)
\]

We consider the numbers \([0, b_1, b_2, \ldots]\) where \( b_i = 1 \) or \( 2 \) containing no subsequence 1, 2, 1 as formed by Cantor subdivision of an initial interval from \( A \) to \( B \) where

\[
A = [0, 2, 1, 2, 2] = (\sqrt{120} - 8)/8 = 0.3693064,
\]

\[
B = [0, 1, 2, 2, 2] = (\sqrt{120} + 8)/8 = 0.7077787.
\]

The first subdivision is into intervals \( A_1 \) to \( B_1 \) and \( A_2 \) to \( B_2 \) where

\[
A_2 = [0, 2, 1, 2, 2, 2] = (\sqrt{120} - 8)/8 = 0.3693064,
\]

\[
B_2 = [0, 2, 1, 2, 2, 2] = (\sqrt{120} + 8)/8 = 0.4220645,
\]

\[
A_1 = [0, 1, 2, 2, 2, 2] = (\sqrt{120} - 8)/8 = 0.5855599,
\]

\[
B_1 = [0, 1, 2, 2, 2, 2] = (\sqrt{120} + 8)/8 = 0.7077787.
\]
The second subdivision is

\[
\begin{align*}
A_{11} &= [0, 2, 1, 1, 2, 1, 2, 1] = (\sqrt[3]{120} - 8)/8 = 0.3693064, \\
B_{11} &= [0, 2, 1, 1, 2, 1, 2, 1] = (33 - \sqrt[3]{120})/8 = 0.3867486; \\
A_{12} &= [0, 2, 1, 1, 2, 1, 2, 1] = (\sqrt[3]{120} - 8)/12 = 0.4128769, \\
B_{12} &= [0, 2, 1, 1, 2, 1, 2, 1] = (\sqrt[3]{120} - 8)/12 = 0.4296646; \\
A_{11} &= [0, 1, 1, 1, 2, 1, 2, 1] = (\sqrt[3]{120} - 1)/8 = 0.5855595, \\
B_{11} &= [0, 1, 1, 1, 2, 1, 2, 1] = (16 - \sqrt[3]{120})/8 = 0.3693064; \\
A_{22} &= [0, 1, 1, 2, 1, 2, 1] = (\sqrt[3]{120} + 1)/17 = 0.7032080, \\
B_{22} &= [0, 1, 1, 2, 1, 2, 1] = (\sqrt[3]{120} - 6)/17 = 0.7077877.
\end{align*}
\]

We note that in the last interval \( b_1 = 1, b_2 = 2 \) forces \( b_3 = 2 \) because no subsequence \( 1, 2, 1 \) is allowed.

In an interval \( u_1 \) to \( u_2 \)

\[
\begin{align*}
A_{11} &= [0, b_1, b_2, \ldots, b_r, a_r], \\
B_{11} &= [0, b_1, b_2, \ldots, b_r, a_r]
\end{align*}
\]

if \( b_r = 1, a_1 \) and \( a_2 \) are \( [1, 2, 2, 2] = (\sqrt[3]{120} + 6)/12 \) and \( [2, 1, 2, 2] = (\sqrt[3]{120} + 6)/7 \) in this order if \( r \) is odd and reverse order if \( r \) is even. If \( b_r = 2 \) and \( b_{r-1} = 1 \), this forces \( b_{r+1} = 2 \). Hence we need only consider cases with \( b_r = 2 \) and \( b_{r-1} = 2 \). Here \( a_1 \) and \( a_2 \) are \( [1, 2, 2, 2] = (\sqrt[3]{120} + 6)/12 \) and \( [2, 1, 2, 2] = (\sqrt[3]{120} + 6)/7 \). Thus in subdividing we consider two types of intervals, type 1 in which \( b_r = 1 \), and type 2 in which \( b_r = 2, b_{r-1} = 2 \).

Subdivision of type 1 interval, \( b_r = 1 \)

\[
I_1 = \begin{cases}
A_{11} = [0, b_1, b_2, \ldots, b_r, a_r], \\
B_{11} = [0, b_1, b_2, \ldots, b_r, a_r]
\end{cases}
\]

\[
I_2 = I_{22} = \begin{cases}
A_{22} = [0, b_1, b_2, \ldots, b_r, a_r], \\
B_{22} = [0, b_1, b_2, \ldots, b_r, a_r]
\end{cases}
\]

These limits are in increasing order if \( r \) is odd, decreasing if \( r \) is even.

The lengths of the whole interval and the two subintervals are

\[
\begin{align*}
I &= \left( \frac{5\sqrt[3]{120} + 30}{84} \right) / y_2 \left( \frac{\sqrt[3]{120} + 6 + \epsilon}{12} + \epsilon \right) \left( \frac{\sqrt[3]{120} + 6 + \epsilon}{7} + \epsilon \right), \\
I_1 &= \left( \frac{5\sqrt[3]{120} - 30}{84} \right) / y_2 \left( \frac{\sqrt[3]{120} + 6 + \epsilon}{12} + \epsilon \right) \left( \frac{\sqrt[3]{120} + 1 + \epsilon}{7} + \epsilon \right), \\
I_2 &= \left( \frac{\sqrt[3]{120} - 8}{56} \right) / y_2 \left( \frac{\sqrt[3]{120} + 8 + \epsilon}{8} + \epsilon \right) \left( \frac{\sqrt[3]{120} + 6 + \epsilon}{7} + \epsilon \right)
\end{align*}
\]

and the ratios of the subintervals to the interval are

\[
\begin{align*}
I_{1}/I &= \left( \frac{13 - \sqrt[3]{120}}{7} \right) \left( \frac{\sqrt[3]{120} + 6 + \epsilon}{\sqrt[3]{120} + 1 + \epsilon} \right), \\
I_{2}/I &= \left( \frac{12 - \sqrt[3]{120}}{20} \right) \left( \frac{\sqrt[3]{120} + 6 + \epsilon}{\sqrt[3]{120} + 8 + \epsilon} \right)
\end{align*}
\]

Since \( b_r = 1 \) and \( \epsilon = [0, b_1, b_{r-1}, \ldots, b_r] \), \( 1/2 \leq \epsilon \leq 5/7 \). As \( I_1/I \) is a decreasing function of \( \epsilon \) and \( I_2/I \) is an increasing function of \( \epsilon \) we find the largest value for \( I_1/I \) by taking \( \epsilon = 1/2 \) and the largest value for \( I_2/I \) by taking \( \epsilon = 5/7 \). As \( I_1/I \) by taking \( \epsilon = 1/2 \) and the largest value for \( I_2/I \) by taking \( \epsilon = 5/7 \).

\[
\begin{align*}
I_{1}/I &\leq \left( \frac{33 - \sqrt[3]{120}}{57} \right) = \frac{3867600}{387}, \\
I_{2}/I &\leq \left( \frac{1083 - 92\sqrt[3]{120}}{2085} \right) = \frac{38660626}{387}.
\end{align*}
\]

Similarly for a type 2 interval, \( b_{r-1} = 2, b_r = 2 \) we have

\[
\begin{align*}
A_1 &= [0, b_1, b_2, \ldots, b_r, 1, 2, 2, 2], \\
B_1 &= [0, b_1, b_2, \ldots, b_r, 1, 2, 2, 2], \\
A_2 &= [0, b_1, b_2, \ldots, b_r, 2, 2, 1, 2, 2], \\
B_2 &= [0, b_1, b_2, \ldots, b_r, 2, 2, 1, 2, 2].
\end{align*}
\]

Here for the interval lengths we have

\[
\begin{align*}
I &= \left( \frac{5\sqrt[3]{120} + 54}{84} \right) / y_2 \left( \frac{\sqrt[3]{120} + 6 + \epsilon}{12} + \epsilon \right) \left( \frac{\sqrt[3]{120} + 8 + \epsilon}{7} + \epsilon \right), \\
I_1 &= \left( \frac{5\sqrt[3]{120} - 30}{84} \right) / y_2 \left( \frac{\sqrt[3]{120} + 6 + \epsilon}{12} + \epsilon \right) \left( \frac{\sqrt[3]{120} + 1 + \epsilon}{7} + \epsilon \right), \\
I_2 &= \left( \frac{\sqrt[3]{120} + 8}{56} \right) / y_2 \left( \frac{\sqrt[3]{120} + 8 + \epsilon}{8} + \epsilon \right) \left( \frac{\sqrt[3]{120} + 8 + \epsilon}{7} + \epsilon \right)
\end{align*}
\]

The ratios of the subintervals to the interval are

\[
\begin{align*}
I_{1}/I &= \left( \frac{35 - 5\sqrt[3]{120}}{7} \right) \left( \frac{\sqrt[3]{120} + 8 + \epsilon}{\sqrt[3]{120} + 1 + \epsilon} \right), \\
I_{2}/I &= \left( \frac{12 - \sqrt[3]{120}}{12} \right) \left( \frac{\sqrt[3]{120} + 6 + \epsilon}{\sqrt[3]{120} + 8 + \epsilon} \right)
\end{align*}
\]
Here again $L/I$ is a decreasing function of $e$ and $L^2/I$ is an increasing function of $e$. As $b_2 = b_3 = 2$, we have $3/3 \leqslant e \leqslant 3/7$ and we get an upper bound on $L_2/I$ by taking $e = 2/5$ and on $L_3/I$ by taking $e = 3/7$.

These upper bounds are

$$L_2/I < \left(510 - \frac{36}{\sqrt{120}}\right)/377 < 0.3357897 < 0.336,$$

$$L_3/I < \left(55 - 4\sqrt{120}\right)/65 < 0.1720337 < 0.173.$$

Here the interval $L_2$ is of type 1, the interval $L_3$ of type 2. The next stage of subdivision gives:

$$I_4(L_3)/I < (0.387)(0.387) < 0.131,$$

$$I_4(L_3)/I < (0.387)(0.037) < 0.015,$$

$$I_4(L_3)/I < (0.173)(0.336) < 0.059,$$

$$I_4(L_3)/I < (0.173)(0.173) < 0.036.$$

Further subdivision of an interval of type 1 gives:

$$I_4(L_2)/I < (0.387)(0.387) < 0.150,$$

$$I_4(L_2)/I < (0.387)(0.037) < 0.015,$$

$$I_4(L_2)/I < (0.037)(0.336) < 0.013,$$

$$I_4(L_2)/I < (0.037)(0.173) < 0.007.$$

Thus we may subdivide an interval $A$ of length $|A|$ into four subintervals $A_1, A_2, A_3, A_4$ where $|A_1| + |A_2| + |A_3| + |A_4| < 0.333|A|$ or $0.183|A|$ depending on whether $A$ is of type 2 or type 1.

Now we consider a sum set $A + B = C$ consisting of all numbers $c = a + b, a \in A, b \in B$. Then for the lengths of the intervals spanned by these sets we have

$$|C| = |A| + |B|.$$

If we subdivide each of $A$ and $B$ into four subintervals $A_{ij}, B_{ij}, i, j = 1, 2, 3, 4, C$ is the union of the sets $C_{ij} = A_{ij} + B_{ij}$ and we have

$$\sum_{ij} |C_{ij}| = 4 \sum_{ij} |A_{ij}| + 4 \sum_{ij} |B_{ij}| \leqslant 0.382|A| + 0.382|B| < 0.382|C|.$$

Hence subdivision multiplies the total length covered by sum intervals by a factor of at most $0.382$. Subdividing $n$ times gives a total length covered of $(0.382)^n |L|$ if $|L|$ was the initial length covered. Taking $n$ sufficiently large, the total length covered is arbitrarily small and so the sum set of two numbers of the form $[0, b_1, b_2, \ldots]$ is of measure zero where the $b_i$'s are 1's and 2's with no subsequence 1, 2, 1. This proves the theorem and the Markoff spectrum is of measure zero for $m \geqslant 1/\sqrt{10}$.