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Mean value theorems for a class of arithmetic functions

by

H. HALBERSTAM (Nottingham) and H.-E. RICHERT (Marburg and Syracuse)

1. Let ω be a real multiplicative arithmetic function satisfying, for some constant $A_1 \geqslant 1$, the condition

$$(\Omega_1) \qquad 0 \leqslant \frac{\omega(p)}{p} \leqslant 1 - \frac{1}{A_1} \quad \text{for all primes } p;$$

and, on the sequence of squarefree numbers, define the related multiplicative arithmetic function g by

$$g(d) = \frac{\omega(d)}{\prod_{p|d}(p-\omega(p))}, \quad \mu(d) \neq 0. \tag{1.1}$$

With[1] $x > 0$ and $z \geqslant 2$ we form the sums

$$G(z) = \sum_{d<z} \mu^2(d) g(d), \tag{1.2}$$

$$G(x, z) = \sum_{\substack{d<x \\ d|P(z)}} g(d) \tag{1.3}$$

where

$$P(z) = \prod_{p<z} p, \tag{1.4}$$

and also the product

$$W(z) = \prod_{p<z} \left(1 - \frac{\omega(p)}{p}\right). \tag{1.5}$$

The sums $G(z)$ and $G(x, z)$ occur naturally in Selberg sieve theory. In most applications of Selberg's sieve the *basic function* $\omega(p)$ *is, on average over the primes, constant*, and our aim in this paper is to obtain, under

[1] The numbers x and z will satisfy these inequalities throughout the paper.

a weak condition of this kind, asymptotic formulae (with error terms) for both these sums. To be precise, we shall impose on ω the further condition that

(Ω_2) *there exist constants $\varkappa > 0$ and $A_2 \geqslant 1$, and a number $L \geqslant 1$ such that*

$$-L \leqslant \sum_{w \leqslant p < z}{}' \frac{\omega(p)}{p}\log p - \varkappa \log\frac{z}{w} \leqslant A_2 \quad \text{if} \quad 2 \leqslant w \leqslant z.$$

While all constants implied by the use of the O- and $\ll$-notations may, throughout this paper, depend on A_1, A_2 and $\varkappa$, dependence on L will be everywhere explicit. This distinction between L and the constants A_1, A_2 may appear somewhat artificial, but the formulation of a sieve problem usually involves several basic parameters and, while it mostly turns out that A_2 can be chosen independent of these, this is not always the case with the lower bound L.

We shall prove, subject to conditions (Ω_1) and (Ω_2), the following two theorems:

Theorem 1. *We have*

$$G(z)\,W(z) = \frac{e^{-\gamma\varkappa}}{\Gamma(\varkappa+1)} + O\left(\frac{\min(L, \log z)}{\log z}\right), \tag{1.6}$$

where γ is Euler's constant.

Theorem 2. *We have*

$$G(x, z)\,W(z) = \sigma_\varkappa(2\tau) + \left(\frac{L\tau^{2\varkappa+1}}{\log z}\right) \quad \text{if} \quad z \leqslant x, \tag{1.7}$$

where

$$\tau = \frac{\log x}{\log z} \tag{1.8}$$

and $\sigma_\varkappa$ is the solution of the differential-difference problem

$$\sigma_\varkappa(u) = \frac{e^{-\gamma\varkappa}}{\Gamma(\varkappa+1)}\left(\frac{u}{2}\right)^\varkappa \quad \text{if} \quad 0 \leqslant u \leqslant 2,$$

$$\left(u^{-\varkappa}\sigma_\varkappa(u)\right)' = -\varkappa u^{-\varkappa-1}\sigma_\varkappa(u-2) \quad \text{if} \quad u > 2,$$

with $\sigma_\varkappa$ continuous at $u = 2$.

Although many partial or special results of this type occur in the literature, only Ankeny–Onishi [1] state results at our level of generality; they give a theorem like Theorem 1, but without proof, and derive a result similar to Theorem 2 by the use of Buchstab identities. We base the proofs of both theorems on the fundamental lemma which is the subject of Section 3; our method goes back to an idea of Wirsing [4], and in this (important) respect is similar to Levin–Feinleib [3] (where sharper results are proved under stronger conditions). Reference to Ankeny-Onishi [1], or to the discussion in Chapter IV.9 of Halberstam–Roth [2], indicates the important part played in sieve theory by such results (²).

2. Some auxiliary results. The proofs of Lemma 3 (the fundamental lemma) and of the two main theorems require some preparation. We begin by remarking that, by (1.1),

$$g(p) = \frac{\omega(p)/p}{1-\omega(p)/p} = \frac{1}{1-\omega(p)/p} - 1,$$

so that, by (Ω_1), $g(p) \leqslant A_1 - 1$ and

$$\frac{\omega(p)}{p} \leqslant g(p) \leqslant A_1\frac{\omega(p)}{p}. \tag{2.1}$$

Moreover, $g(p) = \dfrac{\omega(p)}{p} + \dfrac{\omega(p)}{p}g(p)$, so that, by (2.1),

$$\frac{\omega(p)}{p} \leqslant g(p) \leqslant \frac{\omega(p)}{p} + A_1\frac{\omega^2(p)}{p^2}.$$

If we take $w = p$ and $z = p+\varepsilon$ in (Ω_2) and then let $\varepsilon \to 0$, we obtain at once

$$\frac{\omega(p)\log p}{p} \leqslant A_2, \tag{2.2}$$

whence also

$$\frac{\omega(p)}{p} \leqslant g(p) \leqslant \frac{\omega(p)}{p} + A_1A_2\frac{\omega(p)}{p\log p}. \tag{2.3}$$

Lemma 1. *If $2 \leqslant w \leqslant z$, then*

$$-\frac{L}{\log w} \leqslant \sum_{w \leqslant p < z}{}' \frac{\omega(p)}{p} - \varkappa\log\frac{\log z}{\log w} \leqslant \frac{A_2}{\log w}, \tag{2.4}$$

$$-\frac{L}{\log w} \leqslant \sum_{w \leqslant p < z} g(p) - \varkappa\log\frac{\log z}{\log w} \leqslant \frac{A_2}{\log w}\left\{1 + A_1\left(\varkappa + \frac{A_2}{\log w}\right)\right\}, \tag{2.5}$$

$$\frac{W(w)}{W(z)} \leqslant \left(\frac{\log z}{\log w}\right)^\varkappa \left\{1 + O\left(\frac{1}{\log w}\right)\right\} \ll \left(\frac{\log z}{\log w}\right)^\varkappa \tag{2.6}$$

and

$$\frac{W(w)}{W(z)} = \left(\frac{\log z}{\log w}\right)^\varkappa \left\{1 + O\left(\frac{L}{\log w}\right)\right\}. \tag{2.7}$$

(²) A comprehensive account of sieve theory is in course of preparation by us, and will be published by Markham, Chicago.

Proof. We have

$$\sum_{w\leqslant p<z}\frac{\omega(p)}{p}=\int_w^z\frac{1}{\log t}\,d\Big(\sum_{w\leqslant p<t}\frac{\omega(p)\log p}{p}\Big)$$
$$=\frac{1}{\log z}\sum_{w\leqslant p<z}\frac{\omega(p)\log p}{p}+\int_w^z\Big(\sum_{w\leqslant p<t}\frac{\omega(p)\log p}{p}\Big)\frac{dt}{t\log^2 t},$$

so that (2.4) follows by an easy calculation from (Ω_2).

We can show in the same way that

$$(2.8)\qquad \sum_{w\leqslant p<z}\frac{\omega(p)}{p\log p}\leqslant\frac{1}{\log w}\Big(\varkappa+\frac{A_2}{\log w}\Big),$$

and (2.5) then follows at once from (2.3), (2.4) and (2.8).

Finally,

$$(2.9)\qquad \frac{W(w)}{W(z)}=\prod_{w\leqslant p<z}\Big(1-\frac{\omega(p)}{p}\Big)^{-1}=\prod_{w\leqslant p<z}(1+g(p))$$
$$=\exp\Big\{\sum_{w\leqslant p<z}\log\big((1+g(p)\big)\Big\},$$

so that, by the right hand inequality in (2.5),

$$\frac{W(w)}{W(z)}\leqslant\exp\Big\{\sum_{w\leqslant p<z}g(p)\Big\}\leqslant\exp\Big\{\log\Big(\frac{\log z}{\log w}\Big)^{\varkappa}+O\Big(\frac{1}{\log w}\Big)\Big\}$$
$$=\Big(\frac{\log z}{\log w}\Big)^{\varkappa}\exp\Big\{O\Big(\frac{1}{\log w}\Big)\Big\};$$

and from this (2.6) follows at once. Moreover, (2.9) actually implies (using that $\log(1+x)=x+O(x^2)$ if $x\geqslant-\frac{1}{2}$) that

$$\frac{W(w)}{W(z)}=\exp\Big\{\sum_{w\leqslant p<z}g(p)+O\Big(\sum_{w\leqslant p<z}g^2(p)\Big)\Big\}$$
$$=\Big(\frac{\log z}{\log w}\Big)^{\varkappa}\exp\Big\{O\Big(\frac{L}{\log w}\Big)+O\Big(\sum_{w\leqslant p<z}g^2(p)\Big)\Big\}$$

by (2.5); and since, by (2.1), (2.2) and (2.8),

$$(2.10)\qquad \sum_{w\leqslant p<z}g^2(p)\leqslant A_1^2\sum_{w\leqslant p\leqslant z}\frac{\omega^2(p)}{p^2}\leqslant A_1^2A_2\sum_{w\leqslant p<z}\frac{\omega(p)}{p\log p}\ll\frac{1}{\log w},$$

we have

$$\frac{W(w)}{W(z)}=\Big(\frac{\log z}{\log w}\Big)^{\varkappa}\exp\Big\{O\Big(\frac{L}{\log w}\Big)\Big\}.$$

If $L/\log w$ is sufficiently small, (2.7) follows at once; otherwise (2.6) gives a better result.

LEMMA 2. *If $2\leqslant w\leqslant z$, we have*

$$(2.11)\qquad \prod_{w\leqslant p<z}\Big(1+\frac{g(p)}{p^s}\Big)\Big(1-\frac{1}{p^{s+1}}\Big)^{\varkappa}=1+O\Big(\frac{L}{\log w}\Big)\quad \textit{uniformly in } s\geqslant 0$$

and

$$(2.12)\qquad W(z)=\prod_{p}\Big(1-\frac{\omega(p)}{p}\Big)\Big(1-\frac{1}{p}\Big)^{-\varkappa}\frac{e^{-\gamma\varkappa}}{\log^{\varkappa}z}\Big\{1+O\Big(\frac{L}{\log z}\Big)\Big\};$$

the product in (2.12) is convergent and uniformly positive — indeed,

$$(2.13)\qquad \prod_{p}\Big(1-\frac{\omega(p)}{p}\Big)\Big(1-\frac{1}{p}\Big)^{-\varkappa}\geqslant\exp\{-A_1A_2(1+\varkappa+A_2)\}>0.$$

Proof. It follows from (2.5) and a standard result from Mertens prime number theory that

$$-\frac{L}{\log w}\ll\sum_{w\leqslant p<z}\Big(g(p)-\frac{\varkappa}{p}\Big)\ll\frac{1}{\log w},$$

and an easy calculation of the kind used at the beginning of Lemma 1 allows us to deduce that

$$(2.14)\qquad -\frac{L}{\log w}\ll\sum_{w\leqslant p<z}\Big(\frac{g(p)}{p^s}-\frac{\varkappa}{p^{1+s}}\Big)\ll\frac{1}{\log w}\quad \text{uniformly in } s\geqslant 0.$$

Hence the product on the left of (2.11) is equal to

$$(2.15)\qquad \exp\Big\{\sum_{w\leqslant p<z}\Big(\frac{g(p)}{p^s}-\frac{\varkappa}{p^{1+s}}+O\big(g^2(p)\big)+O\big(p^{-2}\big)\Big)\Big\}$$
$$=\exp\Big\{\sum_{w\leqslant p<z}\Big(\frac{g(p)}{p^s}-\frac{\varkappa}{p^{1+s}}\Big)+O\Big(\frac{1}{\log w}\Big)\Big\}$$

by (2.10). Using only the right-hand inequality of (2.14), this expression is $\leqslant\exp\Big\{O\Big(\frac{1}{\log w}\Big)\Big\}=1+O\Big(\frac{1}{\log w}\Big)$, which is better than (2.11) if $L/\log w$ is not small. If $L/\log w$ is small enough, (2.11) follows at once from an application of (2.14) in the expression (2.15).

We now take $s = 0$ in (2.11), allow z to tend to infinity and then write z in place of w; we obtain

$$\prod_{z\leqslant p}\big(1+g(p)\big)\left(1-\frac{1}{p}\right)^{\varkappa} = 1+O\left(\frac{L}{\log z}\right),$$

so that

$$W(z) = \prod_{p<z}\left(1-\frac{\omega(p)}{p}\right)\prod_{z\leqslant p}\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\varkappa}\left\{1+O\left(\frac{L}{\log z}\right)\right\}$$

$$= \prod_{p}\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\varkappa}\prod_{p<z}\left(1-\frac{1}{p}\right)^{\varkappa}\left\{1+O\left(\frac{L}{\log z}\right)\right\};$$

and (2.12) follows at once from another well-known result of Mertens prime number theory.

Finally, taking z to be so large that $\sum_{p<z} p^{-1} > \log\log z$ (it is well known that this is possible), we have, by (2.5) (with $w = e$), that

$$\prod_{p<z}\big(1+g(p)\big)\left(1-\frac{1}{p}\right)^{\varkappa} \leqslant \exp\left\{\sum_{p<z} g(p)-\varkappa\sum_{p<z} p^{-1}\right\}$$

$$\leqslant \exp\{g(2)+A_2+A_1A_2(\varkappa+A_2)\}$$

$$\leqslant \exp\{A_1+A_2-1+A_1A_2(\varkappa+A_2)\}$$

$$\leqslant \exp\{A_1A_2(1+\varkappa+A_2)\}$$

since $g(2)\leqslant A_1-1$ and $(A_1-1)(A_2-1)\geqslant 0$; and this proves (2.13).

3. The Fundamental Lemma. We define

(3.1)
$$T(x, z) = \int_1^x G(t, z)\frac{dt}{t},$$

so that, by (1.3),

(3.2)
$$T(x, z) = \sum_{\substack{d<x\\ d|P(z)}} g(d)\log\frac{x}{d}.$$

Our object in this section is to prove the following result.

LEMMA 3. *We have*

(3.3)
$$G(x, z)\log x = (\varkappa+1)T(x, z)-\varkappa T\left(\frac{x}{z}, z\right)+O\big(LG(x, z)\big).$$

Proof. If we write

$$G_p(x, z) = \sum_{\substack{d<x\\ d|P(z)\\ (d,p)=1}} g(d),$$

and take p to be any prime divisor of $P(z)$, then, by (1.3),

$$G(x, z) = G_p(x, z)+\sum_{\substack{d<x\\ d|P(z)\\ p|d}} g(d) = G_p(x, z)+g(p)G_p\left(\frac{x}{p}, z\right).$$

We multiply this formula by $1-\dfrac{\omega(p)}{p}$ and then replace x by x/p; after rearrangement we obtain

(3.4)
$$G_p\left(\frac{x}{p}, z\right) = \left(1-\frac{\omega(p)}{p}\right)G\left(\frac{x}{p}, z\right)+\frac{\omega(p)}{p}\left\{G_p\left(\frac{x}{p}, z\right)-G_p\left(\frac{x}{p^2}, z\right)\right\}.$$

Now

$$\sum_{\substack{d<x\\ d|P(z)}} g(d)\log d = \sum_{\substack{d<x\\ d|P(z)}} g(d)\sum_{p|d}\log p = \sum_{p<z} g(p)\log p\cdot G_p\left(\frac{x}{p}, z\right),$$

and if we substitute from (3.4) on the right we have, after obvious interchanges of summation, that

$$\sum_{\substack{d<x\\ d|P(z)}} g(d)\log d = \sum_{\substack{d<x\\ d|P(z)}} g(d)\sum_{p<\min(x/d, z)}\frac{\omega(p)}{p}\log p+$$

$$+\sum_{\substack{xz^{-2}\leqslant d<x\\ d|P(z)}} g(d)\sum_{\substack{\sqrt{x/d}\leqslant p<\min(x/p, z)\\ p\nmid d}}\frac{g(p)\,\omega(p)}{p}\log p.$$

For the first inner sum we use (Ω_2) in the form

$$\sum_{p<y}\frac{\omega(p)}{p}\log p = \varkappa\log y+O(L);$$

but for the inner sum in the second expression on the right, since all the terms are non-negative, we are satisfied, using (Ω_1) and (2.5) (with $w = \sqrt{x/d}$, $z = x/d$) to use

$$\sum_{\substack{\sqrt{x/d}\leqslant p<\min(x/d, z)\\ p\nmid d}}\frac{g(p)\,\omega(p)}{p}\log p \ll \sum_{\sqrt{x/d}\leqslant p<x/d} g(p) \ll 1.$$

Hence

$$\sum_{\substack{d<x\\ d|P(z)}} g(d)\log d = \sum_{\substack{x/z\leqslant d<x\\ d|P(z)}} g(d)\left\{\varkappa\log\frac{x}{d}+O(L)\right\} + \sum_{\substack{d<x/z\\ d|P(z)}} g(d)\{\varkappa\log z+O(L)\} + O\big(G(x,z)\big)$$

$$= \varkappa\sum_{\substack{d<x\\ d|P(z)}} g(d)\log\frac{x}{d} - \varkappa\sum_{\substack{d<x/z\\ d|P(z)}} g(d)\log\frac{x/z}{d} + O\big(LG(x,z)\big);$$

and if we now add

$$\sum_{\substack{d<x\\ d|P(z)}} g(d)\log\frac{x}{d}$$

to both sides and use (3.2), we arrive at (3.3).

If is clear from (1.2) and (1.3) that

(3.5) $$G(x,z) = G(x) \quad \text{if} \quad x\leqslant z,$$

and, in particular, that

(3.6) $$G(z,z) = G(z).$$

Hence, if we define

(3.7) $$T(z) = \int_1^z G(t)\frac{dt}{t},$$

it follows from (3.1) and (3.5) that $T(x,z) = T(z)$ if $x\leqslant z$, and Lemma 3 implies, since $T(1,z)=0$, that

Corollary. *We have*

(3.8) $$G(z)\log z = (\varkappa+1)T(z)+O\big(LG(z)\big).$$

4. Proof of Theorem 1. We set out from (3.8), written for convenience in the form

(4.1) $$G(z)\log z = (\varkappa+1)T(z)+G(z)r(z)\log z$$

where

(4.2) $$r(z) = O\left(\frac{L}{\log z}\right).$$

Evidently

$$G(z)\leqslant\sum_{d|P(z)} g(d) = \prod_{p<z}\big(1+g(p)\big) = W^{-1}(z),$$

so that

(4.3) $$G(z)W(z)\leqslant 1;$$

hence Theorem 1 contains new information only when $\min(L,\log z) = L$, and then only if $L/\log z$ is sufficiently small. Thus we shall lose nothing by assuming that

(4.4) $$L\leqslant\frac{1}{B_1}\log z$$

where B_1 ($\geqslant 2$) is a sufficiently large constant; large enough, in particular, to ensure that

(4.5) $$|r(y)|\leqslant\tfrac{1}{2} \quad \text{if} \quad y\geqslant z.$$

We write (4.1) as

$$G(z) = \frac{1}{1-r(z)}\,\frac{\varkappa+1}{\log z}\,T(z),$$

so that

(4.6) $$\frac{G(z)}{\log^{\varkappa} z} = \frac{1}{1-r(z)}\exp E(z)$$

where

$$E(y) = \log\left\{\frac{\varkappa+1}{\log^{\varkappa+1}y}T(y)\right\}.$$

But

$$E'(y) = \frac{T'(y)}{T(y)} - \frac{\varkappa+1}{y\log y} = \frac{G(y)}{yT(y)} - \frac{\varkappa+1}{y\log y},$$

so that from above, if $y\geqslant z$,

$$E'(y) = \frac{1}{1-r(y)}\,\frac{\varkappa+1}{y\log y} - \frac{\varkappa+1}{y\log y} = \frac{r(y)}{1-r(y)}\,\frac{\varkappa+1}{y\log y} \ll \frac{L}{y\log^2 y}$$

using (4.2) and (4.5). Hence the integral

$$\int_z^\infty E'(y)\,dy$$

converges, and we infer that there exists a constant C such that

$$\exp E(z) = C\exp\left\{-\int_z^\infty E'(y)\,dy\right\} = C\left\{1+O\left(\frac{L}{\log z}\right)\right\};$$

the last step was justified by (4.4). It follows from (4.6) that

$$\frac{G(z)}{\log^{\varkappa} z} = C\left(1+\frac{r(z)}{1-r(z)}\right)\left\{1+O\left(\frac{L}{\log z}\right)\right\} = C\left\{1+O\left(\frac{L}{\log z}\right)\right\}$$

on using (4.5) and (4.2) once again, and so we arrive at the relation

$$G(z) = C\log^{\varkappa} z + (L\log^{\varkappa-1} z), \tag{4.7}$$

which, in view of (2.12), implies Theorem 1 if we can show that

$$C = \frac{1}{\Gamma(\varkappa+1)} \prod_p \left(1-\frac{\omega(p)}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{\varkappa}. \tag{4.8}$$

To prove (4.8) we argue as follows: if $s > 0$, then, by (4.7),

$$\prod_p \left(1+\frac{g(p)}{p^s}\right) = \sum_{d=1}^{\infty} \frac{\mu^2(d)g(d)}{d^s} = s\int_1^{\infty} \frac{G(y)}{y^{s+1}}\,dy$$

$$= s\int_1^{\infty} \frac{C\log^{\varkappa} y + O(L\log^{\varkappa-1} y)}{y^{s+1}}\,dy$$

$$= C\frac{\Gamma(\varkappa+1)}{s^{\varkappa}} + O\left(\frac{L}{s^{\varkappa-1}}\right);$$

in quoting (4.7) we have assumed that (4.7) is true for all $z > 1$; whereas we were able to prove it only subject to z being large enough to satisfy (4.4) — however, by (4.3) and (2.6) we can assert that $G(y) \ll \log^{\varkappa} y$ for all $y > 1$, and the assumption was therefore justified.

Hence

$$C = \frac{1}{\Gamma(\varkappa+1)} \lim_{s\to+0} s^{\varkappa} \prod_p \left(1+\frac{g(p)}{p^s}\right).$$

But if ζ is Riemann's zeta-function, we know that $\lim_{s\to+0} s\zeta(s+1) = 1$, and we may therefore write

$$C = \frac{1}{\Gamma(\varkappa+1)} \lim_{s\to+0} \prod_p \left(1+\frac{g(p)}{p^s}\right)\left(1-\frac{1}{p^{1+s}}\right)^{\varkappa};$$

this implies (4.8) in view of Lemma 2, (2.13); and the proof of the theorem is complete.

5. Proof of Theorem 2; the functions $\sigma_{\varkappa}$ and $\bar\sigma_{\varkappa}$. To prove Theorem 2 we shall need some information about the function $\sigma_{\varkappa}$ which was defined in the statement of the theorem, and also about the related function

$$\bar\sigma_{\varkappa}(u) = \int_0^u \sigma_{\varkappa}(t)\,dt. \tag{5.1}$$

It can be proved(³) that $\sigma_{\varkappa}(u)$ is non-negative, increasing with u, and that

$$\lim_{u\to\infty} \sigma_{\varkappa}(u) = 1, \tag{5.2}$$

so that

$$\sigma_{\varkappa}(2) = \frac{e^{-\gamma\varkappa}}{\Gamma(\varkappa+1)} \leqslant \sigma_{\varkappa}(u) \leqslant 1 \quad \text{if} \quad u \geqslant 2. \tag{5.3}$$

Then clearly $\bar\sigma_{\varkappa}(u)$ is also non-negative, increasing and

$$\bar\sigma_{\varkappa}(u) = \frac{2^{-\varkappa}e^{-\gamma\varkappa}}{\Gamma(\varkappa+2)} u^{\varkappa+1} \quad \text{if} \quad 0 \leqslant u \leqslant 2, \tag{5.4}$$

while

$$\sigma_{\varkappa}(u) = (\varkappa+1)\frac{\bar\sigma_{\varkappa}(u)}{u} - \varkappa\frac{\bar\sigma_{\varkappa}(u)}{u} \quad \text{if} \quad u > 2. \tag{5.5}$$

If we multiply (5.5) by $u^{-\varkappa-1}$ and rearrange the terms suitably, we find that $\bar\sigma_{\varkappa}$ satisfies the differential-difference equation

$$\left(u^{-\varkappa-1}\bar\sigma_{\varkappa}(u)\right)' = -\varkappa u^{-\varkappa-2}\bar\sigma_{\varkappa}(u-2), \quad u > 2,$$

and from this we deduce that

$$\frac{\bar\sigma_{\varkappa}(2\tau)}{\tau^{\varkappa+1}} = \frac{\bar\sigma_{\varkappa}(2u)}{u^{\varkappa+1}} - \varkappa\int_u^{\tau} \frac{\bar\sigma_{\varkappa}(2t-2)}{t^{\varkappa+2}}\,dt, \quad 1 \leqslant u \leqslant \tau. \tag{5.6}$$

6. Proof of Theorem 2. We begin with the remark (cf. (4.3)) that

$$G(x, z)\,W(z) \leqslant W(z) \sum_{d|P(z)} g(d) = 1, \tag{6.1}$$

so that Theorem 2 gives new information only if $L\tau^{2\varkappa+1}/\log z$ is sufficiently small. As was the case with Theorem 1, we therefore lose nothing by assuming from now on (cf. (4.4)) that

$$L\tau^{2\varkappa+1} \leqslant \frac{1}{B_2}\log z, \tag{6.2}$$

where B_2 is a sufficiently large positive constant.

If $x = z$, Theorem 2 then follows at once from Theorem 1, and we may assume henceforward that $x > z$; in other words, that

$$\tau = \frac{\log x}{\log z} > 1. \tag{6.3}$$

(³) See [1]. Note that $\sigma_{\varkappa}(u) = \dfrac{e^{-\gamma\varkappa}}{\Gamma(\varkappa)} J_{\varkappa}\left(\dfrac{u}{2}\right)$ in the notation of [1].

Lemma 3 provides the foundation for our argument. By (4.3) (with t in place of z) and (2.6) (with $w = 2$ and t in place of z) we have that

$$G(t, z) \leqslant G(t) \ll \log^{\varkappa} t, \tag{6.4}$$

so that we may write (3.3) in the form (using t in place of x)

$$G(t, z)\log t = (\varkappa+1)T(t, z) - \varkappa T(t/z, z) + (L\log^{\varkappa} t). \tag{6.5}$$

We divide (6.5) throughout by $t\log^{\varkappa+2} t$ and integrate with respect to t from w to ξ, to obtain

$$\int_w^\xi \frac{G(t, z)}{t\log^{\varkappa+1} t}\,dt = (\varkappa+1)\int_w^\xi \frac{T(t, z)}{t\log^{\varkappa+2} t}\,dt - \varkappa\int_w^\xi \frac{T(t/z, z)}{t\log^{\varkappa+2} t} + O\left(\frac{L}{\log w}\right), \quad 2 \leqslant w \leqslant \xi;$$

but since

$$\frac{\partial}{\partial t}\left\{\frac{T(t, z)}{\log^{\varkappa+1} t}\right\} = \frac{G(t, z)}{t\log^{\varkappa+1} t} - (\varkappa+1)\frac{T(t, z)}{t\log^{\varkappa+2} t},$$

we arrive at the 'reduction' formula

$$\frac{T(\xi, z)}{\log^{\varkappa+1}\xi} = \frac{T(w, z)}{\log^{\varkappa+1} w} - \varkappa\int_w^\xi \frac{T(t/z, z)}{t\log^{\varkappa+2} t}\,dt + O\left(\frac{L}{\log w}\right), \quad 2 \leqslant w \leqslant \xi. \tag{6.6}$$

We now put

$$T(\xi, z) = \tfrac{1}{2} C_0 \bar{\sigma}_\varkappa(2\tau_0)\log^{\varkappa+1} z + R(\xi, z), \qquad \tau_0 = \frac{\log\xi}{\log z} \tag{6.7}$$

where (cf. Lemma 2)

$$C_0 = e^{\gamma\varkappa}\prod_p \left(1 - \frac{\omega(p)}{p}\right)^{-1}\left(1 - \frac{1}{p}\right)^{\varkappa}. \tag{6.8}$$

Our object will be to prove that

$$R(\xi, z) \ll L\tau_0^{2\varkappa+2}\log^{\varkappa} z \quad \text{if} \quad \xi > z. \tag{6.9}$$

We proceed by induction on the range of τ_0; that is, we assume the result to be true for $\nu-1 < \tau_0 \leqslant \nu$ $(\nu \geqslant 2)$ and derive it for $\nu < \tau_0 \leqslant \nu+1$. To carry out the inductive step we introduce (6.7) into (6.6) and make use of (5.6); we find that the leading terms disappear throughout and what remains is a relation between the remainder terms only, namely

$$\frac{R(\xi, z)}{\log^{\varkappa+1}\xi} = \frac{R(w, z)}{\log^{\varkappa+1} w} - \varkappa\int_w^\xi \frac{R(t/z, z)}{t\log^{\varkappa+2} t}\,dt + O\left(\frac{L}{\log w}\right), \quad 2 \leqslant w \leqslant \xi. \tag{6.10}$$

We shall prove (6.9) by deducing from (6.10) that, for all integers $\nu \geqslant 2$,

$$\frac{|R(\xi, z)|}{\log^{\varkappa+1}\xi} \leqslant \frac{BL}{\log z}(\nu-1)^{\varkappa+1} \quad \text{if} \quad \nu-1 < \tau_0 \leqslant \nu; \tag{6.11}$$

since $\log\xi = \tau_0\log z$, it is clear that (6.9) follows from the truth of (6.11) for all $\nu \geqslant 2$. If we take $\nu = 2$, so that $z < \xi \leqslant z^2$, we see that use of (6.10) involves knowledge of $R(t, z)$ for $1 < t \leqslant z$. But in this range of t we have, by (4.7), (4.8) and (6.8) that

$$T(t, z) = T(t) = \int_1^t G(u)\frac{du}{u} = \frac{e^{-\gamma\varkappa}}{\Gamma(\varkappa+2)} C_0\log^{\varkappa+1} t\cdot\left\{1 + O\left(\frac{L}{\log t}\right)\right\} \quad (t > 1),$$

and, in view of (5.4), this is consistent with (6.7) if we take

$$|R(t, z)| \leqslant B_3 L\log^{\varkappa} t, \quad 1 < t \leqslant z. \tag{6.12}$$

We now choose $w = z$ in (6.10) and apply (6.12) on the right of (6.10); we obtain

$$\frac{|R(\xi, z)|}{\log^{\varkappa+1}\xi} \leqslant B_3 L\left\{\frac{1}{\log z} + \varkappa\int_z^\xi \frac{\log^{\varkappa}(t/z)}{t\log^{\varkappa+2} t}\,dt + \frac{B_4}{B_3}\frac{1}{\log z}\right\},$$

where B_4 is the constant implied by the O-symbol on the right of (6.10). Since we may choose $B_3 \geqslant B_4$, and

$$\int_z^\xi \frac{\log^{\varkappa}(t/z)}{t\log^{\varkappa+2} t}\,dt = \frac{1}{\log z}\int_1^{\tau_0} \frac{(u-1)^{\varkappa}}{u^{\varkappa+2}}\,du \leqslant \frac{1}{\log z},$$

we have

$$\frac{|R(\xi, z)|}{\log^{\varkappa+1}\xi} \leqslant B_3(\varkappa+2)\frac{L}{\log z},$$

which confirms (6.11) with $\nu = 2$ on taking $B = B_3(\varkappa+2)$.

Suppose now that $\nu \geqslant 2$ and that (6.11) is true. Let ξ satisfy

$$z^{\nu} < \xi \leqslant z^{\nu+1},$$

and take $w = z^{\nu}$ in (6.10). Then, by (6.11),

$$\frac{|R(\xi, z)|}{\log^{\varkappa+1}\xi} \leqslant \frac{BL}{\log z}\left\{(\nu-1)^{\varkappa+1} + \varkappa(\nu-1)^{\varkappa+1}\int_{z^{\nu}}^{\xi} \frac{\log^{\varkappa+1}(t/z)}{t\log^{\varkappa+2} t}\,dt + \frac{B_4}{B\nu}\right\},$$

and

$$\int_{z^{\nu}}^{\xi} \frac{\log^{\varkappa+1}(t/z)}{t\log^{\varkappa+2} t}\,dt = \int_{\nu}^{\tau_0} \frac{(u-1)^{\varkappa+1}}{u^{\varkappa+2}}\,du \leqslant \frac{1}{\nu};$$

hence, using the fact that $B > B_4$,

$$\frac{|R(\xi, z)|}{\log^{\varkappa+1}\xi} \leqslant \frac{BL}{\log z}\left\{(\nu-1)^{\varkappa+1}\left(1+\frac{\varkappa}{\nu}\right)+\frac{1}{\nu}\right\} < \frac{BL}{\log z}(\nu-1)^{\varkappa+1}\left(1+\frac{\varkappa+1}{\nu}\right),$$

and since $(\nu-1)^{\varkappa+1}\left(1+\frac{\varkappa+1}{\nu}\right) \leqslant \nu^{\varkappa+1}$ (as may easily be verified), we obtain

$$\frac{|R(\xi, z)|}{\log^{\varkappa+1}\xi} \leqslant \frac{BL}{\log z}\nu^{\varkappa+1} \quad \text{if} \quad z^{\nu} < \xi \leqslant z^{\nu+1},$$

and thereby confirm the truth of (6.11) with $\nu+1$ in place of ν.

This completes the proof of (6.11) and hence also of (6.9).

To complete the proof of Theorem 2, we substitute (6.9) in (6.7), and use this composite relation, with $\xi = x$ and $\xi = x/z$ in turn, to evaluate $G(x, z)$ from (6.5) (with $t = x$): we obtain

$$G(x, z) = (\varkappa+1)C_0\frac{\overline{\sigma}_\varkappa(2\tau)}{2\tau}\log^\varkappa z - \varkappa C_0\frac{\overline{\sigma}_\varkappa(2\tau-2)}{2\tau}\log^\varkappa z + O(L\tau^{2\varkappa+1}\log^{\varkappa-1}z)$$

$$= C_0\sigma_\varkappa(2\tau)\log^\varkappa z + O(L\tau^{2\varkappa+1}\log^{\varkappa-1}z)$$

by (5.5). Theorem 2 follows at once from this and (2.12).

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A theorem on chains of finite sets, II

by

R. Rado (Reading)

Dedicated to the memory of Harold Davenport

1. Introduction. E. Harzheim [1] proved the following theorem:

Theorem A. *Given a positive integer n, there is a positive integer n^* such that the following statement holds. If S is a set of n^* elements, and if $f(X)$, for every non-empty subset X of S, is an element of X, then there always are subsets $X_0, X_1, \ldots, X_n$ of S such that*[1] $X_0 \subset X_1 \subset \ldots \subset X_n$ *and*

$$f(X_0) = f(X_1) = \ldots = f(X_n).$$

The following theorem is a generalization of Theorem A ([4], Theorem 3):

Theorem B. *Given a positive integer n, there is a positive integer n^* such that the following statement holds. If S is a set of n^* elements, and if $f(X)$, for every subset X of S, is a subset of X, then there always are subsets $X_0, \ldots, X_n$ of S such that $X_0 \subset \ldots \subset X_n$ and $f(X_0) \subseteq \ldots \subseteq f(X_n)$.*

In the present note Theorem B will be further generalized. No knowledge of the earlier papers [1], [4] will be assumed. In fact, the proof of the still more general Theorem C given below is simpler than that of Theorem B as given in [4], thanks to an application of an idea used by D. J. White [6] which makes it unnecessary to appeal to a theorem of G. Higman [3] which was needed in [4].

2. Notation and terminology. We put $N = \{0, 1, 2, \ldots\}$. Lower case letters other than $f, g, h, \varphi, \psi, \chi, \pi$ denote elements of N, and capital letters denote subsets of N. If nothing is said to the contrary these sets are finite. The cardinal of A is denoted by $|A|$, and for every S, finite or infinite, we put

$$[S]^r = \{X\colon X \subseteq S;\ |X| = r\}.$$

Also, $[0, m) = \{0, 1, \ldots, m-1\}$.

(1) $A \subset B$ denotes set inclusion in the strict sense.