Addendum during proof correction (2.5.1971). As I have learn from a latter by Dr. Morris Newman, Washington, D. C., a simple argument, based on § 27, (3), of my monography quoted on p. 275, allows to conclude that also the *first* sums in (2) are all different from zero, simply because between them and the *second* sums in (2) the following elementary connection holds:

$$p\chi(2)\sum_{s=1}^{n}\chi(s) = (1-2\chi(2))\sum_{s=1}^{n}s\chi(s).$$

Hence *the answer to Chowla's question is in the positive also for the tan-values.*

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The distribution of Farey points, I

by

M. N. Huxley (Cambridge)

Dedicated to the memory of
Professor H. Davenport

The "Farey sequence" of rational points with denominators not exceeding some bound Q has many amusing properties (see [2], chapter III). We are concerned here with the uniform distribution of the sequence modulo 1 as Q tends to infinity. By Weyl's principle the distribution is uniform if and only if certain exponential sums are small; for the Farey sequence these exponential sums transform by way of Ramanujan sums into expressions involving the Möbius function. In 1924 J. Franel [1] produced a quantitative form of this equivalence: he found an elegant identity for the sum of the squares of the values of the discrepancy function corresponding to our $E(a)$ below at the Farey points in terms of the sum-function of the Möbius function. In fact he showed that the infimum of θ for which (in our notation, for which see below)

$$\sum_{i=1}^{F}|E(f_i)|^2 \ll Q^{2+2\theta}$$

was the supremum of real parts of zeros of the Riemann zeta-function. (We use $\ll$ to indicate an inequality with an unspecified absolute constant.)

Davenport proposed in his problems list that an analogous result should hold for the zeros of a fixed Dirichlet L-function. In this note we supply the analogue by elementary arguments. We state our theorem with a general weight $\lambda(q)$, not necessarily a Dirichlet character, on the Farey point a/q. First we introduce the notation.

Let $f_1, \ldots, f_F$, where $f_i = a_i/q_i$, be the Farey sequence of order Q, that is, the sequence of rationals a/q with $(a, q) = 1$, $0 < a \leqslant q$ and $0 < q \leqslant Q$, arranged in ascending order. Let $\lambda(1), \ldots, \lambda(Q)$ be any complex numbers, and for each integer m write

$$L(m) = \sum_{n\leqslant Q/m}\lambda(mn)\sum_{d|n}\frac{\mu(d)\,d}{n}, \tag{1}$$

where $\mu(d)$ is the Möbius function. We define a measure of irregularity of the Farey sequence with weights $\lambda(q_i)$. If $0 < a \leqslant 1$ we put

$$E(a) = aG - \sum_{f_i \leqslant a}' \lambda(q_i) - \tfrac{1}{2}\lambda(1), \tag{2}$$

where G is the total of the weights:

$$G = \sum_{q \leqslant Q} \lambda(q)\varphi(q) \tag{3}$$

and the dash indicates that if $a = f_i$ for some i then the term $\lambda(q_i)$ is replaced by $\frac{1}{2}\lambda(q_i)$. Outside the range $0 < a \leqslant 1$ we define $E(a)$ by stipulating that it be periodic with period 1. We can now state our result.

THEOREM. *We have*

$$\sum_{i=1}^{F} |E(f_i)|^2 \ll Q^2 \sum_{m=1}^{Q} \sum_{k|m} \frac{\mu(k)}{k^2} |L(m)|^2 + \sum_{q \leqslant Q} |\lambda(q)|^2 \varphi(q), \tag{4}$$

and

$$Q^2 \sum_{m=1}^{Q} \sum_{k|m} \frac{\mu(k)}{k^2} |L(m)|^2 \ll \sum_{i=1}^{F} |E(f_i)|^2 + Q^2 \sum_{q \leqslant Q} |\lambda(q)|^2 \log Q. \tag{5}$$

If further we have $|\lambda(q)| \leqslant 1$ *for each* $q \leqslant Q$, *then*

$$Q^2 \sum_{m=1}^{Q} \sum_{k|m} \frac{\mu(k)}{k^2} |L(m)|^2 \ll \sum_{i=1}^{F} |E(f_i)|^2 + Q^3. \tag{6}$$

In the special case when $\lambda(q)$ is a Dirichlet character $\chi(q)$, we can write

$$L(m) = \chi(m)\, M(Q/m),$$

where for each integer x

$$M(x) = \sum_{n \leqslant x} \chi(n) \sum_{d|n} \frac{\mu(d)\, d}{n}. \tag{7}$$

We see that for any $\theta > \frac{1}{2}$ the bounds

$$|M(x)| \ll x^{\theta} \quad \text{for each } x, \tag{8}$$

and

$$\sum_{i=1}^{F} |E(f_i)|^2 \ll Q^{2+2\theta} \quad \text{for each } Q \text{ and the corresponding } F \tag{9}$$

are equivalent. Since the infimum of θ for which (8) holds is the supremum of real parts of zeros of the Dirichlet L-function formed with the character χ, we have an intimate connection between the zeros of that L-function and the distribution of Farey fractions a/q with weights $\chi(q)$.

We can also apply our theorem to the subset of Farey points with prime denominators. Here $\lambda(q) = 1$ if q is a prime and 0 otherwise; $L(m)$ is 0 if m is composite, 1 if m is prime and of order $\pi(Q)$ if $m = 1$. By considering the possible denominators one at a time we see that $O(\pi(Q))$ is a trivial upper bound for $E(a)$; and (6) shows that

$$|E(f_i)| \gg \pi(Q)$$

for a positive proportion of the Farey points f_i, so that no better inequality for $E(a)$ can be true in this case.

In the second part of this paper we shall discuss a definition of a Farey sequence for an algebraic number field.

Franel's identity will generalise to allow a system of weights of our type, but it then gives an expression for

$$\sum_{i=1}^{F} \lambda(q_i)\, |E(f_i)|^2,$$

which is of little interest unless the weights are real and positive. Landau's ingenious argument in [3], VII, Kap. 13, which gives an inequality from the irregularity to the sum of the Möbius function does not appear to generalise at all. We argue directly from the basic lemma underlying Franel's work, an expression of Weyl's principle.

LEMMA. *We have*

$$\int_0^1 |E(a)|^2\, da = \frac{1}{12} \sum_{m=1}^{Q} \sum_{k|m} \frac{\mu(k)}{k^2} |L(m)|^2. \tag{10}$$

Proof. Writing

$$g(\beta) = \beta - [\beta] - \tfrac{1}{2} = \sum_{\substack{r=-\infty \\ r \neq 0}}^{\infty} \frac{i}{2\pi r} e(r\beta),$$

where β is any real number, $[\beta]$ is the integer part of β and $e(a)$, $e_q(a)$ denote for any real a the complex exponentials $\exp(2\pi i a)$, $\exp(2\pi i a/q)$, then we have

$$E(a) = \sum_{n=1}^{F} \lambda(q_n)\, g(a - f_n) = \sum_{\substack{r=-\infty \\ r \neq 0}}^{\infty} \frac{i}{2\pi r} e(ra) \sum_{q \leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \lambda(q)\, e_q(-ar).$$

Using properties of Ramanujan's sum, we see that the sum over q gives

$$\sum_{q \leqslant Q} \lambda(q) \sum_{\substack{d|q \\ d|r}} d\mu\left(\frac{q}{d}\right) = \sum_{\substack{d \leqslant Q \\ d|r}} d \sum_{g \leqslant Q/d} \mu(g)\, \lambda(dg).$$

By Parseval's formula we can now write

$$\int_0^1 |E(a)|^2 da = 2\sum_{r=1}^{\infty}\frac{1}{4\pi^2 r^2}\Big|\sum_{d|r}\sum_{g\leqslant Q/d}\lambda(dg)\mu(g)\Big|^2$$

$$= \sum_{d_1}\sum_{d_2} d_1 d_2 \sum_{g_1\leqslant Q/d_1}\lambda(d_1 g_1)\mu(g_1)\sum_{g_2\leqslant Q/d_2}\overline{\lambda}(d_2 g_2)\mu(g_2)\sum_{r}\frac{1}{2\pi^2 r^2}$$

where the last sum is over r which are multiples both of d_1 and d_2, and so is $d^2/(12 d_1^2 d_2^2)$ where $d = (d_1, d_2)$. We write $d_1 = dh_1$, $d_2 = dh_2$ and express the condition that $(h_1, h_2) = 1$ by means of the Möbius function. The whole expression above can now be written as

$$\frac{1}{12}\sum_{d}\sum_{h_1}\sum_{h_2}\sum_{\substack{k|h_1\\ k|h_2}}\frac{\mu(k)}{h_1 h_2}\sum_{g_1\leqslant Q/(dh_1)}\lambda(dg_1h_1)\mu(g_1)\sum_{g_2\leqslant Q/(dh_2)}\overline{\lambda}(dg_2h_2)\mu(g_2).$$

When we put $m = dk$, $h_1 = kl_1$, $h_2 = kl_2$ this becomes

$$\frac{1}{12}\sum_{m}\sum_{k|m}\frac{\mu(k)}{k^2}\sum_{g_1}\sum_{l_1\leqslant Q/(g_1m_1)}\frac{\lambda(g_1l_1m)\mu(g_1)}{l_1}\sum_{g_2}\sum_{l_2\leqslant Q/(g_2m_2)}\frac{\lambda(g_2l_2m)\mu(g_2)}{l_2}.$$

By the definition (1) of $L(m)$ we have proved (10).

To make use of the lemma we divide the unit interval $[1/(Q+1), (Q+2)/(Q+1)]$ into F arcs $I_1, \ldots, I_F$, the point of division between I_i and I_{i+1} for $i = 1, 2, \ldots, F-1$ being the mediant

$$\frac{a_i + a_{i+1}}{q_i + q_{i+1}}.$$

We note that the length δ_i of the interval I_i satisfies

$$(q_iQ)^{-1} \leqslant \delta_i \leqslant 4(q_iQ)^{-1}, \tag{11}$$

and that the interval $J_i = [f_i - \frac{1}{2}Q^{-2}, f_i + \frac{1}{2}Q^{-2}]$ lies entirely within I_i. Now if a is in J_i, then

$$|E(a) - E(f_i)| \leqslant \tfrac{1}{2}|\lambda(q_i)| + \tfrac{1}{2}|G|Q^{-2},$$

and so we have

$$|E(f_i)|^2 \leqslant 3Q^2\int_{J_i}\big(|E(a)|^2 + \tfrac{1}{4}|\lambda(q_i)|^2 + \tfrac{1}{4}|G|^2Q^{-4}\big)da$$

$$\leqslant 3Q^2\int_{I_i}|E(a)|^2 da + \tfrac{3}{4}|\lambda(q_i)|^2 + \tfrac{3}{4}|G|^2Q^{-4}.$$

We sum this inequality over i and note that Cauchy's inequality gives

$$|G|^2 \leqslant F\sum_{q=1}^{Q}\varphi(q)|\lambda(q)|^2 \leqslant Q^2\sum_{q=1}^{Q}\varphi(q)|\lambda(q)|^2. \tag{12}$$

Hence

$$\sum_{i=1}^{F}|E(f_i)|^2 \leqslant 3Q^2\int_{1/(Q+1)}^{(Q+2)/(Q+1)}|E(a)|^2 da + \tfrac{3}{2}\sum_{q=1}^{Q}\varphi(q)|\lambda(q)|^2.$$

The integral is now over a complete period of $E(a)$, and so by (10) of the lemma we have (4).

To prove (5) we must argue a little less crudely. We note first that

$$\int_{I_i}|E(a)|^2 da \leqslant 3\int_{I_i}\big(|E(f_i)|^2 + \delta_i^2|G|^2 + \tfrac{1}{4}|\lambda(q_i)|^2\big)da \tag{13}$$

$$\leqslant 3\int_{I_i}|E(f_i)|^2 da + 3\delta_i^3|G|^2 + \tfrac{3}{4}\delta_i|\lambda(q_i)|^2.$$

We now distinguish two cases. If $q_i < \frac{1}{4}Q$, we let m/n be the fraction next below a_i/q_i in the Farey sequence of order $[\frac{1}{2}Q]$. Then $n > \frac{1}{4}Q$ and there are fractions of the Farey sequence of order Q which lie between m/n and a_i/q_i and are of the form

$$\frac{m + ra_i}{n + rq_i} \tag{14}$$

with $r \leqslant (Q-n)/q_i$. Each of these must be in its lowest terms, or it would already occur in the Farey sequence of order $[\frac{1}{2}Q]$. We now see that there are at least $[\frac{1}{2}Q/q_i]+1$ other fractions of the Farey sequence of order Q within $4(q_iQ)^{-1}$ of a_i/q_i.

If f_j is any one of these, we have

$$|E(f_i) - E(f_j)| \leqslant \sum_{k}|\lambda(q_k)| + \frac{4|G|}{q_iQ},$$

the sum being over all f_k of the form (14), and thus it has at most Q/q_i terms. We now deduce

$$|E(f_i)|^2 \leqslant 3|E(f_j)|^2 + \frac{3Q}{q_i}\sum_{k}|\lambda(q_k)|^2 + \frac{48|G|^2}{q_i^2Q^2}$$

which we average to obtain

$$|E(f_i)|^2 \leqslant \frac{Gq_i}{Q}\sum_{k}|E(f_k)|^2 + \frac{3Q}{q_i}\sum_{k}|\lambda(q_k)|^2 + \frac{48|G|^2}{q_i^2Q^2}.$$

Using (11), we see that

$$\int_{I_i}|E(f_i)|^2 da \leqslant \frac{24}{Q^2}\sum_{k}|E(f_k)|^2 + \frac{12}{q_i^2}\sum_{k}|\lambda(q_k)|^2 + \frac{192|G|^2}{q_i^3Q^3}. \tag{15}$$

Our second case is $q_i \geqslant \frac{1}{4}Q$, so that by (11) $\delta_i \leqslant 16Q^{-2}$ and hence

(16) $$\int_{I_i} |E(f_i)|^2 d\alpha \leqslant 16 Q^{-2} |E(f_i)|^2.$$

It remains to substitute (15) and (16) into (13). We note that any f_k can be represented in the form (14) with $n < \frac{1}{2}Q$ and $q_i < \frac{1}{4}Q$ in at most one way. Hence

(17) $$\sum_{i=1}^{F} \int_{I_i} |E(f_i)|^2 d\alpha$$
$$\leqslant \frac{40}{Q^2} \sum_{i=1}^{F} |E(f_i)|^2 + 12 \sum_{q \leqslant Q} |\lambda(q)| \sum_{\substack{f_i \\ q_i < \frac{1}{4}Q}} \frac{1}{q_i^2} + \frac{192\,|G|^2}{Q^3} \sum_{q=1}^{\infty} \frac{\varphi(q)}{q^3}$$
$$\leqslant 40 Q^{-2} \sum_{i=1}^{F} |E(f_i)|^2 + 12 \log Q \sum_{q \leqslant Q} |\lambda(q)|^2 + 384 \sum_{q \leqslant Q} |\lambda(q)|^2,$$

where we have used the estimate (12) for $|G|^2$. We note that in the special case when $|\lambda(q)| \leqslant 1$ for all q we can replace the second term by

$$\sum_{\substack{f_i \\ q_i < \frac{1}{4}Q}} \frac{12}{q_i^2} \frac{Q}{q_i} \leqslant 24Q,$$

and so we have

(18) $$\sum_{i=1}^{F} \int_{I_i} |E(f_i)|^2 d\alpha \leqslant 40 Q^{-2} \sum_{i=1}^{F} |E(f_i)|^2 + 408 Q.$$

We use (12) and (11) to sum (13), so that

(19) $$\sum_{i=1}^{F} \delta_i^3 |G|^2 \leqslant \sum_{i=1}^{F} \frac{64 Q^3}{q_i^3 Q^3} \sum_{q \leqslant Q} |\lambda(q)|^2 \leqslant 128 \sum_{q \leqslant Q} |\lambda(q)|^2,$$

and

(20) $$\sum_{i=1}^{F} \delta_i |\lambda(q_i)|^2 \leqslant \sum_{q \leqslant Q} Q^{-1} |\lambda(q_i)|^2.$$

We substitute (19), (20) and (17) into

$$\int_0^1 |E(\alpha)|^2 d\alpha = \sum_{i=1}^{F} \int_{I_i} |E(\alpha)|^2 d\alpha$$

and use (10) of the lemma to obtain (5) of the theorem. Replacing (17) by (18) gives (6).

References

[1] J. Franel, *Les suites de Farey et les problèms des nombres premiers*, Göttinger Nachrichten 1924, pp. 198–201.

[2] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, fourth edition, Oxford 1960.

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CHURCHILL COLLEGE
Cambridge, England

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