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A problem in comparative prime number theory

by

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In memory of Harold Davenport

1. Introduction. Let $\pi(y, k, a)$ denote the number of primes $\leqslant y$ that are congruent to $a \pmod k$. In a series of papers, Knapowski and Turán [3] considered among other questions the problem of whether $\pi(y, k, a)-\pi(y, k, b)$ changes sign infinitely often. The first result of this nature is due to Littlewood who showed that $\pi(y, 4, 1)-\pi(y, 4, 3)$ changes sign infinitely often. Knapowski and Turán were able to handle many other cases under the assumption that no L-series with a character $\bmod k$ has a real zero strictly between 0 and 1 (an assumption that has been checked for $k \leqslant 24$ and is quite possibly true for all k). For example, under this assumption they were able to show that $\pi(y, k, 1)-\pi(y, k, a)$ changes sign infinitely often. However the general problem is still open.

The first unknown case is that of $\pi(y, 5, 4)-\pi(y, 5, 2)$. We prove below (Theorem 2 and 6) that there are positive constants c_1 and c_2 such that

$$\liminf_{y\to\infty} \frac{\pi(y, 5, 4)-\pi(y, 5, 2)}{\sqrt{y}/\log y} \leqslant -c_1,$$

$$\limsup_{y\to\infty} \frac{\pi(y, 5, 4)-\pi(y, 5, 2)}{\sqrt{y}/\log y} \geqslant c_2.$$

The first inequality is actually easy; the real difficulty is in the second. The "correct" values of c_1 and c_2 are undoubtedly $+\infty$, but this remains unestablished.

More generally, one can consider sign changes of $\varphi(k)\pi(y, k, a)-\varphi(K)\pi(y, K, A)$. We prove below a general result (Theorem 1) that applies to this situation. Unfortunately most cases will require a numerical calculation to reach the desired conclusion. This will be discussed in Sections 5 and 6.

2. Notation and other preliminaries. Throughout, k, K, a, A will be positive integers and if $k = K$, we will assume that $a \not\equiv A \pmod k$.

We will let χ and X denote characters $(\bmod k)$ and $(\bmod K)$ respectively. The principal characters to these moduli will be denoted by χ_0 and X_0. Real characters (other than the principal characters) will be denoted by χ_1 and X_1 respectively. The combination

$$r = r(k, a; K, A) = \sum_{X_1} X_1(A) - \sum_{\chi_1} \chi_1(a)$$

will be of great importance; when $r = 0$ things are much simpler. We have already introduced $\pi(y, k, a)$; the related function

$$\psi_0(y, k, a) = \sideset{}{'}\sum_{\substack{p^n \leqslant y \\ p^n \equiv a \pmod k}} \log p$$

will be useful (here $\sum'$ means that if $p^n = y$, only $\frac{1}{2}\log p$ should be included in the sum).

As usual we write $s = \sigma + it$. The Dirichlet L-functions

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

and Riemann zeta function $\zeta(s)$ are natural tools in this sort of problem. We will use $\varrho_\chi = \beta_\chi + i\gamma_\chi$ to denote a zero of $L(s, \chi)$; ϱ_χ is said to be trivial if $\beta_\chi \leqslant 0$, otherwise it is non-trivial.

Next, for real u and positive T, the following related sums will be used,

$$A_T(u) = A_T(u; k, a; K, A)$$
$$= \sum_{X \neq X_0} \sum_{\substack{\varrho_X \\ \beta_X > 0, |\gamma_X| < T}} \frac{\overline{X}(A)}{\varrho_X} e^{(\varrho_X - \frac{1}{2})u} - \sum_{\chi \neq \chi_0} \sum_{\substack{\varrho_\chi \\ \beta_\chi > 0, |\gamma_\chi| < T}} \frac{\overline{\chi}(a)}{\varrho_\chi} e^{(\varrho_\chi - \frac{1}{2})u},$$

$$A_T^*(u) = A_T^*(u; k, a; K, A)$$
$$= r + \sum_{X \neq X_0} \sum_{\substack{\varrho_X \\ \beta_X > 0, |\gamma_X| < T}} \frac{\overline{X}(A)}{\varrho_X} \left(1 - \frac{|\gamma_X|}{T}\right) e^{(\varrho_X - \frac{1}{2})u} -$$
$$- \sum_{\chi \neq \chi_0} \sum_{\substack{\varrho_\chi \\ \beta_\chi > 0, |\gamma_\chi| < T}} \frac{\overline{\chi}(a)}{\varrho_\chi} \left(1 - \frac{|\gamma_\chi|}{T}\right) e^{(\varrho_\chi - \frac{1}{2})u}.$$

The relation between them is easily seen to be

$$A_T^*(u) = r + \frac{1}{T} \int_0^T A_t(u)\, dt. \tag{1}$$

Wherever the limit exists, we will define

$$A_\infty(u) = A_\infty(u; k, a; K, A) = \lim_{T \to \infty} A_T(u; k, a; K, A).$$

Lastly, for convenience we will use the notation $f_a(s)$ (a real) to denote an analytic function of s for $\sigma > a$; even with the same value of a it will usually denote different functions in different appearances. Any function $f_{1/2}(s)$ will be slightly better behaved: $f_{1/2}(s)$ will also be analytic for $\sigma = \frac{1}{2}$.

Lemma 1. *For* $\sigma > 1$,

$$\int_0^\infty [\varphi(k)\pi(e^u, k, a) - \varphi(K)\pi(e^u, K, A)] e^{-us} du \tag{2}$$
$$= \sum_{\chi \neq \chi_0} \frac{\overline{\chi}(a)}{s} \log L(s, \chi) - \sum_{X \neq X_0} \frac{\overline{X}(A)}{s} \log L(s, X) - \frac{r}{2s} \log(s - \tfrac{1}{2}) + f_{1/2}(s).$$

Proof. For $\sigma > 1$,

$$\log L(s, \chi) = \sum_p \sum_n \frac{1}{n} \chi(p^n) p^{-ns}$$
$$= \sum_p \chi(p) p^{-s} + \tfrac{1}{2} \log L(2s, \chi^2) + f_{1/3}(s).$$

Hence if $\sigma > 1$,

$$\varphi(k) \sum_{p \equiv a \pmod k} p^{-s} = \sum_\chi \overline{\chi}(a) [\log L(s, \chi) - \tfrac{1}{2} \log L(2s, \chi^2)] + f_{1/3}(s)$$
$$= \log \zeta(s) - \tfrac{1}{2} \log \zeta(2s) + \sum_{\chi \neq \chi_0} \overline{\chi}(a) \log L(s, \chi) +$$
$$+ \tfrac{1}{2} \sum_{\chi_1} \chi_1(a) \log(s - \tfrac{1}{2}) + f_{1/2}(s).$$

The lemma follows since for $\sigma > 1$,

$$\int_0^\infty \varphi(k)\pi(e^u, k, a) e^{-us} du = \frac{1}{s} \varphi(k) \sum_{p \equiv a \pmod k} p^{-s}.$$

Lemma 2. *Suppose that none of the* $L(s, \chi)$ *and* $L(s, X)$ *have real zeros strictly between* $1/2$ *and* 1. *Suppose further that*

$$\limsup_{y \to \infty} \frac{\varphi(k)\pi(y, k, a) - \varphi(K)\pi(y, K, A)}{\sqrt{y}/\log y} \neq \infty.$$

If $\varrho = \beta + i\gamma$, $\beta > 0$ *is such that*

$$\sum_{X \neq X_0} \sum_{\substack{\varrho_X \\ \varrho_X = \varrho}} \overline{X}(A) - \sum_{\chi \neq \chi_0} \sum_{\substack{\varrho_\chi \\ \varrho_\chi = \varrho}} \overline{\chi}(a) \neq 0, \tag{3}$$

then $\beta = \frac{1}{2}$. *Thus the right side of* (2) *is analytic for* $\sigma > 1/2$.

Proof. By hypothesis there is a constant $c > 0$ such that for $u > \log 2$,

$$(4)\qquad g(u) = c\frac{e^{u/2}}{u} - [\varphi(k)\pi(e^u, k, a) - \varphi(K)\pi(e^u, K, A)] > 0.$$

But $\int_{\log 2}^{\infty} g(u)e^{-us}du$ is analytic for real $s > \frac{1}{2}$ by hypothesis and Lemma 1 and now, thanks to (4) and Landau's theorem on Laplace transforms of positive functions, $\int_{\log 2}^{\infty} g(u)e^{-us}du$ is analytic in the half plane $\sigma > 1/2$. Hence the right side of (2) is analytic for $\sigma > 1/2$. The lemma now follows for $\beta \geqslant 1/2$. Finally if $0 < \beta \leqslant 1/2$, we note that the expression on the left of (3) is unchanged if ϱ is replaced by $1-\beta+i\gamma$. Thus again $\beta = 1/2$.

Lemma 3. *The function*

$$\sum_{\chi\neq\chi_0}\frac{\overline{\chi}(a)}{s}\log L(s,\chi) - \sum_{X\neq X_0}\frac{\overline{X}(A)}{s}\log L(s,X)$$

has a singularity in the half plane $\sigma \geqslant 1/2$ (if the hypotheses of Lemma 2 are satisfied, any such singularity is on $\sigma = 1/2$).

It would be amazing if Lemma 3 were not true; we defer the proof to Section 4.

3. Application of a Tauberian theorem. Ingham [2] proved a Tauberian theorem about functions whose Laplace transforms have simple poles. This theorem has been generalized in [5] to include other types of singularities. For convenience, we state the special case of this generalization that applies here.

Theorem. *Let*

$$\frac{F(s)}{s} = \int_0^{\infty} A(u)e^{-su}du,$$

where $A(u)$ is real valued and absolutely integrable on every interval $0 \leqslant u \leqslant U$, and the integral is absolutely convergent for $\sigma > 1$. Set

$$F_1(s) = \sum_{n=-N}^{N} a_n\log(s-\tfrac{1}{2}-i\gamma_n)$$

where $a_{-n} = \overline{a}_n$, γ_n is real, $\gamma_{-n} = -\gamma_n$. Suppose that for some $T > 0$, $F'(s)-F_1'(s)$ is continuous in the region $\sigma \geqslant 1/2$, $-T \leqslant t \leqslant T$ and analytic in the interior of this region. Then for any u_0,

$$\limsup_{u\to\infty}\frac{uA(u)}{e^{u/2}} \geqslant -\sum_{|\gamma_n|<T}\frac{a_n}{\frac{1}{2}+i\gamma_n}\left(1-\frac{|\gamma_n|}{T}\right)e^{i\gamma_n u_0}.$$

In fact this particular result follows from Ingham's theorem (with slightly more general hypotheses but the same proof) applied to the derivative with respect to s of

$$\frac{F(s)}{s} - \sum_{n=-N}^{N}\frac{a_n}{\frac{1}{2}+i\gamma_n}\log(s-\tfrac{1}{2}-i\gamma_n)+f_{-\infty}(s)$$
$$= \int_1^{\infty}\left[A(u)+\frac{1}{u}\sum_{n=-N}^{N}\frac{a_n}{\frac{1}{2}+i\gamma_n}e^{(\frac{1}{2}+i\gamma_n)u}\right]e^{-su}du.$$

An immediate consequence of this theorem and Lemmas 1 and 2 is our main result,

Theorem 1. *If none of the $L(s,\chi)$ and $L(s,X)$ have real zeros strictly between $1/2$ and 1 then for any $T > 0$ and any u_0,*

$$\limsup_{y\to\infty}\frac{\varphi(k)\pi(y,k,a)-\varphi(K)\pi(y,K,A)}{\sqrt{y}/\log y} \geqslant A_T^*(u_0).$$

The rest of this paper will consist of applications of Theorem 1. As our first example, we have the simple corollary,

Theorem 2. *Suppose that none of the functions $L(s,\chi)$ and $L(s,X)$ have real zeros in the range $0 < s < 1$.*

(i) *If $r(k,a;K,A) = 0$, then there is a constant $c > 0$ such that*

$$\limsup_{y\to\infty}\frac{\varphi(k)\pi(y,k,a)-\varphi(K)\pi(y,K,A)}{\sqrt{y}/\log y} \geqslant c.$$

(ii) *If $r(k,a;K,A) > 0$, then the result of* (i) *is true with $c = r$.*

Proof. Part (ii) follows from Theorem 1 with any value of u_0 and very small T. Part (i) comes by picking T just larger than the imaginary part of the first singularity on the line $\sigma = 1/2$ represented in the sum $A_T^*(u_0)$. Such a singularity exists by Lemmas 3 and 2 (otherwise $c = \infty$ is already correct). $A_T^*(u_0)$ is now of the form $\mathrm{Re}(ae^{i\gamma u_0})$ and it is easy to pick a value of u_0 that makes this positive.

The reason for separating parts (i) and (ii) of Theorem 2 is that the condition of (i), $r = 0$, is symmetric in $k, a; K, A$ while the condition in (ii) isn't. Thus for example, it follows from (i) that $\pi(y,5,2)-\pi(y,5,3)$ changes sign infinitely often while (ii) tells us only that $\pi(y,5,2)-\pi(y,5,4)$ is positive infinitely often. Thus we see that the real challenge is to derive the result of (i) when $r < 0$. This has not yet been done in general. Incidentally, Theorem 2 can be derived solely from Landau's theorem.

4. An application of some exact formulae. There is an exact formula for $\psi_0(y, k, a)$ [1] for $y > 1$,

$$\varphi(k)\psi_0(y, k, a) = y - \sum_{\chi} \bar{\chi}(a) \sum_{\varrho_\chi} \frac{y^{\varrho_\chi}}{\varrho_\chi} - \sum_{\chi} \bar{\chi}(a) \cdot \operatorname{res}_{s=0} \left[\frac{L'(s, \chi)}{L(s, \chi)} \cdot \frac{y^s}{s}\right]. \tag{5}$$

Here the sum over ϱ_χ is over both the trivial and non-trivial zeros of $L(s, \chi)$, other than $s = 0$. When $\chi(-1) = 1$, the sum over the real negative trivial zeros is

$$\sum_{\beta_\chi < 0} \frac{y^{\varrho_\chi}}{\varrho_\chi} = \sum_{n=1}^{\infty} \frac{y^{-2n}}{-2n} = \frac{1}{2} \log\left(\frac{y^2-1}{y^2}\right), \tag{6}$$

while when $\chi(-1) = -1$, this sum is

$$\sum_{\beta_\chi < 0} \frac{y^{\varrho_\chi}}{\varrho_\chi} = \sum_{n=1}^{\infty} \frac{y^{-2n+1}}{-2n+1} = \frac{1}{2} \log\left(\frac{y-1}{y+1}\right). \tag{7}$$

There are also certain imprimitive characters $\chi(\operatorname{mod} k)$ for which $L(s, \chi)$ has other trivial zeros. These come from the presence of factors in $L(s, \chi)$ of the form $1 - \alpha p^{-s}$ where p is fixed and $|\alpha| = 1$. The zeros of $1 - \alpha p^{-s}$ are zeros of $L(s, \chi)$ and are on the line $\sigma = 0$. If we sum over only the zeros, ϱ, of $1 - \alpha p^{-s}$ then we get for $y > 1$,

$$\sum_{\varrho} \frac{y^\varrho}{\varrho} = (\log p)\left[\frac{\alpha}{\alpha - 1} + \sum_{\substack{n \geq 1 \\ p^n \leq y}}{}' \alpha^n\right] \quad (\alpha \neq 1),$$

$$\sum_{\varrho \neq 0} \frac{y^\varrho}{\varrho} = -\log y + (\log p)\left[\frac{1}{2} + \sum_{\substack{n \geq 1 \\ p^n \leq y}}{}' \alpha^n\right] \quad (\alpha = 1).$$

Proof of Lemma 3. If Lemma 3 were false, then all of the non-trivial zeros of all the $L(s, \chi)$ and $L(s, X)$ would cancel out in the exact formula for $\varphi(k)\psi_0(y, k, a) - \varphi(K)\psi_0(y, K, A)$. But this function has occasional jumps on the order of $\log y$ while the sums over the trivial zeros of $L(s, \chi)$ and $L(s, X)$ cannot produce such jumps.

The exact formula (5) shows us that for $u > 0$, $A_\infty(u)$ exists and can be found in other terms. In fact if $y > 1$, we have

$$\begin{aligned}\sqrt{y} A_\infty(\log y; k, a; K, A) &= \varphi(k)\psi_0(y, k, a) - \varphi(K)\psi_0(y, K, A) + \\ &+ \sum_{\chi} \bar{\chi}(a) \sum_{\beta_\chi \leq 0} \frac{y^{\varrho_\chi}}{\varrho_\chi} - \sum_{X} \bar{X}(A) \sum_{\beta_X \leq 0} \frac{y^{\varrho_X}}{\varrho_X} + \\ &+ \sum_{\chi} \bar{\chi}(a) \cdot \operatorname{res}_{s=0}\left[\frac{L'(s, \chi)}{L(s, \chi)} \cdot \frac{y^s}{s}\right] - \sum_{X} \bar{X}(A) \cdot \operatorname{res}_{s=0}\left[\frac{L'(s, X)}{L(s, X)} \cdot \frac{y^s}{s}\right].\end{aligned} \tag{8}$$

Another exact formula to be given in (9) will show us that $A_\infty(u)$ exists for $u < 0$ also.

THEOREM 3. *Suppose that none of the functions $L(s, \chi)$, $L(s, X)$ have real zeros strictly between $1/2$ and 1. Then for any $u_0 \neq 0$,*

$$\limsup_{y \to \infty} \frac{\varphi(k)\pi(y, k, a) - \varphi(K)\pi(y, K, A)}{\sqrt{y}/\log y} \geq r + A_\infty(u_0).$$

Proof. This follows from Theorem 1 and (1). As an application of Theorem 3, we have

THEOREM 4. *Suppose that none of the functions $L(s, \chi)$, $L(s, X)$ have real zeros strictly between $1/2$ and 1. If $a \not\equiv 1 (\operatorname{mod} k)$ then*

$$\limsup_{y \to \infty} \frac{\varphi(k)\pi(y, k, a) - \varphi(K)\pi(y, K, 1)}{\sqrt{y}/\log y} = \infty.$$

Proof. The trivial terms from the characters X in (8) give an infinite contribution as $y \to 1^+$ when $A = 1$ while those corresponding to the characters χ remain bounded when $a \not\equiv 1 (\operatorname{mod} k)$. Hence when $a \not\equiv 1 (\operatorname{mod} k)$, we see from (8) that $\lim_{u \to 0^+} A_\infty(u; k, a; K, 1) = \infty$. The result follows from Theorem 3.

THEOREM 5. *Suppose that none of the functions $L(s, \chi)$, $L(s, X)$ have real zeros strictly between $1/2$ and 1. If $A \not\equiv 1 (\operatorname{mod} K)$ then*

$$\limsup_{y \to \infty} \frac{\varphi(k)\pi(y, k, 1) - \varphi(K)\pi(y, K, A)}{\sqrt{y}/\log y} = \infty.$$

Proof. This follows from the exact formula for

$$\varphi(k) \sum_{\substack{p^n \leq y^{-1} \\ ap^n \equiv 1 (\operatorname{mod} k)}}{}' \frac{\log p}{p^n} = \sum_{\substack{p^n \\ p^n \leq y^{-1}}}{}' \sum_{\chi} \chi(a) \frac{\chi(p^n)}{p^n} \log p$$

where $0 < y < 1$:

$$\begin{aligned}\varphi(k) \sum_{\substack{p^n \leq y^{-1} \\ ap^n \equiv 1 (\operatorname{mod} k)}}{}' \frac{\log p}{p^n} &= \frac{-1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \sum_{\chi} \chi(a) \frac{L'(s+1, \chi)}{L(s+1, \chi)} \frac{y^{-s}}{s} ds \\ &= -\operatorname{res}_{s=1} \sum_{\chi} \chi(a) \frac{L'(s, \chi)}{L(s, \chi)} \frac{y^{1-s}}{s-1} - \sum_{\chi} \chi(a) \sum_{\varrho_\chi} \frac{y^{1-\varrho_\chi}}{\varrho_\chi - 1} \\ &= \sum_{\chi} \bar{\chi}(a) \sum_{\substack{\varrho_\chi \\ \beta_\chi > 0}} \frac{y^{\varrho_\chi}}{\varrho_\chi} + \sum_{\chi} \chi(a) \sum_{\substack{\varrho_\chi \\ \beta_\chi \leq 0}} \frac{y^{1-\varrho_\chi}}{1 - \varrho_\chi} - \\ &\quad - \operatorname{res}_{s=1} \sum_{\chi} \chi(a) \frac{L'(s, \chi)}{L(s, \chi)} \frac{y^{1-s}}{s-1}.\end{aligned}$$

This shows us that for $u < 0$, $A_\infty(u)$ exists and can be found in other terms. For $0 < y < 1$, we get

(9) $\sqrt{y}A_\infty(\log y; k, a; K, A)$

$$= -\varphi(k) \sum_{\substack{p^n \leqslant y^{-1} \\ ap^n \equiv 1 \pmod{k}}} \frac{\log p}{p^n} + \varphi(K) \sum_{\substack{p^n \leqslant y^{-1} \\ Ap^n \equiv 1 \pmod{K}}} \frac{\log p}{p^n} +$$

$$+ \sum_{\chi} \chi(a) \sum_{\beta_\chi \leqslant 0} \frac{y^{1-\varrho_\chi}}{1-\varrho_\chi} - \sum_{X} X(A) \sum_{\beta_X \leqslant 0} \frac{y^{1-\varrho_X}}{1-\varrho_X} -$$

$$- \sum_{\chi} \chi(a) \cdot \operatorname*{res}_{s=1} \left[\frac{L'(s, \chi)}{L(s, \chi)} \frac{y^{1-s}}{s-1}\right] + \sum_{X} X(A) \cdot \operatorname*{res}_{s=1} \left[\frac{L'(s, X)}{L(s, X)} \frac{y^{1-s}}{s-1}\right].$$

The trivial terms from the characters χ in (9) give an infinite contribution as $y \to 1^-$ when $a = 1$ while those corresponding to the characters X remain bounded when $A \not\equiv 1 \pmod{K}$. Thus

$$\lim_{u \to 0^-} A_\infty(u; k, 1; K, A) = \infty.$$

Theorem 5 now follows from Theorem 3.

5. The case of $\pi(y, 5, 4) - \pi(y, 5, 2)$. Theorems 2, 4, 5 (and more) are already contained in the work of Knapowski and Turán. They have been briefly included here to show the application of Theorem 1. In this section, we specialize to the case $K = k$, a prime number. Theorem 2 applies unless a is a quadratic residue of k and A a non-residue. Even in this latter case if $a \equiv 1 \pmod{k}$, Theorem 5 handles the situation. Thus for the rest of this section, we assume that

$$\left(\frac{a}{k}\right) = 1, \quad \left(\frac{A}{k}\right) = -1, \quad a \not\equiv 1 \pmod{k}, \quad k \text{ prime}.$$

Here

(10) $A_T^*(u; k, a; k, A)$

$$= -2 + \sum_{\chi \neq \chi_0} \sum_{\substack{\varrho_\chi \\ \beta_\chi > 0, |\gamma_\chi| < T}} \frac{\bar\chi(A) - \bar\chi(a)}{\varrho_\chi} \left(1 - \frac{|\gamma_\chi|}{T}\right) e^{(\varrho_\chi - \frac{1}{2})u}.$$

Spira [4] has given all the non-trivial zeros of all $\varphi(k)$ L-functions $(\operatorname{mod} k)$, $3 \leqslant k \leqslant 24$, in the range $|\gamma_\chi| < 25$ (in each case $\beta_\chi = \frac{1}{2}$). There are enough zeros in this region for $k = 5$, $a = 4$, $A = 2$ that if their imaginary parts were linearly independent, we would have

(11) $$\sup A_{25}^*(u; 5, 4; 5, 2) > 0.$$

A computer search would undoubtedly yield a value of u to demonstrate (11), but this would be rather tedious by hand. A brief hand survey turned up a value of $u \approx 9.34$ which does a very good job on the first seven zeros (measured up from the real axis). Had the value of T been larger, these seven zeros would have contributed more than enough to cancel out the $r = -2$ term in A_T^*. In fact, had Spira furnished more zeros of the L-functions (mod 5) (so that larger values of T could be used), we could have successfully applied Theorem 1 with $u_0 \approx 9.34$.

In this sort of problem, a large value of $A_T^*(u)$ often means that something interesting is happening near $y = e^u$. In fact at $y = 10949 = e^{9.20\ldots}$, $\pi(y, 5, 4)$ is as large as $\pi(y, 5, 2)$ for the first time since $y = 2^-$. Set $y_0 = 10949^+$. Our exact formula is then

(12) $\sqrt{y_0}A_\infty(\log y_0; 5, 4; 5, 2)$

$$= 4[\psi_0(y_0, 5, 4) - \psi_0(y_0, 5, 2)] + \sum_{\chi \neq \chi_0} (\bar\chi(4) - \bar\chi(2)) \operatorname*{res}_{s=0} \left[\frac{L'(s, \chi)}{L(s, \chi)} \frac{y_0^s}{s}\right] + \theta$$

where

$$|\theta| < 10^{-3}.$$

Let the 4 characters (mod 5) be given by

$$\chi_0(2) = 1, \quad \chi_1(2) = -1, \quad \chi_2(2) = i, \quad \chi_3(2) = -i$$

and let

$$L_j(s) = L(s, \chi_j).$$

The second term on the right of (12) is

$$(-1+i)\frac{L_2'(0)}{L_2(0)} + (-1-i)\frac{L_3'(0)}{L_3(0)} + 2\left(\log y_0 + \frac{L_1''(0)}{2L_1'(0)}\right).$$

It is a pleasant surprise that the $\log y_0$ works for us rather than against us. The 18.40 ... that it contributes to (12) lets us be quite sloppy in estimating the first term on the right of (12) (this being useful since the estimate was done by hand).

Since $\sqrt{y_0} < 105$, there are relatively few powers of primes greater than the first power contributing to (12); if we take these and first 5 primes $\equiv 2 \pmod 5$ and the first 5 primes $\equiv 4 \pmod 5$ then we find that

$$\psi_0(y_0, 5, 4) - \psi_0(y_0, 5, 2)$$
$$> \log(2^{-1} \cdot 19^2 \cdot 23 \cdot 29 \cdot 43 \cdot 53 \cdot 59 \cdot 67 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97 \cdot 103)$$
$$= 54.489 \ldots$$

The reason is that the right side has omitted from the left side the terms

$$\sum_{\substack{p \equiv 4 \pmod 5 \\ 90 < p \leqslant 10949}} \log p - \sum_{\substack{p \equiv 2 \pmod 5 \\ 48 < p \leqslant 10949}} \log p.$$

This last expression is positive since there are equally many terms in each sum and those $\equiv 4 (\mathrm{mod} 5)$ are pairwise larger than those $\equiv 2 (\mathrm{mod} 5)$ since $\pi(y, 5, 4) - \pi(y, 5, 2)$ is never positive for $y < 10949$. Thus (12) yields

$$(13)\quad A_\infty(\log y_0)\sqrt{y_0} > 236 + (-1+i)\frac{L_2'(0)}{L_2(0)} + (-1-i)\frac{L_3'(0)}{L_3(0)} + \frac{L_1''(0)}{L_1'(0)}.$$

We want only to show that $A_\infty(\log y_0)\sqrt{y_0} > 2\sqrt{y_0} = 209.27\ldots$ Any reasonable sort of estimate of $L_2'(0)$ and $L_1''(0)$ accomplishes this

$$\left[L_3'(0) = \overline{L_2'(0)},\ L_2(0) = \frac{3+i}{5},\ L_3(0) = \frac{3-i}{5},\ L_1'(0) = \log\left(\frac{1+\sqrt{5}}{2}\right)\right].$$

The result is

THEOREM 6.

$$\limsup_{y\to\infty} \frac{\pi(y, 5, 4) - \pi(y, 5, 2)}{\sqrt{y}/\log y} > 0.$$

6. Concluding remarks. Now that we have seen Theorem 1 applied, we may ask how easy it would be to apply it in other cases. As k and K grow, there will be more values of ϱ_χ and ϱ_X with small imaginary parts. This should enable one to take smaller values of T and fewer zeros into the calculations. In a sense, things should be even easier when $k \neq K$. For large values of k and K, it might be possible to make the first one or two zeros closest to the real axis do almost all the work. Alternatively, one can use the exact formulae (8) and (9) along with Theorem 3. While this saves one from computing the complex zeros of L-functions, one must now search for a value of y that will allow Theorem 3 to work. Both of these approaches have been illustrated in Section 5.

I would like to express my thanks to Robert Spira for giving me a table of zeros of L-functions (mod 5) before they were published.

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The thinnest double lattice covering of three-spheres

by

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1. Introduction. Let L be a lattice in three-dimensional euclidean space such that the system of closed unit balls with centres at all points of L forms a covering of R_3. Bambah [1] proved that the maximum possible value of the determinant $d(L)$ of L is $32/5\sqrt{5}$, and several other proofs have since been found (Barnes [2], Few [3]). It is still an open question as to whether the density of any point set yielding such a covering can be smaller than that for the best possible lattice. In this direction it is proved here that if D is a double lattice, so that D is the union of a lattice L and a single translate of L, which provides a covering of R_3 by closed unit balls then

$$d(L) \leqslant 64/5\sqrt{5}.$$

Hence no double lattice can yield a thinner such covering than the best possible lattice.

2. For an arbitrary point X of R_3 denote by $S(X)$ the closed unit ball with centre X.

Let L and $L+X$ be a lattice and its translate in R_3 such that $S(Y)$, $Y \in L \cup (L+X)$ taken together cover R_3. The objective is to prove $d(L) \leqslant 64/5\sqrt{5}$. To this end we assume that $d(L) > 64/5\sqrt{5}$ and derive a contradiction.

LEMMA 1. *Let P be any plane containing a two-dimensional sub-lattice of L. Then the collection $S(X)$, $X \in P \cap L$ does not cover P.*

Proof. Assume the assertion is false so that P is completely covered by such balls. By classical theory

$$d(L \cap P) \leqslant 3\sqrt{3}/2.$$

Let the distance of P to a next lattice plane of L be d. Then

$$64/5\sqrt{5} < d(L) \leqslant 3\sqrt{3}d/2$$