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The Markoff spectrum*

by

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Dedicated to Harold Davenport

1. Introduction. An indefinite binary quadratic form $f(x, y) = ax^2 + bxy + cy^2$ of positive discriminant d for integers (x, y) not $(0, 0)$ always takes on a minimum less than or equal to $\sqrt{d}/\sqrt{5}$, and equality is necessary for forms equivalent to $ax^2 - axy - ay^2$, and for all other forms the minimum is at most $\sqrt{d}/\sqrt{8}$. Markoff [6] showed for $M(f)$ the lower bound of $|f(x, y)|$ over integers $(x, y) \neq (0, 0)$, with $M(f) = m\sqrt{d}$, that only a countable number of values greater than $1/3$ are possible for m, and that in these cases the minimum is attained. He also describes exactly these forms and their minima, which are called the Markoff chain. There are excellent accounts of this by Cassels [1], [2] and by Dickson [4].

The set of values of $m = M(f)/\sqrt{d}$ is called the Markoff spectrum. In this paper it is shown that if $M(f)$ is not attained for a form f, there is another form f^* of the same discriminant with $M(f^*) = M(f)$ for which $M(f^*)$ is attained. Hence in studying the spectrum we may consider only those forms which attain their minimum. It is also shown that the spectrum contains every positive number $m \leqslant 1/5.1007$. In addition it is shown that minima $m \geqslant 1/\sqrt{10}$ form a set of measure zero. Between $1/\sqrt{10}$ and $1/\sqrt{21}$ there are gaps in the spectrum. For instance it has long been known that there is a gap between $1/\sqrt{12}$ and $1/\sqrt{13}$, but there are further gaps between $1/\sqrt{13}$ and $1/5.1007$.

2. Let $f(x, y) = ax^2 + bxy + cy^2$ be a real indefinite binary quadratic form of positive discriminant $d = b^2 - 4ac$. We are interested in the minimum of f, $M(f)$ defined as

$$M(f) = \inf_{(x,y)\neq(0,0)} |f(x, y)|, \qquad x, y \text{ integers.} \tag{2.1}$$

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If $t \neq 0$ is a real number then $M(tf) = |t|M(f)$ and the discriminant of tf is $t^2 d$. Hence the quantity $M(f)/\sqrt{d}$ is the same for forms differing by a constant factor of proportionality, and it is this ratio $M(f)/\sqrt{d}$ which we consider the minimum of f.

As is customary we say that $f(x, y) = ax^2+bxy+cy^2$ and $f_1(x_1, y_1) = a_1x_1^2+b_1x_1y_1+c_1y_1^2$ are *equivalent* if there is a transformation T

$$(2.2)\qquad T:\quad \begin{aligned} x &= rx_1+sy_1, \\ y &= tx_1+uy_1, \end{aligned} \qquad ru-st = \pm 1,$$

where r, s, t, u are integers such that T transforms $f(x, y)$ into $f_1(x_1, y_1)$. Clearly, $M(f_1) = M(f)$.

Associated with the form f are its two roots θ_1, θ_2 defined by

$$(2.3)\qquad f(x, y) = a(x-\theta_1 y)(x-\theta_2 y)$$

where θ_1 and θ_2 are real and distinct since $d > 0$. If either root is rational, then from (2.3) we can find integers $(x, y) \neq (0, 0)$ such that $f(x, y) = 0$ and so $M(f) = 0$. From now on we shall assume that neither root is rational. Following Dickson [3], [4] we say that $f(x, y)$ is *reduced* if numbering θ_1, θ_2 appropriately

$$(2.4)\qquad \theta_1 > 1, \quad -1 < \theta_2 < 0.$$

Hence θ_1 and θ_2 can be represented by infinite continued fractions (being irrational) and

$$(2.5)\qquad \begin{aligned} \theta_1 &= [b_0, b_1, b_2, \ldots], \\ -\theta_2 &= [0, b_{-1}, b_{-2}, \ldots], \end{aligned}$$

where the b_i are positive integers.

If we apply the transformation T to f where

$$(2.6)\qquad T:\quad \begin{aligned} x &= b_0 x_1+y_1, \\ y &= x_1, \end{aligned}$$

then $f_1(x_1, y_1)$ is also reduced and for its roots

$$(2.7)\qquad \begin{aligned} \varphi_1 &= [b_1, b_2, b_3, \ldots], \\ -\varphi_2 &= [0, b_0, b_{-1}, b_{-2}, \ldots]. \end{aligned}$$

The general theory, due originally to Lagrange may be found in Dickson [3], [4] and asserts that if $f(x, y)$ and $g(x, y)$ are equivalent reduced forms, then if the roots of $f(x, y)$ are θ_1, θ_2 as given by (2.5) and if the roots of $g(x, y)$ are ψ_1, ψ_2 given by

$$(2.8)\qquad \begin{aligned} \psi_1 &= [c_0, c_1, c_2, \ldots], \\ -\psi_2 &= [0, c_{-1}, c_{-2}, \ldots] \end{aligned}$$

then necessarily there is an integer n_0 such that

$$(2.9)\qquad c_i = b_{i+n_0}, \quad \text{for all } i.$$

Application of transformations T as in (2.6) shows the converse to be true.

Furthermore from (2.3)

$$(2.10)\qquad d = a^2(\theta_1-\theta_2)^2$$

so that

$$(2.11)\qquad |a| = \sqrt{d}/(\theta_1-\theta_2).$$

The general theory asserts that every number m properly represented by f where $|m| < \sqrt{d}/2$ is the leading coefficient of a reduced form equivalent to f, and so $|m| = |a|$ as represented in (2.11).

The following theorem summarizes what we shall need of the general theory.

THEOREM 2.1. *Let $f(x, y) = ax^2+bxy+cy^2$ be a real binary quadratic form with positive discriminant $d = b^2-4ac$, and suppose also that $f(x, y) \neq 0$ for integers $(x, y) \neq (0, 0)$, and let $M(f) = \inf|f(x, y)|$, for integers $(x, y) \neq (0, 0)$. Then there is a doubly infinite sequence S of positive integers*

$$S: (\ldots, b_{-j}, \ldots, b_{-1}, b_0, b_1, \ldots, b_i, \ldots)$$

such that if we form the sum S_i of the two continued fractions

$$S_i = [b_i, b_{i+1}, \ldots]+[0, b_{i-1}, b_{i-2}, \ldots]$$

for every i, then $M(f)/\sqrt{d} = \inf_i (1/S_i)$. Conversely a sequence S defines a class of equivalent forms.

We can now show that in studying the values of $M(f)/\sqrt{d}$ we may restrict our attention to forms which attain their minimum.

THEOREM 2.2. *If $m = M(f)/\sqrt{d}$ is the minimum of a form f, there is a form f^* which attains the minimum m.*

Proof. If $m = 0$, the form $f^*(x, y) = x^2-y^2$ attains this minimum. If $m > 0$ then the b's in the sequence S are bounded. Now if $f(x, y)$ does not attain its minimum for a particular choice of i in the sum S_i, then there is an infinite sequence E_n: $i_1, i_2, \ldots, i_n, \ldots$, such that

$$\lim_{n\to\infty} S_{i_n} = \frac{1}{m}.$$

As the b's are bounded integers, there is an infinite subsequence E_n^1 of E_n in which the central integer b_{i_n} has the same value c_0. Then E_n^1 has a subsequence E_n^2 in which $b_{i_n} = c_0$ and $b_{i_n+1} = c_1$ for a fixed c_1. Then in turn E_n^2 has an infinite subsequence E_n^3 in which $b_{i_{n-1}} = c_{-1}$, $b_{i_n} = c_0$, and $b_{i_{n+1}} = c_1$. Continuing there is a doubly infinite sequence

$$S^*: (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots)$$

with the property that $b_{i_{n-r}} = c_{-r}, \ldots, b_{i_{n-1}} = c_{-1}, b_{i_n} = c_0, \ldots, b_{i_{n+r}} = c_r$, for every r, occurs in an infinite subsequence of E_n. Hence

(2.12) $$1/m = [c_0, c_1, \ldots] + [0, c_{-1}, c_{-2}, \ldots]$$

and for the form $f^*(x, y)$ associated with S^* the value $m = f^*(x, y)/\sqrt{d}$ is attained. Furthermore as every finite section of S^* $c_{j-t}, \ldots, c_{j-1}, c_j, \ldots, c_{j+t}$ is also a section of S we have $S_j^* \leqslant 1/m$ in every case so that m is in fact the minimum of $f^*(x, y)/\sqrt{d}$.

3. The Markoff spectrum

DEFINITION. The *Markoff spectrum* is the set of real numbers $m = M(f)/\sqrt{d}$ corresponding to all real indefinite binary quadratic forms $f(x, y)$.

The Markoff chain is a sequence of forms for which m takes on its largest values, namely all $m > 1/3$.

THEOREM 3.1 (Markoff [6]). *Let $f(x, y) = ax^2 + bxy + cy^2$ be an indefinite quadratic form with real coefficients and discriminant $d = b^2 - 4ac$, and let $M = M(f)$ be the lower bound of $|f(x, y)|$ over all integer pairs $(x, y) \neq (0, 0)$. Then $M \leqslant \sqrt{d}/\sqrt{5}$, the sign of equality being necessary for*

$$f_0 = M(x^2 - xy - y^2).$$

If f is not equivalent to f_0, then $M \leqslant \sqrt{d}/\sqrt{8}$, the sign of equality being necessary for

$$f_1 = M(x^2 - 2xy - y^2).$$

If f is not equivalent to f_0 or f_1, then $M \leqslant 5\sqrt{d}/\sqrt{221}$, the sign of equality being necessary for

$$f_2 = M/5(5x^2 - 11xy - 5y^2).$$

If f is not equivalent to f_0, f_1, or f_2, then $M \leqslant 13\sqrt{d}/\sqrt{1517}$, the sign of equality being necessary for

$$f_3 = M/13(13x^2 - 29xy - 13y^2),$$

and so on. The set $f_0, f_1, f_2, \ldots, f_i$ continues indefinitely and every f such that $M > \sqrt{d}/3$ is equivalent to some f_i.

For the proof of this result, the reader is referred to Dickson [3], [4], or Cassels [1], [2]. The forms are exhibited explicitly and every form attains its lower bound.

The Markoff chain describes that part of the Markoff spectrum for which $m > 1/3$. It will be proved here that every positive number below 1/5.1007 is in the Markoff spectrum.

THEOREM 3.2. *If $0 < m \leqslant 1/s$, $s = 4 + \dfrac{\sqrt{21} - 3}{2} + \dfrac{\sqrt{48} - 6}{3} = 5.1006890$ then there is a real indefinite form $f(x, y)$ for which $M(f)/\sqrt{d} = m$.*

Proof. Freiman and Yudin [9] have shown that if D is the set of continued fractions

$$[0, a_1, a_2, \ldots]$$

with $a_i = 1, 2, 3, 4$ such that there is no sequence a_i, a_{i+1} of the form 1, 4 or 2, 4, then every number z with $5 - \sqrt{21} \leqslant z \leqslant \sqrt{21} - 3$ is of the form

(3.1) $$z = a_1 + a_2, \quad a_1, a_2 \in D.$$

From this they were able to show that the Markoff spectrum contained all numbers $0 < m \leqslant 1/s$, with $s = 5.118$. This theorem is a slight improvement on theirs.

Choose a positive integer a_0 and define the doubly infinite sequence S as

$$S = (\ldots, a_{-i}, \ldots, a_{-1}, a_0, a_1, \ldots)$$

where each of $[0, a_1, a_2, \ldots]$ and $[0, a_{-1}, a_{-2}, \ldots]$ is a continued fraction of the set D. Then with

$$S_i = [a_i, a_{i+1}, \ldots] + [0, a_{i-1}, a_{i-2}, \ldots]$$

we have

$$m = M(f)/\sqrt{d} = \min_i (1/S_i).$$

Choosing $a_0 = n$, then from (3.1) we may choose a_1 and a_2 for D so that S_0 is any number in the interval

$$n + 5 - \sqrt{21} \leqslant S_0 \leqslant n + \sqrt{21} - 3$$

and we note that $5 - \sqrt{21} = .4174242$, $\sqrt{21} - 3 = 1.5825757$ so that this interval is of length greater than 1, so that taking $n = 5, 6, 7, \ldots$ we obtain every number greater than $11 - \sqrt{21} = 5.4174242$ as a value of S_0. If $i \neq 0$ and $a_i = 1, 2$, or 3, then $S_i < 3 + 1 + 1 = 5$, while if $a_i = 4$ then since there is no sequence, 1, 4 or 2, 4 in D, then $a_{i-1} \neq 1, 2$ if i is positive while, $a_{i+1} \neq 1, 2$ if i is negative, and so

$$S_i \leqslant 4 + [0, 1, \ldots] + [0, 3, \ldots] = 5.3333 < S_0.$$

Thus in all these cases $m = 1/S_0$ and we have every m possible $0 < m \leqslant 1/s$ $s = 11-\sqrt{21}$. Now let D_1 be the subset of D for which $a_1 = 1$ and D_2 the subset of D for which $a_1 = 2$. Then for $\alpha \in D_1$

$$[0,1,1,\overline{3,1}] = \frac{\sqrt{21}+1}{10} = .5582576,$$

$$(3.2)\qquad [0,1,3,\overline{1,3}] = \frac{\sqrt{21}-3}{2} = .7912878,$$

$$\frac{\sqrt{21}+1}{10} \leqslant \alpha \leqslant \frac{\sqrt{21}-3}{2}$$

and for $\alpha \in D_2$

$$[0,2,1,\overline{3,1}] = \frac{\sqrt{21}-1}{10} = .3582576,$$

$$(3.3)\qquad [0,2,3,\overline{1,3}] = \frac{9-\sqrt{21}}{10} = .4417424,$$

$$\frac{\sqrt{21}-1}{10} \leqslant \alpha \leqslant \frac{9-\sqrt{21}}{10}.$$

We now form the doubly infinite sequence S by taking $a_0 = 4$, and $\alpha_1 = [0, a_1, a_2, \ldots]$ from D_1 and $\alpha_2 = [0, a_{-1}, a_{-2}, \ldots]$ in the first instance from D_1 and in the second instance from D_2. Alternatively we might take α_1 from D_2 and α_2 from D_1. In the first instance the values of S_0 are in the interval from $(\sqrt{21}+21)/5 = 5.1165151$ to $1+\sqrt{21} = 5.5825757$. In the second instance the values are in the interval from $(20+\sqrt{21})/5 = 4.9165151$ to $(17+2\sqrt{21})/5 = 5.2330303$. Freiman and Yudin have shown that D_1 and D_2 may be obtained by Cantor subdivisions in which the length of the middle interval removed is shorter than either interval remaining. Since the length of D_2 (.0834848) is greater than one-third the length of D_1 (.2330302), it follows from Theorem 2.2 of the author's paper [5] that $S_0 = 4+\alpha_1+\alpha_2$ takes on every value in the interval in both instances.

To complete the proof of the theorem we need to show that if

$$S_0 \geqslant 4+\frac{\sqrt{21}-3}{2}+\frac{\sqrt{48}-6}{3} = 5.1006890$$

then for $i \neq 0$, $S_i \leqslant S_0$. Here if $a_i = 1, 2$, or 3 then $S_i < 3+1+1 = 5 < S_0$ and so we need only consider cases with $a_i = 4$. Again if $a_{i+1} \geqslant 2$ and $a_{i-1} \geqslant 2$ then $S_i \leqslant 4+\frac{1}{2}+\frac{1}{2} = 5 < S_0$. Thus we need only consider cases in which a_{i-1} or a_{i+1} is 1. But as 1, 4 does not arise in D_1 or D_2, this must have arisen from a sequence 4, 1 in D_1 or D_2. Hence if i is positive a_{i+1} may be 1 but not a_{i-1}, while if i is negative a_{i-1} may be 1 but not a_{i+1}. Let us suppose that i is positive since replacing a_j by a_{-j} throughout in S does not alter the values of S_i to be considered. Thus $a_i = 4$, a_{i+1} with $i > 0$. Since a_1 is 1 or 2, and D_1 and D_2 do not contain a sequence 1, 4 or 2, $i \neq 1$ and $i \neq 2$, so that $i \geqslant 3$. Now

$$(3.4)\qquad [4, 1, a_{i+2}, \ldots] \leqslant [4, \overline{1, 3}] = 4+\frac{\sqrt{21}-3}{2} = 4.7912879.$$

Since $i \geqslant 3$ $a_{i-1}, 4$ is a sequence in D_1 or D_2 so that $a_{i-1} \neq 1, 2$. If $a_{i-1} = 4$, then

$$(3.5)\qquad [0, a_{i-1}, \ldots] = [0, 4, \ldots] < .25$$

and here $S_i \leqslant 5.0412789 < S_0$. If $a_{i-1} = 3$, then I assert

$$(3.6)\qquad [0, 3, a_{i-2}, \ldots] < [0, \overline{3, 4}] = \frac{\sqrt{48}-6}{3}.$$

This is certainly true if $a_{i-2} = 1, 2$, or 3. If $a_{i-2} = 4$, then $a_{i-2} \neq a_0$ since a_1 is 1 or 2 and here $a_{i-1} = 3$. Thus with $a_{i-2} = 4$, and $i-2 \geqslant 2$ then $a_{i-3} = 4$ is a sequence in D_1 or D_2 and so $a_{i-3} = 4$. If $a_{i-3} = 4$ the inequality (3.6) certainly holds. Hence suppose $a_{i-3} = 3$. Continuing suppose

$$(3.7)\qquad [0, a_{i-1}, a_{i-2}, \ldots, a_{i-n}, a_{i-n-1}, \ldots] = [0, 3, 4, 3, \ldots, 4, 3, a_{i-n-1}]$$

with $a_{i-n-1} \neq 4$, n odd, or

$$(3.8)\qquad [0, a_{i-1}, a_{i-2}, \ldots, a_{i-n}, a_{i-n-1}, \ldots] = [0, 3, 4, \ldots, 3, 4, a_{i-n-1}]$$

with $a_{i-n-1} \neq 3$, n even. In (3.7) with $a_{i-n-1} = 1, 2$, or 3 the inequality (3.6) certainly holds. Since a_{-1}, a_0, a_1 are 1, 4, 1, or 2, 4, 1, or 1, 4, 2, then in (3.8) $i-n$ is positive and as $a_{i-n} = 4$, and D_1 and D_2 do not contain a sequence 1, 4 or 2, 4, then $i-n \geqslant 3$ and hence $a_{i-n-1} = 3$ or 4. As we assumed $a_{i-n-1} \neq 3$, we must have $a_{i-n-1} = 4$ and so the inequality (3.6) holds.

Thus in every case in which $S_0 > 4+\dfrac{\sqrt{21}-3}{2}+\dfrac{\sqrt{48}-6}{3}$ we have $S_i < S_0$ for $i \neq 0$ and we have completed the proof of the theorem.

THEOREM 3.3. *The Markoff spectrum is of measure zero for* $m \geqslant 1/\sqrt{10}$.

Proof. In a doubly infinite sequence

$$(3.9)\qquad S = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)$$

if there is a 3, the smalles possible value $\max S_i = [b_i, b_{i+1}, \ldots]+[0, b_{i-1}, \ldots]$ is attained by $[3, \overline{3}]+[0, \overline{3}] = \sqrt{13}$. If the b's are 1's and 2's the greatest value is $S = [2, \overline{1, 2}]+[0, \overline{1, 2}] = \sqrt{12}$. Thus there is a gap in the spectrum between $1/\sqrt{12}$ and $1/\sqrt{13}$.

Thus for $m > 1/\sqrt{13}$ we may suppose that S consists entirely of 1's and 2's. Now suppose that S has a sequence 1, 2, 1, 2 and we take this first 2 as b_0. Then

$$S_0 \geqslant [2, 1, 2, \overline{2, 1}]+[0, 1, \overline{1, 2}] = (84+14\sqrt{3})/33 = 3.2802639. \tag{3.10}$$

Now suppose that S has a sequence 1, 2, 1, 1, 1. Then

$$S_0 \geqslant [2, 1, 1, 1, \overline{1, 2}]+[0, 1, \overline{1, 2}] = (81+14\sqrt{3})/33 = 3.1893548. \tag{3.11}$$

Now suppose that S has a sequence 1, 2, 1, 1, 2, 2. Then

$$S_0 \geqslant [2, 1, 1, 2, 2, \overline{2, 1}]+[0, 1, \overline{1, 2}] = 2+(82+\sqrt{3})/143+\sqrt{3}/3 = 3.1628890. \tag{3.12}$$

Now we suppose that S consists entirely of 1's and 2's and does have a sequence 1, 2, 1 but none of the sequences 1, 2, 1, 2 or 1, 2, 1, 1, 1 or 1, 2, 1, 1, 2, 2 or their reverses. Then the only possibility is that S is the sequence 2, 1, 1 repeated infinitely often. Here

$$S_0 = [2, \overline{1, 1, 2}]+[0, \overline{1, 1, 2}] = \sqrt{10} = 3.1622777. \tag{3.13}$$

As $\sqrt{10}$ is less than the values in (3.10), (3.11) and (3.12) it follows that $1/\sqrt{10}$ is the largest minimum corresponding to sequences of 1's and 2's containing a subsequence 1, 2, 1. The form corresponding to (3.13) is

$$f(x, y) = 2x^2-4xy-3y^2 \tag{3.14}$$

for which $d = 40$. It is easily seen that $\inf|f(x, y)|$ for $(x, y) \neq (0, 0)$ and integral is 2 so that $m = M(f)/\sqrt{d} = 2/\sqrt{40} = 1/\sqrt{10}$. Furthermore from (3.10), (3.11) and (3.12) we see that $1/\sqrt{10}$ is an isolated value of m.

The largest S_0 which can be constructed from a sequence of 1's and 2's which does not contain a subsequence 1, 2, 1 is easily found to be

$$\begin{aligned} S_0 &= [2, \overline{1, 2, 2, 2}]+[0, \overline{2, 2, 1, 2}] \\ &= (\sqrt{120}+8)/7+(\sqrt{120}-8)/7 \\ &= 2\sqrt{120}/7 = 3.1298432 < \sqrt{10}. \end{aligned} \tag{3.15}$$

A corresponding quadratic form is

$$f(x, y) = 7x^2-16xy-8y^2 \tag{3.16}$$

for which $M(f) = 7$ and $d = 480$ and $m = 7/\sqrt{480}$. Thus there is a gap in the Markoff spectrum between $7/\sqrt{480}$ and $1/\sqrt{10}$, and all sequences of 1's and 2's not containing a subsequence 1, 2, 1 have $m \geqslant 7/\sqrt{480}$.

Let

$$u_1 = [b_0, b_1, \ldots, b_r, a_1], \quad u_2 = [b_0, b_1, \ldots, b_r, a_2]. \tag{3.17}$$

Then if x_{r-1}/y_{r-1} and x_r/y_r are the last two convergents to $[b_0, b_1, \ldots, b_r]$

$$u_i = \frac{a_i x_r + x_{r-1}}{a_i y_r + y_{r-1}}, \quad i = 1, 2 \tag{3.18}$$

and since, as is shown in Perron [7]

$$\varepsilon = y_{r-1}/y_r = [0, b_r, b_{r-1}, \ldots, b_1], \qquad x_r y_{r-1} - x_{r-1} y_r = (-1)^{r-1}, \tag{3.19}$$

we will have from (3.17), (3.18) and (3.19)

$$u_2-u_1 = \frac{(-1)^{r-1}(a_2-a_1)}{y_r^2(a_1+\varepsilon)(a_2+\varepsilon)}. \tag{3.20}$$

We consider the numbers $[0, b_1, b_2, \ldots]$ where $b_i = 1$ or 2 containing no subsequence 1, 2, 1 as formed by Cantor subdivision of an initial interval from A to B where

$$\begin{aligned} A &= [0, \overline{2, 1, 2, 2}] = (\sqrt{120}-8)/8 = .3693064, \\ B &= [0, \overline{1, 2, 2, 2}] = (\sqrt{120}-6)/7 = .7077787. \end{aligned} \tag{3.21}$$

The first subdivision is into intervals A_1 to B_1 and A_2 to B_2 where

$$\begin{cases} A_2 = [0, 2, \overline{1, 2, 2, 2}] = (\sqrt{120}-8)/8 = .3693064, \\ B_2 = [0, 2, \overline{2, 1, 2, 2}] = (\sqrt{120}-8)/7 = .4220645, \end{cases} \qquad \begin{cases} A_1 = [0, 1, \overline{1, 2, 2, 2}] = (\sqrt{120}-1)/17 = .5855559, \\ B_1 = [0, 1, \overline{2, 2, 2, 1}] = (\sqrt{120}-6)/7 = .7077787. \end{cases} \tag{3.22}$$

The second subdivision is

$$(3.23)\quad \begin{aligned} A_{21} &= [0,2,1,\overline{2,2,2,1}] = (\sqrt{120}-8)/8 = .3693064,\\ B_{21} &= [0,2,1,\overline{1,2,2,2}] = (33-\sqrt{120})/57 = .3867640;\\ A_{22} &= [0,2,2,\overline{2,1,2,2}] = (\sqrt{120}-6)/12 = .4128709,\\ B_{22} &= [0,2,2,\overline{1,2,2,2}] = (\sqrt{120}-8)/7 = .4220645;\\ A_{11} &= [0,1,1,\overline{2,2,2,1}] = (\sqrt{120}-1)/17 = .5855559,\\ B_{11} &= [0,1,1,\overline{1,2,2,2}] = (16-\sqrt{120})/8 = .6306936;\\ A_{122} &= [0,1,2,2,\overline{1,2,2,2}] = (\sqrt{120}+1)/17 = .7032030,\\ B_{122} &= [0,1,2,2,\overline{2,1,2,2}] = (\sqrt{120}-6)/7 = .7077787. \end{aligned}$$

We note that in the last interval $b_1 = 1$, $b_2 = 2$ forces $b_3 = 2$ because no subsequence 1, 2, 1 is allowed.

In an interval u_1 to u_2

$$(3.24)\quad \begin{aligned} u_1 &= [0, b_1, b_2, \ldots, b_r, \alpha_1],\\ u_2 &= [0, b_1, b_2, \ldots, b_r, \alpha_2] \end{aligned}$$

if $b_r = 1$, α_1 and α_2 are $[1,\overline{2,2,2}] = (\sqrt{120}+6)/12$ and $[\overline{2,2,2,1}] = (\sqrt{120}+6)/7$ in this order if r is odd and reverse order if r is even. If $b_r = 2$ and $b_{r-1} = 1$, this forces $b_{r+1} = 2$. Hence we need only consider cases with $b_r = 2$ and $b_{r-1} = 2$. Here α_1 and α_2 are $[\overline{1,2,2,2}] = (\sqrt{120}+6)/12$ and $[\overline{2,1,2,2}] = (\sqrt{120}+8)/7$. Thus in subdividing we consider two types of intervals, type 1 in which $b_r = 1$, and type 2 in which $b_r = 2$, $b_{r-1} = 2$.

Subdivision of type 1 interval, $b_r = 1$

$$(3.25)\quad \begin{aligned} I_1&: \begin{cases} A_1[0, b_1, b_2, \ldots, b_r, 1, \overline{2,2,2,1}],\\ B_1[0, b_1, b_2, \ldots, b_r, 1, \overline{1,2,2,2}]; \end{cases}\\ I_2 = I_{22}&\begin{cases} A_{22}[0, b_1, b_2, \ldots, b_r, 2, 2, \overline{1,2,2,2}],\\ B_{22}[0, b_1, b_2, \ldots, b_r, 2, 2, \overline{2,1,2,2}]. \end{cases} \end{aligned}$$

These limits are in increasing order if r is odd, decreasing if r is even. The lengths of the whole interval and the two subintervals are

$$(3.26)\quad \begin{aligned} I &= \left(\frac{5\sqrt{120}+30}{84}\right)\Big/ y_r^2\left(\frac{\sqrt{120}+6}{12}+\varepsilon\right)\left(\frac{\sqrt{120}+6}{7}+\varepsilon\right),\\ I_1 &= \left(\frac{5\sqrt{120}-30}{84}\right)\Big/ y_r^2\left(\frac{\sqrt{120}+6}{12}+\varepsilon\right)\left(\frac{\sqrt{120}+1}{7}+\varepsilon\right),\\ I_2 &= \left(\frac{\sqrt{120}-8}{56}\right)\Big/ y_r^2\left(\frac{\sqrt{120}+8}{8}+\varepsilon\right)\left(\frac{\sqrt{120}+6}{7}+\varepsilon\right), \end{aligned}$$

and the ratios of the subintervals to the interval are,

$$(3.27)\quad \begin{aligned} I_1/I &= \left(\frac{13-\sqrt{120}}{7}\right)\left(\frac{\sqrt{120}+6}{7}+\varepsilon\right)\Big/\left(\frac{\sqrt{120}+1}{7}+\varepsilon\right),\\ I_2/I &= \left(\frac{12-\sqrt{120}}{20}\right)\left(\frac{\sqrt{120}+6}{12}+\varepsilon\right)\Big/\left(\frac{\sqrt{120}+8}{8}+\varepsilon\right). \end{aligned}$$

Since $b_r = 1$ and $\varepsilon = [0, b_r, b_{r-1}, \ldots, b_1]$, $1/2 \leqslant \varepsilon \leqslant 5/7$. As I_1/I is a decreasing function of ε and I_2/I is an increasing function of ε we find the largest value for I_1/I by taking $\varepsilon = 1/2$ and the largest value for I_2/I by taking $\varepsilon = 1/2$ and the largest value for I_2/I by taking $\varepsilon = 5/7$.

$$(3.28)\quad \begin{aligned} I_1/I &\leqslant (33-\sqrt{120})/57 = .3867640 < .387,\\ I_2/I &\leqslant (1083-92\sqrt{120})/2085 = .0360626 < .037. \end{aligned}$$

Similarly for a type 2 interval, $b_{r-1} = 2$, $b_r = 2$ we have

$$(3.29)\quad \begin{aligned} I_1&\begin{cases} A_1 = [0, b_1, b_2, \ldots, b_r, 1, \overline{2,2,2,1}],\\ B_1 = [0, b_1, b_2, \ldots, b_r, 1, \overline{1,2,2,2}]; \end{cases}\\ I_2&\begin{cases} A_2 = [0, b_1, b_2, \ldots, b_r, 2, \overline{2,1,2,2}],\\ B_2 = [0, b_1, b_2, \ldots, b_r, 2, \overline{1,2,2,2}]. \end{cases} \end{aligned}$$

Here for the interval lengths we have

$$(3.30)\quad \begin{aligned} I &= \frac{(5\sqrt{120}+54)}{84}\Big/ y_r^2\left(\frac{\sqrt{120}+6}{12}+\varepsilon\right)\left(\frac{\sqrt{120}+8}{7}+\varepsilon\right),\\ I_1 &= \frac{(5\sqrt{120}-30)}{84}\Big/ y_r^2\left(\frac{\sqrt{120}+6}{12}+\varepsilon\right)\left(\frac{\sqrt{120}+1}{7}+\varepsilon\right),\\ I_2 &= \frac{(\sqrt{120}+8)}{56}\Big/ y_r^2\left(\frac{\sqrt{120}+8}{8}+\varepsilon\right)\left(\frac{\sqrt{120}+8}{7}+\varepsilon\right) \end{aligned}$$

The ratios of the subintervals to the interval are

$$(3.31)\quad \begin{aligned} I_1/I &= (55-5\sqrt{120})\left(\frac{\sqrt{120}+8}{7}+\varepsilon\right)\Big/\left(\frac{\sqrt{120}+1}{7}+\varepsilon\right),\\ I_2/I &= \left(\frac{12-\sqrt{120}}{4}\right)\left(\frac{\sqrt{120}+6}{12}+\varepsilon\right)\Big/\left(\frac{\sqrt{120}+8}{8}+\varepsilon\right). \end{aligned}$$

Here again I_1/I is a decreasing function of ε and I_2/I is an increasing function of ε. As $b_r = b_{r-1} = 2$, we have $2/5 \leqslant \varepsilon \leqslant 3/7$ and we get an upper bound on I_1/I by taking $\varepsilon = 2/5$ and on I_2/I by taking $\varepsilon = 3/7$. These upper bounds are

$$(3.32)\qquad \begin{aligned} I_1/I &\leqslant (510 - 35\sqrt{120})/377 = .3357937 < .336, \\ I_2/I &\leqslant (55 - 4\sqrt{120})/65 = .1720337 < .173. \end{aligned}$$

Here the interval I_1 is of type 1, the interval I_2 of type 2. The next stage of subdivision gives

$$(3.33)\qquad \begin{aligned} I_1(I_1)/I &< (.336)(.387) < .131, \\ I_2(I_1)/I &< (.336)(.037) < .013, \\ I_1(I_2)/I &< (.173)(.336) < .059, \\ I_2(I_2)/I &< (.173)(.173) < .030. \end{aligned}$$

Further subdivision of an interval of type 1 gives

$$(3.34)\qquad \begin{aligned} I_1(I_1)/I &< (.387)(.387) < .150, \\ I_2(I_1)/I &< (.387)(.037) < .015, \\ I_1(I_2)/I &< (.037)(.336) < .013, \\ I_2(I_2)/I &< (.037)(.173) < .007. \end{aligned}$$

Thus we may subdivide an interval A of length $|A|$ into four subintervals A_1, A_2, A_3, A_4 where $|A_1|+|A_2|+|A_3|+|A_4| < .233|A|$ or $.185|A|$ depending on whether A is of type 2 or type 1.

Now we consider a sum set $A+B = C$ consisting of all numbers $c = a+b$, $a \in A$, $b \in B$. Then for the lengths of the intervals spanned by these sets we have

$$(3.35)\qquad |C| = |A|+|B|.$$

If we subdivide each of A and B into four subintervals A_i, B_j, $i, j = 1, 2, 3, 4$, C is the union of the sets $C_{ij} = A_i + B_j$ and we have

$$(3.36)\qquad \sum_{i,j} |C_{ij}| = 4\sum_i |A_i| + 4\sum_j |B_j| \leqslant .932|A| + .932|B| < .932|C|.$$

Hence subdivision multiplies the total length covered by sum intervals by a factor of at most .932. Subdividing n times gives a total length covered of $(.932)^n|L|$ if $|L|$ was the initial length covered. Taking n sufficiently large, the total length covered is arbitrarily small and so the sum set of two numbers of the form $[0, b_1, b_2, \ldots]$ is of measure zero where the b's are 1's and 2's with no subsequence 1, 2, 1. This proves the theorem and the Markoff spectrum is of measure zero for $m \geqslant 1/\sqrt{10}$.

References

[1] J. W. S. Cassels, *The Markoff chain*, Ann. of Math. 50 (1949), pp. 676–685.
[2] — *An Introduction to Diophantine Approximation*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 45 (1957).
[3] L. E. Dickson, *Introduction to the Theory of Numbers*, Chicago 1929.
[4] — *Studies in the Theory of Numbers*, Chicago 1930.
[5] Marshall Hall, Jr., *On the sum and product of continued fractions*, Ann. of Math. 48 (1947), pp. 966–993.
[6] A. Markoff, *Sur les formes quadratiques binaire indéfinies*, Math. Ann. 15 (1879), pp. 381–409.
[7] O. Perron, *Die Lehre von den Kettenbrüchen*, Leipzig and Berlin; 3rd edition, Stuttgart 1954.
[8] R. Remak, *Über indefinite binäre quadratische minimalformen*, Math. Ann. 92 (1924), pp. 155–182.
[9] G. A. Freiman and A. A. Yudin, *On Markovian spectrum* (Russian), Litovskiĭ Mat. Sbornik 6 (1966), pp. 443–447.

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