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# A brief survey of the work of Harold Davenport

by

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In the previous note Professor L. J. Mordell has given an account of some aspects of Davenport's work; in this note I give a more general but very brief survey of his mathematical work. Davenport wrote nearly two hundred papers, they are listed in the bibliography with the numbers that he assigned to them; these numbers do not correspond with the order of publication, but do, I believe, correspond to the order in which the manuscripts were completed. It is not possible here to discuss many of these papers in detail; it will be necessary to concentrate on the main themes of his work and to ignore many important papers that did not contribute to their development. I am most grateful to Mrs. Davenport for giving me the list of papers and for letting me see some notes that Davenport wrote about his earlier work.

Much of Davenport's earliest work [1, 2, 3, 5, 6] centred round the related problems of estimating character sums and exponential sums. This prepared him for his joint work with H. Hasse [8] on the zeros of the congruence zeta functions of certain algebraic function fields. This work was greatly improved by Weil's applications of algebraic geometry to obtain general results. Davenport in [27] gave a proof of Hasse's conjectured functional equation for some of his $L$-functions and the best estimates for character sums that could be obtained by "elementary" methods.

After Vinogradov's improvement of the Hardy–Littlewood method, T. Estermann, working in London, and Davenport and Heilbronn, working together in Cambridge, independently proved that every sufficiently large positive integer is the sum of seventeen fourth powers [11, 14]. Further work on the additive theory of numbers, some in collaboration with Heilbronn, followed [18, 19, 31, 33, 34, 35, 36, 37, 38, 49]; in particular, Davenport proves in [35] that every sufficiently large number is representable as the sum of 14 fourth powers unless it is congruent to 15 or 16 (mod 16), in which cases 15 or 16 fourth powers are necessary.

When Davenport returned to Manchester to join Mordell's staff, he began to contribute to the Geometry of Numbers, a subject in which Mordell had been greatly interested. His first major contribution was to the problem of the product of three linear forms, proving in [25] that, if $L_1, L_2, L_3$ are real linear forms in variables $u_1, u_2, u_3$, with determinant $\Delta \neq 0$, then there are integral values of the variables, not all zero satisfying

$$|L_1 L_2 L_3| \leqslant \frac{1}{7} |\Delta|;$$

and in [26] that, if $L_1$ is a real form and $L_2, L_3$ are conjugate complex forms in $u_1, u_2, u_3$, then the corresponding inequality

$$|L_1 L_2 L_3| \leqslant \frac{1}{\sqrt{23}} |\Delta|$$

can be satisfied. Later he gave in [39] a much simpler proof of the first result by a method, which he elaborated to give in [42] a proof that the stronger inequality

$$|L_1 L_2 L_3| \leqslant \frac{1}{9.1} |\Delta|$$

can be satisfied, except only in the cases when $L_1 L_2 L_3$ is equivalent to a multiple of the norm form of a totally real cubic field of discriminant 49 or 81. In this way he obtained the first "isolation" result since the work of Markoff on quadratic forms. He later investigated these isolation results jointly with Rogers [79]. At about this time he made a study of the product of four linear forms; but he found that he needed to split the problem up into so many cases, all requiring detailed calculations, that he gave up the attempt. Much later he took up this problem in collaboration with H. P. F. Swinnerton-Dyer, and, if the necessary computations work out in the expected way, the solution will be published.

In [29] he gave a simple proof of the three dimensional case of Minkowski's conjecture for the product of three non-homogeneous linear forms; a result that had been proved with very great difficulty by Remak. This paper formed the basis for F. J. Dyson's later very difficult proof of the four dimensional case of Minkowski's conjecture (Ann. Math. (2), 49 (1948), pp. 82–109). We recall this conjecture of Minkowski, that, if $L_1, L_2, \ldots, L_n$ are non-homogeneous linear forms in the variables $u_1, u_2, \ldots, u_n$, i.e. if

$$L_i = a_{i1} u_1 + a_{i2} u_2 + \ldots + a_{in} u_n + b_i, \quad i = 1, 2, \ldots, n,$$

with determinant

$$\Delta = \det(a_{ij}) \neq 0,$$



then there are integers $u_1, u_2, \ldots, u_n$ with

$$|L_1 L_2 \ldots L_n| \leqslant (\tfrac{1}{2})^n |\Delta|.$$

A series of papers [41, 45, 46, 89, 90, 126, 190, 191], the last two being joint papers with Heilbronn, discuss the minima, the reduction, and the density of binary cubic forms, and in the last papers, the density of cubic algebraic number fields.

In another series of papers [51, 52, 61, 62, 63, 66] he proved a series of results on the non-homogeneous minima of binary quadratic forms, of ternary norm forms and of ternary quadratic forms. This work led through an intermediate result [68] to his paper [70] (see also [77]) which gives an intelligible reason for the fact that Euclid's algorithm holds in only finitely many real quadratic fields. This enabled him in the first place, with Chatland [74], to show that Euclid's algorithm holds in no real quadratic fields beyond the last known example (believed at the time to be $\sqrt{97}$ but now known to be $\sqrt{73}$); and secondly to show in [76, 82] that Euclid's algorithm holds only in a finite number of cubic fields with negative discriminant and in only a finite number of complex quartic fields with complex conjugate fields. (J. V. Armitage has recently extended some of these results to certain fields of series.)

A series of papers [47, 100, 104, 105, 106, 133, 144, 153, 157, 185, 189] discusses problems of simultaneous Diophantine approximation. In particular [104] shows that there are continuum many pairs $\theta, \varphi$ of numbers that are badly approximable in that, for some $C$, there are only finitely many integral solutions of the inequalities

$$\left|\theta - \frac{u}{q}\right| < \frac{C}{q^{3/2}}, \quad \left|\varphi - \frac{v}{q}\right| < \frac{C}{q^{3/2}}, \quad q > 0.$$

Again [144] shows that if $f_1(\alpha, \beta), \ldots, f_r(\alpha, \beta)$ and $g_1(\alpha, \beta), \ldots, g_r(\alpha, \beta)$ are any real valued functions with continuous first order partial differential coefficients, the Jacobians $\partial(f_q, g_q)/\partial(\alpha, \beta)$, $q = 1, \ldots, r$, being non-zero at some point, then there are continuum many choices for $\alpha, \beta$ that make all the pairs $f_q(\alpha, \beta), g_q(\alpha, \beta)$, $q = 1, \ldots, r$, badly approximable in the above sense. The last two papers, written jointly with W. M. Schmidt discuss in great depth the circumstances in which Dirichlet's theorem on Diophantine approximation can or cannot be improved.

A long and difficult series of papers, written in only two years, [111, 112, 114, 115, 118, 119, 120] gives the account of the work Davenport initiated on quadratic forms in many variables. This was carried on in collaboration with B. J. Birch and concluded in collaboration with D. Ridout. The work combines the Hardy–Littlewood method with the theory of successive minima from the geometry of numbers and appeals to results of Cassels and Oppenheim. The final result asserts that: an indefinite quadratic form with real coefficients in 21 or more variables either vanishes or takes arbitrarily small values, for non-zero integral values of the variables; further, if the coefficients of the form are not all in rational ratios, then the values assumed by the form, for integral values of the variables, are everywhere dense on the real line.

A series of longer and more difficult papers [116, 130, 134, 135, 148] the last two written jointly with D. J. Lewis, and the last incorporating results due to G. L. Watson, discuss the solubility of homogeneous and non-homogeneous cubic equations with integral coefficients. The first major contribution to this subject was made by D. J. Lewis (Mathematika, 4 (1957), pp. 97–101) who proved that cubic forms, with coefficients in an algebraic number field of fixed degree, have non-trivial zeros provided only that they have enough variables. A short time later, Davenport working independently, showed that cubic forms with integral coefficients and 32 or more variables have non-trivial zeros. B. J. Birch (Mathematika, 4 (1957), pp. 102–105), shortly afterwards, and again independently, obtained the corresponding results for the simultaneous representation of zero by a system of forms of odd degree, with algebraic coefficients and with sufficiently many variables. Lewis and Birch made essential use of earlier work of R. Brauer and L. G. Peck; Davenport used sophisticated extensions of the ideas he had used in the discussion of quadratic forms. The final results of the series were that homogeneous cubic equations with integral coefficients and 16 or more variables have non-trivial solutions [134] and that non-homogeneous cubic equations in 19 or more variables have solutions provided a certain invariant $h$ (in general greater than 4) is at least 4 [148], this second result being obtained with Lewis and Watson (see G. L. Watson, Proc. London Math. Soc. (3), 17 (1967), pp. 271–295).

Yet another series of papers, written in collaboration with S. Chowla, B. J. Birch, and D. J. Lewis [124, 128, 131, 138, 167, 169, 184] discuss the solution of equations of various additive types. The culmination of the series is the paper [184] with D. J. Lewis. This shows, in particular, that, if $k$ is an odd positive integer, the equations

$$\sum_{j=1}^{N} a_{ij} x_j^k = 0, \qquad i = 1, 2, \ldots, R,$$

have a solution in integers $x_1, \ldots, x_N$, not all zero, if

$$N \geqslant [9R^2 k \log 3Rk];$$

it also obtains similar results for forms of even degree under appropriate conditions.



Davenport had a continuing interest in the properties of polynomials and wrote a varied series of papers, partly in collaboration with D. J. Lewis and A. Schinzel on this topic [129, 140, 145, 146, 158, 159].

Davenport, working in collaboration with W. M. Schmidt, took up the problem studied by E. Wirsing (Journ. Math. 206 (1960), pp. 67–77) of the approximation of irrational and algebraic numbers by algebraic numbers, or algebraic integers [173, 177, 183, 188]. For example, they proved that, if $n \geqslant 3$ and $\xi$ is real but is not an algebraic number of degree at most $\frac{1}{2}(n-1)$, then there are infinitely many real algebraic integers $\alpha$ of degree at most $n$ which satisfy

$$0 < |\xi - \alpha| \ll H(\alpha)^{-[\frac{1}{2}(n+1)]}.$$

One joint paper [163] with E. Bombieri deserves special mention. It uses the results of the large sieve (of Linnik, Rényi, Roth and Bombieri) to show that the sequence of primes satisfies

$$\liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} \leqslant \tfrac{1}{8}(2+\sqrt{3}) = 0.46650\ldots$$

A series of papers written jointly with H. Halberstam and E. Bombieri [168, 170, 172, 181, 187] simplify and refine the large sieve results.

Davenport was always concerned to obtain effective or constructive results. This concern motivated his work on Euclid's algorithm, on Dirichlet's $L$-functions [171], on Thue's theorem [180], and his paper with A. Baker [182] showing that the equations

$$3x^2 - 2 = y^2, \qquad 8x^2 - 7 = z^2$$

have no solution sets other than (1, 1, 1) and (11, 19, 31).