This last expression is positive since there are equally many terms in each sum and those $\equiv 4 \pmod 5$ are pairwise larger than those $\equiv 2 \pmod 5$ since $\pi(y,5,4)-\pi(y,5,2)$ is never positive for $y < 10949$. Thus (12) yields

$$A_\infty(\log y_0)\sqrt{y_0} > 236+(-1+i)\frac{L_2'(0)}{L_2(0)}+(-1-i)\frac{L_3'(0)}{L_3(0)}+\frac{L_1''(0)}{L_1'(0)}. \tag{13}$$

We want only to show that $A_\infty(\log y_0)\sqrt{y_0} > 2\sqrt{y_0} = 209.27\ldots$ Any reasonable sort of estimate of $L_2'(0)$ and $L_1''(0)$ accomplishes this

$$\left[L_3'(0) = \overline{L_2'(0)},\ L_2(0) = \frac{3+i}{5},\ L_3(0) = \frac{3-i}{5},\ L_1'(0) = \log\left(\frac{1+\sqrt{5}}{2}\right)\right].$$

The result is

THEOREM 6.

$$\limsup_{y\to\infty} \frac{\pi(y,5,4)-\pi(y,5,2)}{\sqrt{y}/\log y} > 0.$$

6. Concluding remarks. Now that we have seen Theorem 1 applied, we may ask how easy it would be to apply it in other cases. As k and K grow, there will be more values of ϱ_χ and ϱ_X with small imaginary parts. This should enable one to take smaller values of T and fewer zeros into the calculations. In a sense, things should be even easier when $k \neq K$. For large values of k and K, it might be possible to make the first one or two zeros closest to the real axis do almost all the work. Alternatively, one can use the exact formulae (8) and (9) along with Theorem 3. While this saves one from computing the complex zeros of L-functions, one must now search for a value of y that will allow Theorem 3 to work. Both of these approaches have been illustrated in Section 5.

I would like to express my thanks to Robert Spira for giving me a table of zeros of L-functions (mod 5) before they were published.

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The thinnest double lattice covering of three-spheres

by

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1. Introduction. Let L be a lattice in three-dimensional euclidean space such that the system of closed unit balls with centres at all points of L forms a covering of R_3. Bambah [1] proved that the maximum possible value of the determinant $d(L)$ of L is $32/5\sqrt{5}$, and several other proofs have since been found (Barnes [2], Few [3]). It is still an open question as to whether the density of any point set yielding such a covering can be smaller than that for the best possible lattice. In this direction it is proved here that if D is a double lattice, so that D is the union of a lattice L and a single translate of L, which provides a covering of R_3 by closed unit balls then

$$d(L) \leqslant 64/5\sqrt{5}.$$

Hence no double lattice can yield a thinner such covering than the best possible lattice.

2. For an arbitrary point X of R_3 denote by $S(X)$ the closed unit ball with centre X.

Let L and $L+X$ be a lattice and its translate in R_3 such that $S(Y)$, $Y \in L \cup (L+X)$ taken together cover R_3. The objective is to prove $d(L) \leqslant 64/5\sqrt{5}$. To this end we assume that $d(L) > 64/5\sqrt{5}$ and derive a contradiction.

LEMMA 1. *Let P be any plane containing a two-dimensional sub-lattice of L. Then the collection $S(X)$, $X \in P \cap L$ does not cover P.*

Proof. Assume the assertion is false so that P is completely covered by such balls. By classical theory

$$d(L\cap P) \leqslant 3\sqrt{3}/2.$$

Let the distance of P to a next lattice plane of L be d. Then

$$64/5\sqrt{5} < d(L) \leqslant 3\sqrt{3}d/2$$

so that
$$d > 128/15\sqrt{15} > 2.$$
There is thus a unique plane of points $(P \cap L)+Y$ of $L+X$ between these two planes of L. Reselect if necessary the plane P as the other plane so that the distance of P to $P+Y$ is at least $\frac{1}{2}d$. We assert now that the part of space between P and $P+Y$ is covered by the balls

(1) $$S(Z), \quad Z \in (P \cap L) \cup \big((P \cap L)+Y\big).$$

For assume that this assertion is false. Then there is an open set not covered. We first ask which spheres of the whole covering can possibly meet this open set. As $d > 2$ these are restricted to those whose centres lie in one of the four planes
$$P, \quad P+Y, \quad P+Y-dN \quad \text{and} \quad P+dN$$
where N is the normal to P of unit length drawn in the direction of $P+Y$. Thus the open set must be met by a sphere whose centre lies in one of the two planes
$$P+Y-dN \quad \text{and} \quad P+dN.$$
Suppose first that such a ball has its centre in $P+dN$ and suppose that it meets a point Z of the open set. Then the normal to P through Z meets the boundary of the ball at a unique point T between P and $P+Y$. Now T must lie in another ball of the covering but since $d > 2$ the only such candidates are those whose centres lie in $P+Y$. However in that case Z would also be covered by such a sphere which is impossible. A similar argument applied to $P+Y-dN$ again leads to a contradiction and so we infer that the part of space between P and $P+Y$ is covered by the collection (1).

It follows that the collection
$$S(Z), \quad Z \in (P \cap L)+nY, \quad n = 0, \pm 1, \pm 2, \ldots$$
covers R_3 since the sub-collection
$$S(Z), \quad Z \in \big((P \cap L)+nY\big) \cup \big((P \cap L)+(n+1)Y\big)$$
covers that part of space between $P+nY$ and $P+(n+1)Y$. This is therefore a lattice covering of R_3 by unit balls of determinant at least $\frac{1}{2}d(L) > 32/5\sqrt{5}$ contrary to the known result. This proves the lemma.

LEMMA 2. *There exist linearly independent points A, B, C of L such that $S(A), S(B), S(C)$ all meet $S(0)$.*

Proof. Assume by way of contradiction that the lemma is false. For a set of points K let $S(K)$ denote the union of the balls $S(Z), Z \in K$. Then by Lemma 1, $R_3 - S(L)$ is a connected region of infinite extent in every direction which must be covered by $S(L+X)$ which is a disconnected region whose components are bounded in at least one direction. This absurdity proves the lemma.

LEMMA 3. (a) *If $0 \neq Y \in L$ is such that $S(Y) \cap S(0) \neq \emptyset$ then Y is a primitive point of L.*

(b) *No three $S(Z), Z \in L$ have a point in common.*

Proof. (a) Assume that this assertion is false so that Y is not a primitive point of L. Then there is an integer $n \geqslant 2$ such that $(1/n)Y \in L$. Using Lemma 2 let $(1/n)Y, B, C$ be linearly independent points of L such that $S\big((1/n)Y\big), S(B), S(C)$ all meet $S(0)$. Then
$$d(L) \leqslant |\det\{(1/n)Y, B, C\}| \leqslant 8/n \leqslant 4 < 64/5\sqrt{5}.$$
This contradiction proves (a).

(b) Suppose on the contrary that $0, A, B$ are distinct points of L such that $S(0) \cap S(A) \cap S(B) \neq \emptyset$. If X, Y are linearly dependent we arrive at a contradiction to (a) hence they must be linearly independent. But then by classical theory if P denotes the plane of $0, X, Y$ then
$$S(Z), \quad Z \in P \cap L$$
covers P contrary to Lemma 1. This proves (b).

Let $U \in L$ be such that $S(U) \cap S(0) \neq \emptyset$ and such that U is the furthest point from 0 with this property. Take rectangular coordinates so that $0U$ is the positive z-axis. The projection of L onto the xy-plane parallel to the z-axis is a two-dimensional lattice l say. Further if we denote the length of the segment $0U$ by $2d$ with $d \leqslant 1$ then
$$d(L) = 2dd(l) > 64/5\sqrt{5}$$
so that in particular $d(l) > 32/5\sqrt{5}$.

For a point Z in the xy-plane let $C(Z)$ denote the closed unit disc in the xy-plane centred at Z. By Lemma 2 there exist linearly independent points A, B of l such that $C(A)$ and $C(B)$ meet $C(0)$.

LEMMA 4. *No three of the discs $C(Z), Z \in l$ have a common point.*

Proof. Assume the lemma is false so that there exist three distinct points $0, Z_1, Z_2$ of l such that $C(0) \cap C(Z_1) \cap C(Z_2) \neq \emptyset$.

Suppose that Z_1, Z_2 are not linearly independent. By what has already been said there exists a point $X \in l$ such that $C(0) \cap C(X) \neq \emptyset$ and X is linearly independent from Z_1. Assume without loss of generality that Z_2 is further from 0 than Z_1 so that the index of X, Z_2 in l is at least 2. Then
$$d(l) \leqslant \tfrac{1}{2}\det(X, Z_2) \leqslant 2 < 32/5\sqrt{5}$$

which is a contradiction. Hence Z_1, Z_2 must be linearly independent. Thus the set $C(X)$, $X \in l$ covers the x, y plane. By classical theory it follows that

$$d(l) \leqslant 3\sqrt{3}/2 < 32/5\sqrt{5}$$

the contradiction that proves the lemma.

LEMMA 5. *At most six of the discs $C(X), X \in l, X \neq 0$, meet $C(0)$.*

Proof. Partition l into congruence classes modulo 2. There are exactly four such classes. One of these contains 0 and if another point Z in this class is such that $C(Z) \cap C(0) \neq \emptyset$ then also

$$C(Z) \cap C(\tfrac{1}{2}Z) \cap C(0) \neq \emptyset$$

and $\frac{1}{2}Z \in l$ contrary to Lemma 4. Hence none of the points we are looking for can be in this class. If there are three or more of these points lying in another class then at least two of these A, B say are such that $\frac{1}{2}(A \pm B) \neq 0$. Hence the index of these two points in l is either 0 or at least 2. If it were 0 Lemma 4 would be contradicted. Hence the index is at least 2 so that

$$d(l) \leqslant 2 < 32/5\sqrt{5}$$

the contradiction that proves the lemma.

COROLLARY. *Exactly 6 or exactly 4 of the discs $C(X), X \in l, X \neq 0$, meet $C(0)$.*

Proof. This follows from the symmetry and the preceding lemmas.

The part of the x, y plane not covered by the collection of discs $C(X), X \in l$ now consists of a set of disconnected open regions. Let Y be a point of such a region. Then the whole line of points through Y and parallel to the z-axis lies outside the collection of spheres $S(Z), Z \in L$ and therefore must be completely covered by a translate of this collection. Hence this line of points corresponds to a line of points t say parallel to this line and lying within the collection.

LEMMA 6. *The spheres $S(X), X \in L$ for which $S(X) \cap t \neq \emptyset$ may be linearly ordered*

$$\ldots < S(X_i) < S(X_{i+1}) < S(X_{i+2}) < \ldots$$

so that

(i) *at least one point of $t \cap S(X_i)$ lies in no other sphere $S(X), X \in L$;*

(ii) $S(X_i) \cap S(X_{i+1}) \cap t \neq \emptyset$;

(iii) $S(X_i) \cap S(X_{i+2}) \cap t = \emptyset$.

Proof. We first prove (i). Assume by way of contradiction that $t \cap S(X) \neq \emptyset$ and $t \cap S(X)$ lies in a collection of sets

$$t \cap S(X') \cap S(X), \quad X \neq X' \in L.$$

Then this collection can consist of only one set $t \cap S(X')$, otherwise three spheres $S(X)$ must intersect which contradicts Lemma 3. Hence

$$t \cap S(X) \subset t \cap S(X').$$

Now $t \cap S(X')$ is a closed line segment. If A is an endpoint then A must lie in another sphere $S(X'')$ distinct from $S(X)$ and $S(X')$. Denote by $p(Z)$ the projection of a point Z onto the x, y plane parallel to the z axis, so that in particular $C(p(X)) \cap C(p(X')) \cap C(p(X'')) \neq \emptyset$. By Lemma 4, $p(X), p(X'), p(X'')$ are not all distinct. But $p(X) \neq p(X')$ since $X \neq X'$. Hence either $p(X) = p(X'')$ or $p(X') = p(X'')$. Suppose first that $p(X) = p(X'')$. We have

$$S(X') \cap S(X'') \cap t \neq \emptyset,$$

$$S(X'') \cap t \subset S(X'+X''-X)$$

so that

$$S(X') \cap S(X'') \cap S(X'+X''-X) \neq \emptyset$$

which contradicts Lemma 3.

Hence $p(X') = p(X'')$. It follows that t lies completely in the collection of spheres $S(Y), Y \in L$ for which Y lies on the line through X' parallel to t. Since these spheres are exactly $2d$ apart it is immediate that the projection $p(t)$ of t onto the x, y plane is within a distance $\sqrt{(1-d^2)}$ of $p(X')$. However $C(p(X))$ also contains $p(t)$ so that the distance between $p(X)$ and $p(X')$ is at most

$$1+\sqrt{(1-d^2)}.$$

Hence

$$d(L) \leqslant 4d\left(1+\sqrt{(1-d^2)}\right).$$

For $0 \leqslant d \leqslant 1$ this function has a single local maximum at $d = \sqrt{3}/2$. Therefore

$$d(L) \leqslant 3\sqrt{3} < 64/5\sqrt{5}$$

a contradiction that proves (i).

Assume now that $S(X) \cap t \neq \emptyset \neq S(Y) \cap t$ with $X, Y \in L$. Denoting by $z(Z)$ the z-coordinate of a point Z we conclude that $z(X) \neq z(Y)$ for otherwise $S(X) \cap t \subset S(Y) \cap t$ or vice versa contrary to what we have just proved. Therefore define $S(X) < S(Y)$ to mean $z(X) < z(Y)$. This yields a linear ordering of $S(X)$ for which $S(X) \cap t \neq \emptyset$ and $X \in L$. We may thus subscript them as

$$\ldots < S(X_i) < S(X_{i+1}) < S(X_{i+2}) < \ldots$$

It remains to prove (ii) and (iii).

Assume that $S(X_i) \cap S(X_{i+1}) \cap t = \emptyset$. Then by (i)

$$S(X_i) \cap S(X_j) \cap t = \emptyset \quad \text{for all } j > i.$$

The point $Z \in S(X_i) \cap t$ for which $z(Z)$ is maximal must lie in another $S(X_j)$ so that $Z \in S(X_j) \cap t$ for some $j < i$. But then

$$S(X_j) \cap t \supset S(X_i) \cap t$$

contrary to (i). Therefore (ii) must hold.

Now assume that $S(X_i) \cap S(X_{i+2}) \cap t \neq \emptyset$. The by (i)

$$S(X_{i+1}) \cap t \subset (S(X_i) \cap t) \cup (S(X_{i+2}) \cap t)$$

contrary to (i) and so (iii) follows. This proves the lemma.

LEMMA 7. *With the notation of Lemma 6 either*

(i) $X_{i+1} = X_i + (0, 0, 2d)$ *for all* i;

or

(ii) $p(X_{i+1}) \neq p(X_i)$ *and then* $X_{i+2} = X_i + (0, 0, 2d)$ *for all* i.

Proof. We assume that (i) is false so that for some i we must have

$$p(X_{i+1}) \neq p(X_i).$$

It follows from Lemma 4 that for any j,

$$\text{either } p(X_j) = p(X_i) \text{ or } p(X_j) = p(X_{i+1}),$$

so that

$$\text{either } X_j = X_i + n(0, 0, 2d) \text{ or } X_j = X_{i+1} + n(0, 0, 2d)$$

for some integer n. It is however clear that for any integer n such points are members of the sequence $\{X_j\}$ and the lemma follows.

Using Lemma 7 we classify the line t as type A if (i) holds and as type B if (ii) holds. Let $p(A)$ denote the projection of all the lines t of type A onto the x, y plane and $p(B)$ similarly the projection of all lines t of type B.

LEMMA 8. *If S denotes the projection of $S(0) \cap S((0, 0, 2d))$ onto the x, y plane then*

(i) $p(A) = S + l$,

(ii) $(S+X) \cap (S+Y) = \emptyset$ *for all points* $X \neq Y$ *of* l.

Proof. (i) is immediate whereas if (ii) is not true then three spheres $S(X)$, $X \in L$ have a common point contrary to Lemma 3.

In order to say something about $p(B)$ we let T be the set of lines of type B that are contained in a collection of spheres

$$\ldots S(X_i),\ S(X_{i+1}),\ S(X_{i+2}), \ldots$$

as in Lemma 6 and 7 so that

$$p(X_{i+1}) \neq p(X_i) \quad \text{for any } i,$$
$$X_{i+2} = X_i + (0, 0, 2d) \quad \text{for all } i$$

and

$$z(X_{i+1}) > z(X_i) \quad \text{for all } i.$$

LEMMA 8. (i) $p(T) \subset p(S(X_i) \cap S(X_{i+1}))$ *for all* i;

(ii) *If* $z(X_{i+1}) - z(X_i) \geqslant z(X_{i+2}) - z(X_{i+1})$ *for some* i *then*

$$p(S(X_i) \cap S(X_{i+1}))$$

is the elliptic domain bounded by the projection of the circle of intersection of the surfaces of $S(X_i)$ and $S(X_{i+1})$ onto the x, y plane.

Proof. (i) is immediate. For (ii) without loss of generality we may assume that $X_i = 0$. We claim that the circle of intersection of $S(X_i)$ and $S(X_{i+1})$ lies in the upper half-plane, for assume that this assertion is false. We may then take coordinates so that the point $(1, 0, 0) \in S(X_{i+1})$. However by hypothesis $z(X_{i+1}) - z(X_i) \geqslant d$ so that also $(1, 0, 2d) \in S(X_{i+1})$. Thus if $X_{i+1} = (x, y, z)$ we have

$$(x-1)^2 + y^2 + z^2 \leqslant 1$$

and

$$(x-1)^2 + y^2 + (2d - z)^2 \leqslant 1 \quad \text{where} \quad 0 \leqslant z \leqslant 2d.$$

Hence $(x-1)^2 + y^2 \leqslant 1 - d^2$ so that $\sqrt{(x^2+y^2)} \leqslant 1 + \sqrt{(1-d^2)}$. Again it follows that $d(L) \leqslant 4d(1 + \sqrt{(1-d^2)})$ which leads to a contradiction as in Lemma 6. This proves the lemma.

LEMMA 9. *With the hypotheses of Lemma 8, $p(T)$ is contained in the projection C say of intersection of the surfaces of $S(X_i) = S(0)$ with $S(X_{i+1} - (0, 0, z(X_{i+1})) + (0, 0, d))$.*

Proof. If C denotes the circle of intersection of the surfaces of $S(X_i)$ and $S(X_{i+1} - (0, 0, z(X_{i+1})) + (0, 0, d))$ then it follows exactly as in the last lemma that with $X_i = 0$ so C lies in the upper half-plane. Now if $z(X_{i+1}) - z(X_i) \geqslant z(X_{i+2}) - z(X_{i+1})$ then since C is in the upper half-plane

$$S(X_{i+1}) \cap S(X_i) \subset S(X_i) \cap S(X_{i+1} - (0, 0, z(X_{i+1})) + (0, 0, d))$$

and the lemma is true. If on the other hand the reverse inequality holds then by reversing the direction of the z-axis the same argument holds. This proves the lemma.

LEMMA 10. *$p(B)$ is the union of disconnected open regions such that each component lies inside an ellipse of the type described in Lemma 9 and no such ellipse meets $p(A)$.*

Proof. We have already shown everything except that no ellipse meets $p(A)$. Suppose then that such an ellipse meets $p(A)$. We may then suppose that such an ellipse meets $p\big(S(0)\cap S((0,0,2d))\big)$. This implies the existence of a circle $C(X)$, $X\in l$, $X\neq 0$, $X=(x,y,0)$ such that

$$x^2+y^2\leqslant 1+\sqrt{(1-d^2)}$$

which leads to a contradiction as before.

The proof of the main theorem is now divided into two cases.

Case 1: Exactly four circles $C(X)$, $X\in l$, $X\neq 0$, intersect $C(0)$.

The centers of these circles may be labelled and coordinates chosen so that they become

$$\pm(2\cos a,\,0,\,0)\pm(2\cos c\cos b,\,2\cos c\sin b,\,0),\qquad 0\leqslant a,\,b,\,c\leqslant \pi/2.$$

The part of the x, y plane not covered by the circles $C(X)$, $X\in l$ is of the form $U+l$ where U is the open region bounded by the four circles $C(0)$, $C\big((2\cos a,0,0)\big)$, $C\big((2\cos c\cos b,2\cos c\sin b,0)\big)$ and $C\big((2\cos a+2\cos c\cos b,$ $2\cos c\sin b,0)\big)$ but lying in none of them. Now since a translation of the system $S(X)$, $X\in L$ must cover that part of the space not covered by the system itself it follows that a translation of the lines t parallel to the z-axis covered by $S(X)$, $X\in L$ must cover $U+l$. For this to be the case since U is connected it is necessary that a translation of S or of an ellipse of the form in Lemma 10 covers U.

Subcase (i): A translation of S covers U. It follows that the diameter of S is not less than the diameter of U. The diameter of S is evidently $2\sqrt{(1-d^2)}$. Now U contains the four points

$$(\cos a,\sin a),\qquad \big(\cos(b-c),\,\sin(b-c)\big),$$
$$\big(\cos(b+c),\,\sin(b+c)\big)+(2\cos a,\,0),$$
$$(\cos a,\,-\sin a)+(2\cos c\cos b,\,2\cos c\sin b).$$

Hence if d^* denotes the diameter of U then

$$2d^{*2}\geqslant\big(\cos(b+c)-\cos(b-c)+2\cos a\big)^2+\big(\sin(b+c)-\sin(b-c)\big)^2+$$
$$+(2\cos c\cos b)^2+(2\cos c\sin b-2\sin a)^2$$
$$=8\big(1-\sin b\sin(a+c)\big).$$

Hence

$$d^2\leqslant\sin b\sin(a+c).$$

But

$$d(L)=2d2\cos a\cdot 2\cos c\sin b$$

so that

$$d^2(L)\leqslant 64\cos^2 a\cos^2 c\sin^3 b\sin(a+c)$$
$$\leqslant 64\cos^2 a\cos^2 c\sin(a+c)=M,\ \text{say}.$$

Now $S\big((0,0,2d)\big)$ is the sphere furthest from 0 that intersects $S(0)$ by construction hence

$$d^2\geqslant\cos^2 a\quad\text{ and }\quad d^2\geqslant\cos^2 c$$

which yields

$$\text{(2)}\qquad \cos^2 a\leqslant\sin(a+c)\quad\text{ and }\quad\cos^2 c\leqslant\sin(a+c).$$

Holding $a+c$ fixed we can make a and c approach one another and thus increase M while not violating (2). Therefore

$$M\leqslant 64\cos^4 a\sin 2a=N\ \text{say},$$

for some a satisfying

$$\cos a\leqslant 2\sin a\quad\text{ and }\quad 0<a<\pi/2.$$

If N has a local maximum in the range $0<a<\pi/2$, $\cos a<2\sin a$, then by differentiation for this a we must have $\sin^2 a=1/6$, $\cos^2 a=5/6$ which lies outside the range $\cos a<2\sin a$ and so a contradiction. Thus for a maximum $\cos a=2\sin a$ or $\cos^2 a=4/5$ so that

$$N\leqslant 64(4/5)^2(4/5)<d^2(L)$$

a contradiction which shows that subcase (i) is impossible.

Subcase (ii): A translation of an ellipse of the form in Lemma 9 covers U. Such an ellipse must arise from the intersection of $S(0)$ with $S(2\cos a,0,d)$ or from the intersection of $S(0)$ with $S(2\cos c\cos b,$ $2\cos c\sin b,d)$ or from the symmetric opposites of these in 0 or from a lattice translate of one of these. There is thus no loss of generality in assuming that the ellipse arises from the intersection of $S(0)$ with $S\big((2\cos a,0,d)\big)$. The circle of intersection of the surfaces of these two spheres is given by the two equations

$$x^2+y^2+z^2=1\quad\text{ and }\quad(x-2\cos a)^2+y^2+(z-d)^2=1.$$

These two equations imply

$$(x-\cos a)^2+d^2y^2/(d^2+4\cos^2 a)=d^2(4\sin^2 a-d^2)/4(d^2+4\cos^2 a)$$

which is the equation of the projection of the circle of intersection onto the x, y plane. Now a translation of this ellipse covers U. Hence the x width of U does not exceed the x width of the ellipse. The x width of the ellipse is evidently

$$2\sqrt{d^2(4\sin^2 a-d^2)/4(d^2+4\cos^2 a)}.$$

Further, U contains the points

$$(\cos(b-c),\ \sin(b-c)) \quad \text{and} \quad (\cos(b+c)+2\cos a,\ \sin(b-c))$$

so that the x width of U is at least

$$\cos(b+c)-\cos(b-c)+2\cos a = 2(\cos a-\sin b\sin c).$$

Thus we obtain

$$4(\cos a-\sin b\sin c)^2 \leqslant d^2(4\sin^2 a-d^2)/(d^2+4\cos^2 a)$$
$$\leqslant (4\sin^2 a-d^2)/(1+4\cos^2 a)$$

which implies that

$$d^2 \leqslant 4\sin^2 a-4(1+4\cos^2 a)(\cos a-\sin b\sin c)^2$$

and

$$d^2(L) = 64d^2\cos^2 a\sin^2 b\cos^2 c \leqslant 64\cos^2 a\sin^2 b\cos^2 c\times$$
$$\times\min\left(1,\ 4\left(\sin^2 a-(1+4\cos^2 a)(\cos a-\sin b\sin c)^2\right)\right).$$

Since $d^2(L) > 64^2/125$ so we must have $\cos^2 a > \frac{1}{2}$ and $\cos^2 c > \frac{1}{2}$ and therefore also $\cos a > \sin c$. This implies that $\cos a-\sin b\sin c$ decreases as $\sin b$ increases and so

$$d^2(L) \leqslant 64\cos^2 a\cos^2 c\cdot\min\left(1,\ 4\left(\sin^2 a-(1+4\cos^2 a)(\cos a-\sin c)^2\right)\right).$$

Subcase (ia): $1 \leqslant 4\left(\sin^2 a-(1+4\cos^2 a)(\cos a-\sin c)^2\right)$. This implies that

$$(\cos a-\sin c)^2 \leqslant (\sin^2 a-\tfrac{1}{4})/(1+4\cos^2 a) = (3/4-\cos^2 a)(1+4\cos^2 a)$$

which increases as $\cos a$ decreases. Thus since $\cos^2 a > \frac{1}{2}$,

$$(\cos a-\sin c)^2 \leqslant 1/12.$$

In this case $d^2(L) \leqslant 64\cos^2 a\cos^2 c$, so we look for the maximum of $\cos a\cos c$ subject to $0 < \cos a-\sin c \leqslant 1/\sqrt{12}$ and $\cos c > 1/\sqrt{2}$. This is clearly achieved when $\cos a-\sin c = 1/\sqrt{12}$ and then

$$\cos a\cos c = (1/\sqrt{12}+\sin c)\cos c.$$

The maximum of this function is found by differentiation to be attained when

$$\sin c = (\sqrt{97}-1)/4\sqrt{12}$$

and so

$$\cos^2 a\cos^2 c \leqslant \left(1/\sqrt{12}+(\sqrt{97}-1)/4\sqrt{12}\right)^2\left((94+2\sqrt{97})/192\right)$$
$$= (1/192^2)(11,128+776\sqrt{97}).$$

Hence

$$d^2(L) \leqslant 64(1/192^2)(11,128+776\sqrt{97})$$

which is a contradiction.

Subcase (ib): $1 > 4\left(\sin^2 a-(1+4\cos^2 a)(\cos a-\sin c)^2\right)$. Setting

$$f(x) = x^2\left(1-x^2-(1+4x^2)(x-\sin c)^2\right)$$

we obtain

$$f'(x) = 2x\left(1-x^2-(1+4x^2)(x-\sin c)^2-x^2-4x^2(x-\sin c)^2-\right.$$
$$\left.-x(1+4x^2)(x-\sin c)\right) < 0$$

if $x > 1/\sqrt{2}$ and $\sin c < 1/\sqrt{2}$.

Hence $d^2(L) \leqslant 256\cos^2 c\cdot f(\cos a)$ decreases as $\cos a$ increases with $\cos a \geqslant 1/\sqrt{2}$ and $\cos c > 1/\sqrt{2}$. If now

$$4\left(\sin^2 a-(1+4\cos^2 a)(\cos a-\sin c)^2\right) < 1 \quad \text{when} \quad \cos a = 1/\sqrt{2}$$

this is an immediate contradiction so we must have

$$4\left(\tfrac{1}{2}-3(1/\sqrt{2}-\sin c)^2\right) \geqslant 1.$$

But in this case we may decrease $\cos a$ to a value at least $1/\sqrt{2}$ to obtain

$$4\left(\sin^2 a-(1+4\cos^2 a)(\cos a-\sin c)^2\right) = 1$$

without violating the inequality on $d^2(L)$ and retaining the conditions $\cos^2 a > \frac{1}{2}$, $\cos^2 c > \frac{1}{2}$. Thus we can apply without alteration the argument of subcase (ia) to again obtain a contradiction. This completes the first case.

Case 2: Exactly six circles $C(X)$, $X\in l$ intersect $C(0)$. By the classical theory of the geometry of numbers the centres of these circles are of the form $\pm P$, $\pm Q$, $\pm(P+Q)$.

Let D denote the bounded region bounded by the circles $C(0)$, $C(P)$, $C(P+Q)$ but external to all these circles. Similarly let D' denote the bounded region bounded by $C(0)$, $C(Q)$, $C(P+Q)$ but external to all these circles. Then the set of points of the x, y plane not covered by the collection of circles $C(X)$, $X\in l$ is simply

$$(D+l)\cup(D'+l).$$

This set must then be covered by a translate of $S+l$ together with a translate of the ellipses specified in Lemmas 9 and 10.

Subcase (i): $(D+l)\cup(D'+l)$ is covered by a translate of $S+l$ alone. A translate of S cannot cover two connected members of $D+l$ since all $S+X$, $X\in l$ are disjoint. It therefore follows that a translate of S must cover both one member of $D+l$ and one member of $D'+l$.

Without loss of generality we may assume that this translate covers D. The projections $C(0), C(P), C(P+Q)$ are of columns of spheres of the form $S(P')+nZ, S(P'')+nZ, S(P''')+nZ$ where $Z=(0,0,2d)$ and P', P'', P''' project to $0, P, P+Q$ respectively and n is an integer. Suppose for the moment that no $S(P')+nZ$ intersects an $S(P'')+nZ$. By Lemma 2 it follows that an $S(P'')+nZ$ must meet an $S(P''')+nZ$ and also an $S(P')+nZ$ must meet an $S(P''')+nZ$. Therefore since Z is the furthest point of L from 0 such that $S(0)\cap S(Z)\neq\emptyset$, it follows that the diameters of $C(P)\cap C(P+Q)$ and $C(0)\cap C(P+Q)$ are at least $2\sqrt{(1-d^2)}$ which is the diameter of S. Therefore by reselecting $0, P, Q$ if necessary we may assume that D is bounded by $C(0), C(P), C(Q)$ with $C(0)\cap C(P)$ and $C(0)\cap C(Q)$ of diameter at least the diameter of S. With this notation we now assert that a translate of S must cover D together with the region D'' bounded by $C(P), C(Q), C(P+Q)$ and lying external to all of them. Let E denote the union of the sets

$$C(0),\ C(P),\ C(Q),\ C(P+Q),\ D,\ D''.$$

Our assertion will follow if the distance from some point of D to the complement of E is at least the diameter of S. Now E is bounded by four circular arcs A_1, A_2, A_3, A_4 where $A_1\subset C(0), A_2\subset C(P), A_3\subset C(Q)$, $A_4\subset C(P+Q)$. If the assertion is false then the distance from any point of D to a point X of $A_1\cup A_2\cup A_3\cup A_4$ is less than the diameter of S. Now $D\cap C(0)$ is a circular arc whose minimal distance to A_1 is the smaller of the diameters of $C(0)\cap C(Q)$ and $C(0)\cap C(P)$ both of which are at least the diameter of S. Hence $X\notin A_1$. Further $D\cap C(P)$ is separated from A_2 by $C(0)\cap C(P)$ and $C(P)\cap C(P+Q)$ both of which have diameters at least the diameter of S and therefore as before $X\notin A_2$. Similarly $X\notin A_3$. Thus $X\in A_4$. But any line segment drawn from a point of D to $P+Q$ meets the boundary of $C(P+Q)$ in a point not in A_4 since the rays $P+Q, Q$ produced and $P+Q, P$ produced do not meet D. Hence the closest point of A_4 to D is one of its endpoints and again we have a contradiction. This proves the assertion.

All the conditions of Case 1, subcase (i) are now satisfied together with the additional condition that $C(P)\cap C(Q)\neq\emptyset$. Since this condition does not affect the argument given in this previous case we arrive at a contradiction in the same manner.

Subcase (ii): At least one of D, D' is covered by the translate of an ellipse of the form specified in Lemma 10. Without loss of generality we may assume that D is so covered and that D is bounded by the circles

$$C(0),\ C(P),\ C(Q).$$

Now if an ellipse of the right type arises from intersection of $S(0)+nZ$ with $S(R)+nZ$ then a translation of this ellipse arises from the intersection of $S(0)+nZ$ with $S(-R)+nZ$. Hence the ellipse in question arises from the intersection of two sets of $S(0)+nZ$, $S(P')+nZ$, $S(Q')+nZ$ where P', Q' are points of L that project into P, Q respectively. By relabelling the points $0, P, Q$ if necessary we may assume that the ellipse in question arises from the intersection of the two collections $S(0)+nZ$ and $S(P')+nZ$. Take coordinates as before so that

$$P=(2\cos a, 0, 0) \quad \text{and} \quad Q=(2\cos c\cos b, 2\cos c\sin b, 0)$$

where $0\leqslant a\leqslant\pi/2$, $0\leqslant c\leqslant\pi/2$, and $0\leqslant b\leqslant\pi/2$, the last following from the fact that the three sides of the triangle $0PQ$ are all at most 2 and all at least $\sqrt{2}$ showing that it is acute angled. The ellipse in question is then given as before by

$$(x-\cos a)^2+d^2y^2/(d^2+4\cos^2 a) = d^2(4\sin^2 a-d^2)/4(d^2+4\cos^2 a)$$

with x width

$$2\sqrt{d^2(4\sin^2 a-d^2)/4(d^2+4\cos^2 a)}$$

and this must be at least the x width of D say $x(D)$. Hence

$$4\big(d^2(4\sin^2 a-d^2)/4(d^2+4\cos^2 a)\big)\geqslant x^2(D)$$

hence

$$(4\sin^2 a-d^2)/(1+4\cos^2 a)\geqslant x^2(D),$$

whence

$$d^2\leqslant 4\sin^2 a-(1+4\cos^2 a)x^2(D).$$

Thus

$$d^2(L) = 64d^2\cos^2 a\sin^2 b\cos^2 c$$
$$\leqslant 64\cos^2 a\sin^2 b\cos^2 c\cdot\min\big(1, 4\sin^2 a-(1+4\cos^2 a)x^2(D)\big).$$

In particular $\cos^2 a>\frac{1}{2}$, $\cos^2 c>\frac{1}{2}$ and $\sin^2 b>\frac{1}{2}$.

If we call the angle $0PQ$ by b' and the length PQ by $2\cos c'$ then in exactly the same way we get $\sin^2 b'>\frac{1}{2}$ and $\cos^2 c'>\frac{1}{2}$.

Also we must have $2\cos c\sin b-1>\sin a$ for otherwise

$$d(l)\leqslant 2\cos a(1+\sin a)\leqslant 3\sqrt{3}/2$$

which is too small.

Auxilliary lemma. *Holding a and $\cos c\sin b$ fixed and denoting by D^* the region corresponding to D as b, c vary then $x(D^*)$ is least when*

$$2\cos c\cos b = \cos a.$$

Proof. With the notation as above

$$x(A) = 2\cos a-\cos(b-c)-\cos(b'-c').$$

The variables are connected by the relations

(3) $$\cos a = \cos c' \cos b' + \cos c \cos b$$

and

(4) $$\cos c' \sin b' = \cos c \sin b = \text{constant}.$$

By symmetry we may assume that $b \geqslant b'$ so that $c \geqslant c'$.

Set $f = \cos(b-c) + \cos(b'-c')$. By (3) and (4) we arrive at

$$df = \tfrac{1}{4}(\sin 2c' \sin 2b - \sin 2c \sin 2b')\, dc$$

and since $\pi/4 < b' \leqslant b \leqslant \pi/2$ and $0 < c' \leqslant c \leqslant \pi/4$ it follows that

$$\sin 2c' \sin 2b - \sin 2c \sin 2b' \leqslant 0$$

so that f decreases as c increases in the range in question. Thus as c decreases so $x(D^*)$ decreases and this proves the lemma.

It follows at once from this lemma that

$$d^2(L) \leqslant 16 \cos^2 a (4\cos^2 c - \cos^2 a) \times$$
$$\times \min\big(1,\, 4\big(\sin^2 a - (1 + 4\cos^2 a)\cos^2(b+c)\big)\big),$$

subject to $2\cos c \cos b = \cos a$ and $0 < a, c < \pi/4, \pi/4 < b \leqslant \pi/2$. This implies by our initial assumption that

$$64/125 < \cos^2 a (\cos^2 c - \tfrac{1}{4}\cos^2 a) \times$$
$$\times \min\big(1,\, 4\big(\sin^2 a - (1+4\cos^2 a)\cos^2(b+c)\big)\big) = F, \text{ say}.$$

The function F proved somewhat difficult to handle so we resorted to a computation to show that the above inequality is impossible. We show first that $\cos^2 c > .87$.

Case (i): $\cos^2 a \leqslant 3/4$. With $t = \cos^2 c$ we must have $\cos^2 a (t - \frac{1}{4}\cos^2 a) > .512$. The left hand side is a monotone increasing function of $\cos^2 a$ in the range in question hence

$$3/4 (t - 3/8) > .512$$

whence

$$t > (41.768)/48 > .87.$$

Case (ii): $\cos^2 a > 3/4$. Again with $t = \cos^2 c$ we must have

$$\cos^2 a (t - \tfrac{1}{4}\cos^2 a) 4 (1 - \cos^2 a) > .512.$$

But in the range in question $4\cos^2 a(1-\cos^2 a) \leqslant 3/4$ so that again $3/4(t - 3/16) > .512$ and we obtain the same bound as in case (i).

This proves that $\cos^2 c > .87$.

To describe the computations suppose that within this interval we further restrict $\cos^2 c$ by

$$.87 \leqslant c_1 \leqslant \cos^2 c \leqslant c_2 \leqslant 1.$$

Then it follows that

$$\cos^2 a (c_2 - \tfrac{1}{4}\cos^2 a) > .512.$$

Thus with $t = \cos^2 a$ we obtain

$$0 > t^2 - 4c_2 t + 2.048$$

so that $t > 2c_2 - \sqrt{4c_2^2 - 2.048} > a_1$, say, where a_1 is chosen as a lower approximation to this number and thus

$$a_1(c_2 - \tfrac{1}{4}a_1) < .512 \quad \text{and} \quad \cos^2 a > a_1 > \tfrac{1}{2}.$$

Now since $2\cos c \cos b = \cos a$, it follows that

$$\cos(b+c) = \tfrac{1}{2}\cos a - \sin c \sqrt{(1 - (\cos^2 a / 4\cos^2 c))}$$
$$> \tfrac{1}{2}\sqrt{a_1} - \big(\sqrt{(1-c_1)}\big)\big(\sqrt{(1-(a_1/4c_2))}\big) > b_1, \text{ say},$$

where b_1 is chosen as a lower approximation to this number, $0 < b_1 < 1$. We then have

$$.512 < F \leqslant \cos^2 a (c_2 - \tfrac{1}{4}\cos^2 a) 4 \big(\sin^2 a - (1 + 4\cos^2 a) b_1^2\big).$$

With $t = \cos^2 a$ this becomes

$$f(t) = t(4c_2 - t)\big(1 - t - (1+4t)b_1^2\big) > .512.$$

The roots of the cubic $f(t)$ are

$$0,\ (1-b_1^2)/(1+4b_1^2) < 1 \text{ and } 4c_2 > 1$$

hence if we can show that $f'(\frac{1}{2}) < 0$ and $f'(1) < 0$ then $f(t)$ will be monotone decreasing in our range so that we could conclude that

$$f(a_1) > .512.$$

Now

$$f'(t) = 4c_2(1-b_1^2) - 2\big(1 - b_1^2 + 4c_2(1+4b_1^2)\big)t + 3(1+4b_1^2)t^2$$

so that

$$f'(1) = 4c_2(1-b_1^2) - 2\big(1 - b_1^2 + 4c_2(1+4b_1^2)\big) + 3(1+4b_1^2)$$
$$\leqslant 2c_2(1-b_1^2) + 3(1+4b_1^2) - 6(1+4b_1^2) < 0 \quad \text{since } 3/4 < c_2 \leqslant 1.$$

Further

$$f'(\tfrac{1}{2}) = 4c_2(1-b_1^2) - \big(1 - b_1^2 + 4c_2(1+4b_1^2)\big) + (3/4)(1+4b_1^2)$$
$$\leqslant 3c_2(1-b_1^2) + (3/4)(1+4b_1^2) - 4c_2(1+4b_1^2) < 0$$

and so $f(a_1) > .512$ and therefore also

$$g(a_1) = 4\big(1 - a_1 - (1+4a_1)b_1^2\big) > 1.$$

Now choosing c_1 and c_2 as in the following table we see that this is impossible therefore providing us with the final contradiction that proves the theorem.

c_1	c_2	a_1	b_1^2	$\frac{1}{4}g(a_1)$
.87	.88	.74	.014	.22
.88	.89	.72	.011	.24
.89	.9	.705	.0146	.242
.9	.91	.694	.017	.242
.91	.92	.683	.02	.243
.92	.93	.671	.023	.245
.93	.94	.66	.027	.242
.94	.95	.65	.032	.24
.95	.96	.64	.038	.23
.96	.97	.63	.045	.22
.97	.98	.62	.054	.2
.98	.99	.61	.067	.16
.99	1	.6	.086	.11

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Euclid's algorithm in algebraic function fields, II

by

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In memory of Harold Davenpor

1. Introduction. The theorem that there are only a finite number of Euclidean algebraic number fields with one fundamental unit is a corollary of Davenport's work on the inhomogeneous minima of certain quadratic, cubic and quartic forms ([6], [7], [8]).

In [1], I imitated his arguments and obtained analogues of his theorems for the case of function fields of transcendence degree 1 over finite constant fields. Subsequently, I reformulated the question of the existence of a Euclidean algorithm, in function fields over arbitrary constant fields, in terms of the Riemann–Roch Theorem, [2]. The reformulation led to the solution of the problem (with no restriction on the units) for fields of genus 0 over arbitrary constant fields and later, [3](1), for fields of genus > 0 over infinite constant fields. In this paper, I show that there are only a finite number (in a sense which is made precise in § 3) of function fields, of given genus, over finite constant fields in which Euclid's algorithm holds.

The statement of the main theorem is given in § 3, after the notation has been established in § 2. It is both appropriate and convenient to express part of the argument in the language of the geometry of numbers and the necessary vocabulary is set out in § 4, together with an outline of the proof. The preliminary lemmas are proved in § 5 and the proof of the theorem is completed in § 6.

The results proved below are expressed in terms of the Euclidean algorithm problem, though they can be extended to the case of the inhomogeneous minima of certain forms. The methods are the same as those used here. One replaces a $k[x]$-basis of the ring I defined in § 2 by a set of linear forms with coefficients in the field $k\{x\}$ defined in (17).

(1) There is a mis-print in the displayed formula on p. 5 of the Appendix. The first part should read $\nu_{\mathfrak{p}_1}(\lambda_2^{(1)}-\lambda_1^{(1)}) > \nu_{\mathfrak{p}_1}(\lambda_1^{(1)})$.