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Enumeration of certain sequences

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To the memory of Harold Davenport

1. We shall be concerned with the following problems. Let a be a fixed nonnegative integer and let $f(n, a)$ denote the number of sequences of integers $(a_1, a_2, \ldots, a_n)$ that satisfy

(1.1) $$0 \leqslant a_i \leqslant a \quad (i = 1, 2, \ldots, n)$$

and

(1.2) $$|a_i - a_{i+1}| = 1 \quad (i = 1, 2, \ldots, n-1).$$

Let $g(n, a)$ denote the number of sequences $(a_1, a_2, \ldots, a_n)$ that satisfy (1.1) and

(1.3) $$|a_i - a_{i+1}| \leqslant 1 \quad (i = 1, 2, \ldots, n-1).$$

In evaluating $f(n, a)$ and $g(n, a)$ it is convenient to introduce functions $f_j(n, a)$, $g_j(n, a)$ defined in the following way. Let $f_j(n, a)$ denote the number of sequences $(a_1, a_2, \ldots, a_n)$ that satisfy (1.1) and (1.2) and in addition $a_1 = j$; similarly let $g_j(n, a)$ denote the number of sequences $(a_1, a_2, \ldots, a_n)$ that satisfy (1.1) and (1.3) and in addition $a_1 = j$. It follows at once from the definition that

(1.4) $$f_j(n, a) = f_{a-j}(n, a) \quad (0 \leqslant j \leqslant a)$$

and

(1.5) $$g_j(n, a) = g_{a-j}(n, a) \quad (0 \leqslant j \leqslant a).$$

Also

(1.6) $$f_j(n, a) = f_{j-1}(n-1, a) + f_{j+1}(n-1, a) \quad (0 < j < a),$$

while

(1.7) $$f_0(n, a) = f_1(n-1, a), \quad f_a(n, a) = f_{a-1}(n-1, a).$$

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The corresponding relations for $g_j(n, a)$ are

(1.8) $$g_j(n, a) = g_{j-1}(n-1, a)+g_j(n-1, a)+g_{j+1}(n-1, a) \quad (0 < j < a),$$

(1.9) $$g_0(n, a) = g_0(n-1, a)+g_1(n-1, a),$$
$$g_a(n-1, a) = g_{a-1}(n-1, a)+g_a(n-1, a).$$

Thus for example, for $a = 2$, we find that

$$f_0(n, a) = 2f_0(n-2, 2),$$

which implies

$$f_0(2n+1, 2) = f_0(2n+2, 2) = 2^n.$$

For $a = 3$ we have

$$f_0(n, 3) = f_1(n-1, 3) = f_0(n-2, 3)+f_2(n-2, 3),$$

which reduces to

(1.10) $$f_0(n, 3) = f_0(n-1, 3)+f_0(n-2, 3).$$

Since $f_0(1, 3) = f_0(2, 3) = 1$, it follows from (1.10) that

(1.11) $$f_0(n, 3) = F_n,$$

where F_n is the Fibonacci number defined by

$$F_n = F_{n-1}+F_{n-2}, \quad F_1 = F_2 = 1.$$

Similarly for $a = 4$ we find that

(1.12) $$f_0(2n+1, 4) = f_0(2n+2, 4) = 3^n,$$
$$f_1(2n, 4) = f_1(2n+1, 4) = 3^n.$$

In order to evaluate $f_j(n, a)$, $g_j(n, a)$ for arbitrary a, we define two additional functions $f_j(n)$, $g_j(n)$. We let $f_j(n)$ denote the number of sequences of nonnegative integers $(a_1, a_2, \ldots, a_n)$ that satisfy (1.2) and $a_1 = j$; also let $g_j(n)$ denote the number of such sequences that satisfy (1.3) and $a_1 = j$.

The function $f_j(n)$ is evaluated explicitly by means of (2.12) below or alternatively by means of the following formulas.

(1.13) $$f_k(n+1) = 2^n \quad (k \geqslant n \geqslant 0),$$

(1.14) $$f_{n-2k-1}(n+1) = 2^n-\binom{n}{k}-2\sum_{j=0}^{k-1}\binom{n}{j},$$

(1.15) $$f_{n-2k}(n+1) = 2^n-2\sum_{j=0}^{k-1}\binom{n}{j}.$$

The function $f_0(n, a)$ is shown to satisfy a recurrence of order $\leqslant a$ with initial values

(1.16) $$f_0(n+1, a) = \binom{n}{[n/2]} \quad (0 \leqslant n \leqslant a).$$

The results for $g_j(n)$ and $g_j(n, a)$ are of a similar nature. They are contained in formulas (5.2), (5.3), (5.4) for $g_j(n)$ and (6.6), (6.7) for $g_0(n, a)$. It should be noted that the results for $g_j(n)$ are in terms of the coefficients $c(k, j)$ defined by

(1.17) $$(1+x+x^2)^k = \sum_{j=0}^{\infty} c(k, j)x^j.$$

2. Let the sequence $(a_1, a_2, \ldots, a_n)$ satisfy (1.1), (1.2) and $a_1 = j$. Put

$$b_i = a-a_i \quad (i = 1, 2, \ldots, n).$$

Then the sequence $(b_1, b_2, \ldots, b_n)$ satisfies (1.1), (1.2) and $b_1 = a-j$. This proves (1.4.). The proof of (1.5) is exactly the same. Formulas (1.6) and (1.7) are evidently immediate consequences of the definition of $f_j(n, a)$; similarly for (1.8) and (1.9).

Turning now to $f_j(n)$, it is evident that

$$\begin{aligned} f_0(n) &= f_1(n-1) \\ &= f_0(n-2)+f_2(n-2) \\ &= 2f_1(n-3)+f_3(n-3) \\ &= 2f_0(n-4)+3f_2(n-4)+f_4(n-5) \\ &= 5f_1(n-5)+4f_3(n-5)+f_5(n-6). \end{aligned}$$

Generally we have, for $k \geqslant 0$,

(2.1) $$f_0(n) = \sum_{2j\leqslant k} A_{kj}f_{k-2j}(n-k) \quad (n > k),$$

where the coefficients A_{kj} are independent of n. Moreover, since

$$f_j(n) = f_{j-1}(n-1)+f_{j+1}(n-1) \quad (j > 0),$$

it follows from (2.1) that

$$\begin{aligned} f_0(n) &= \sum_{2j\leqslant k} A_{kj}[f_{k-2j-1}(n-k-1)+f_{k-2j+1}(n-k-1)] \\ &= \sum_{2j\leqslant k+1} (A_{k,j-1}+A_{kj})f_{k-2j+1}(n-k-1). \end{aligned}$$

We have therefore

(2.2) $$A_{k+1,j} = A_{k,j-1}+A_{kj} \quad (0 \leqslant 2j \leqslant k+1).$$

We may also express $f_k(n)$ in terms of $f_0(n+j)$. Clearly

$$\begin{aligned} f_1(n) &= f_0(n+1),\\ f_2(n) &= f_0(n+2)-f_0(n),\\ f_3(n) &= f_0(n+3)-2f_0(n+1),\\ f_4(n) &= f_0(n+4)-3f_0(n+2)+f_0(n),\\ f_5(n) &= f_0(n+5)-4f_0(n+3)+3f_0(n+1). \end{aligned}$$

Generally we have

$$f_k(n) = \sum_{2j\leqslant k}(-1)^j B_{kj} f_0(n+k-2j). \tag{2.3}$$

Moreover the B_{kj} satisfy the recurrence

$$B_{k+1,j} = B_{kj}+B_{k-1,j-1} \qquad (0\leqslant 2j\leqslant k+1). \tag{2.4}$$

It is easily verified, using (2.4), that

$$B_{kj} = \binom{k-j}{j}. \tag{2.5}$$

Thus (2.3) becomes

$$f_k(n) = \sum_{2j\leqslant k}(-1)^j\binom{k-j}{j} f_0(n+k-2j). \tag{2.6}$$

To find the A_{kj} we make use of the following lemma ([3], p. 59). The set of equations

$$v_k = \sum_{2j\leqslant k}(-1)^j\binom{k-j}{j} u_{k-2j} \qquad (k = 0,1,2,\ldots) \tag{2.7}$$

is equivalent to

$$u_k = \sum_{2j\leqslant k}\left[\binom{k}{j}-\binom{k}{j-1}\right] v_{k-2j} \qquad (k = 0,1,2,\ldots). \tag{2.8}$$

It follows therefore that

$$A_{kj} = \binom{k}{j}-\binom{k}{j-1} \qquad (0\leqslant 2j\leqslant k), \tag{2.9}$$

so that (2.1) becomes

$$f_0(n) = \sum_{2j\leqslant k}\left[\binom{k}{j}-\binom{k}{j-1}\right] f_{k-2j}(n-k). \tag{2.10}$$

We now take $n = k+1$ in (2.10). Since $f_j(1) = 1$, (2.10) becomes

$$f_0(k+1) = \sum_{2j\leqslant k}\left[\binom{k}{j}-\binom{k}{j-1}\right].$$

This reduces to

$$f_0(k+1) = \binom{k}{[k/2]}. \tag{2.11}$$

Substituting from (2.11) in (2.6) we get

$$f_k(n+1) = \sum_{2j\leqslant k}(-1)^j\binom{k-j}{j}\binom{n+k-2j}{[(n+k-2j)/2]}. \tag{2.12}$$

It does not seem possible to sum the series on the right of (2.12). For example we have

$$\begin{aligned} f_{2k}(2n+1) &= \sum_{j=0}^{k}(-1)^j\binom{2k-j}{j}\binom{2n+2k-2j}{n+k-j}\\ &= \sum_{j=0}^{k}(-1)^{k-j}\binom{k+j}{2j}\binom{2n+2j}{n+j}\\ &= (-1)^k\binom{2n}{n}\sum_{j=0}^{k}\frac{(-k)_j(k+1)_j(n+\frac{1}{2})}{j!(\frac{1}{2})_j(n+1)_j}, \end{aligned}$$

the ${}_3F_2$ occurring here is not quite Saalschützian ([1], p. 9).

On the other hand it can be verified that (2.12) implies

$$f_n(n+1) = 2^n. \tag{2.13}$$

This result is also evident from the definition. Indeed, we have more generally

$$f_k(n+1) = 2^n \qquad (k\geqslant n\geqslant 0). \tag{2.14}$$

Since

$$f_{n-1}(n+1) = f_{n-2}(n)+f_n(n) = f_{n-2}(n)+2^{n-1},$$

we get

$$f_{n-1}(n+1) = 2^n-1. \tag{2.15}$$

Similarly we find that

$$f_{n-2}(n+1) = 2^n-2 \tag{2.16}$$

and

$$f_{n-3}(n+1) = 2^n-n-2, \tag{2.17}$$

$$f_{n-4}(n+1) = 2^n-2n-2. \tag{2.18}$$

These results suggest that

$$f_{n-k}(n+1) = 2^n-P_k(n) \qquad (0\leqslant k\leqslant n), \tag{2.19}$$

where $P_k(n)$ is a polynomial in n of degree $[(k-1)/2]$. Indeed, it follows from (2.19) that

$$P_k(n+1) = P_k(n)+P_{k-2}(n) \tag{2.20}$$

and therefore $P_k(n)$ is a polynomial of the stated degree. Moreover we readily obtain the following explicit results:

(2.21) $$P_{2k+1}(n) = \binom{n}{k} + 2\sum_{j=0}^{k-1}\binom{n}{j},$$

(2.22) $$P_{2k}(n) = 2\sum_{j=0}^{k-1}\binom{n}{j}.$$

Comparison of (2.12) with (2.21) and (2.22) leads to certain summation formulas that we shall not state.

3. Turning now to $f_j(n, a)$, then exactly as in deriving (2.1) and (2.3) we have

(3.1) $$f_0(n, a) = \sum_{2j\leqslant k} A_{kj} f_{k-2j}(n+k, a) \qquad (0 \leqslant k \leqslant a)$$

and

(3.2) $$f_k(n, a) = \sum_{2j\leqslant k} (-1)^j B_{kj} f_0(n+k-2j, a) \qquad (0 \leqslant k \leqslant a).$$

We again have

$$B_{kj} = \binom{k-j}{j}.$$

Also, since the lemma on the equivalence of (2.7) and (2.8) holds when the parameter k is restricted to $k \leqslant a$, we get

$$A_{kj} = \binom{k}{j} - \binom{k}{j-1}.$$

Thus (3.1) and (3.2) become

(3.3) $$f_0(n, a) = \sum_{2j\leqslant k}\left[\binom{k}{j} - \binom{k}{j-1}\right] f_{k-2j}(n-k, a) \qquad (0 \leqslant k \leqslant a),$$

(3.4) $$f_k(n, a) = \sum_{2j\leqslant k} (-1)^j \binom{k-j}{j} f_0(n+k-2j, a) \qquad (0 \leqslant k \leqslant a).$$

For $k = a$, (3.4) becomes

(3.5) $$f_0(n, a) = \sum_{2j\leqslant a} (-1)^j \binom{a-j}{j} f_0(n+a-2j, a),$$

a recurrence of order a (when $4|a$, the order reduces to $a-2$). For example, for $a = 3$,

(3.6) $$f_0(n, 3) = f_0(n+3, 3) - 2f_0(n+1, 3).$$

It is easily verified that (3.6) is in agreement with (1.10).

Generally, we can obtain a recurrence of order $[(a+2)/2]$. If $a = 2k+1$, it follows from (1.4) that

$$f_k(n, a) = f_{k+1}(n, a).$$

Thus (3.4) gives

(3.7) $$\sum_{2j\leqslant k} (-1)^j \binom{k-j}{j} f_0(n+k-2j, 2k+1) = \sum_{2j\leqslant k+1} (-1)^j \binom{k-j+1}{j} f_0(n+k-2j+1, 2k+1).$$

If $a = 2k$, then

$$f_{k-1}(n, a) = f_{k+1}(n, a)$$

and we get

$$\sum_{2j< k} (-1)^j \binom{k-j-1}{j} f_0(n+k-2j-1, 2k) = \sum_{2j\leqslant k+1} (-1)^j \binom{k-j+1}{j} f_0(n+2k-2j+1, 2k).$$

This may be written in the form

(3.8) $$\sum_{2j\leqslant k+1} (-1)^j \left[\binom{k-j+1}{j} + \binom{k-j}{j-1}\right] f_0(n+k-2j+1, 2k) = 0.$$

The initial values of $f_0(n, a)$ may be obtained by using

(3.9) $$f_0(n, a) = f_0(n) \qquad (1 \leqslant n \leqslant a+1).$$

The proof of (3.9) is immediate from the definition. Therefore, by (2.11),

(3.10) $$f_0(k+1, a) = \binom{k}{[k/2]} \qquad (0 \leqslant k \leqslant a).$$

Finally, since

$$f(n, a) = \sum_{k=0}^{a} f_k(n, a) = \sum_{k=0}^{a} [f_{k-1}(n-1, a) + f_{k+1}(n-1, a)] = 2f(n-1, a) - 2f_0(n-1, a),$$

it follows that

(3.11) $$f(n, a) = 2^{n-1}(a+1) - \sum_{j=1}^{n-1} 2^{n-j} f_0(j, a).$$

4. We now consider $g_j(n)$. It is evident from the definition that

$$\begin{aligned} g_0(n) &= g_0(n-1)+g_1(n-1) \\ &= 2g_0(n-2)+2g_1(n-2)+g_2(n-2) \\ &= 4g_0(n-3)+5g_1(n-3)+3g_2(n-3)+g_3(n-3) \\ &= 9g_0(n-4)+12g_1(n-4)+9g_2(n-4)+4g_3(n-4)+g_3(n-4). \end{aligned}$$

Generally we have

$$g_0(n) = \sum_{j=0}^{k} A'_{kj} g_{k-j}(n-k) \quad (n > k), \tag{4.1}$$

where A'_{kj} satisfies the recurrence

$$A'_{k+1,j} = A'_{k,j-2}+A'_{k,j-1}+A'_{kj}. \tag{4.2}$$

Next, expressing $g_k(n)$ in terms of $g_0(n-j)$, we have

$$\begin{aligned} g_1(n) &= g_0(n-1)-g_0(n), \\ g_2(n) &= g_0(n+2)-2g_0(n+1), \\ g_3(n) &= g_0(n+3)-3g_0(n+2)+g_0(n+1)+g_0(n), \\ g_4(n) &= g_0(n+4)-4g_0(n+3)+3g_0(n+2)+2g_0(n+1)-g_0(n). \end{aligned}$$

Generally we have

$$g_k(n) = \sum_{j=0}^{k} B'_{kj} g_0(n+k-j), \tag{4.3}$$

where B'_{kj} satisfies

$$B'_{kj} = B'_{k-1,j-2}+B'_{k,j-1}+B'_{k+1,j}. \tag{4.4}$$

The first few values of A'_{kj}, B'_{kj} follow.

A'_{kj}

1					
1	1				
1	2	2			
1	3	5	4		
1	4	9	12	9	
1	5	14	25	30	21

B'_{kj}

1					
1	−1				
1	−2	0			
1	−3	1	1		
1	−4	3	2	−1	
1	−5	6	2	−4	0

The coefficients A'_{kj} have occurred in another connection [2].

Put

$$\beta_k(x) = \sum_{j=0}^{\infty} B'_{k+j,j} x^j. \tag{4.5}$$

Then by (4.4)

$$\beta_k(x) = (1+x+x^2)\beta_{k+1}(x). \tag{4.6}$$

Also, if we take $k = j-1$ in (4.4),

$$B'_{j-2,j-2}+B'_{j-1,j-1}+B'_{jj} = 0 \quad (j \geqslant 2).$$

Since

$$B'_{00} = 1, \quad B'_{11} = -1,$$

it follows that

$$\beta_0(x) = (1+x+x^2)^{-1}.$$

Thus (4.6) implies

$$\beta_k(x) = (1+x+x^2)^{-k-1}. \tag{4.7}$$

It is convenient to put

$$(1+x+x^2)^n = \sum_{k=0}^{\infty} c(n,k) x^k, \tag{4.8}$$

where n is an arbitrary integer. Thus

$$B'_{k+j,j} = c(-k-1, j). \tag{4.9}$$

We shall now show that

$$A'_{kj} = c(k,j)-c(k,j-2) \quad (0 \leqslant j \leqslant k), \tag{4.10}$$

with $c(k,j)$ as in (4.8). It follows at once from (4.8) that $c(k,j)$ satisfies the recurrence

$$c(k+1,j) = c(k,j-2)+c(k,j-1)+c(k,j). \tag{4.11}$$

Thus if we assume that (4.10) holds, we get, using (4.2),

$$\begin{aligned} A'_{k-1,j} &= A'_{k,j-2}+A'_{k,j-1}+A_{kj} \\ &= c(k,j-2)+c(k,j-1)+c(k,j)+ \\ &\quad +[c(k,j-4)+c(k,j-3)+c(k,j-2)] \\ &= c(k+1,j)-c(k+1,j-2). \end{aligned}$$

This evidently completes the proof of (4.10).

Analogous to the equivalence of (2.7) and (2.8), we may state the following result. The set of equations

$$u_k = \sum_{j=0}^{k} [c(k,j)-c(k,j-2)] v_{k-j} \quad (k = 0, 1, 2, \ldots) \tag{4.12}$$

is equivalent to

$$v_k = \sum_{j=0}^{k} c(-k+j-1, j) u_{k-j} \quad (k = 0, 1, 2, \ldots). \tag{4.13}$$

5. Returning to (4.1) we take $n = k+1$, so that

$$g_0(k+1) = \sum_{j=0}^{k} A'_{kj} g_{k-j}(1).$$

Since $g_j(1) = 1$, it follows from (4.10) that

$$g_0(k+1) = \sum_{j=0}^{k} [c(k, j) - c(k, j-2)],$$

which reduces to

$$g_0(k+1) = c(k, k) + c(k, k+1). \tag{5.1}$$

Then, by (4.3) and (4.9)

$$g_k(n+1) = \sum_{j=0}^{k} B'_{k,k-j} g_0(n+j+1),$$

so that

$$g_k(n+1) = \sum_{j=0}^{k} c(-j-1, k-1)[c(n+j, n+j) + c(n+j, n+j+1)]. \tag{5.2}$$

It follows from the definition that (compare (2.14))

$$g_k(n+1) = 3^n \quad (k \geqslant n \geqslant 0). \tag{5.3}$$

Corresponding to (2.20) we have

$$g_{n-k}(n+1) = 3^n - Q_k(n) \quad (0 \leqslant k \leqslant n), \tag{5.4}$$

where $Q_k(n)$ is a polynomial in n of degree $k-1$. Indeed, it follows from

$$g_{n-k}(n+1) = g_{n-k-1}(n) + g_{n-k}(n) + g_{n-k+1}(n)$$

that

$$Q_k(n+1) = Q_k(n) + Q_{k-1}(n) + Q_{k-2}(n) \tag{5.5}$$

and therefore $Q_k(n)$ is a polynomial of degree $k-1$. In particular

$$Q_0(n) = 0, \quad Q_1(n) = 1, \quad Q_2(n) = n+2.$$

We shall show that

$$Q_{k+1}(n) = 2\sum_{j=0}^{k-1} c(n, j) + c(n, k). \tag{5.6}$$

The formula is evidently correct for $k = 0, 1$. Assuming that it holds up to and including the value $k-1$, we have by (5.5),

$$\begin{aligned} Q_{k+1}(n+1) - Q_{k+1}(n) &= Q_k(n) + Q_{k-1}(n) \\ &= c(n, k-1) + c(n, k-2) + 2\sum_{j=1}^{k-1} [c(n, j-1) + c(n, j-2)] \\ &= c(n+1, k) - c(n, k) + 2\sum_{j=0}^{k-1} [c(n+1, j) - c(n, j)], \end{aligned}$$

so that

$$Q_{k+1}(n) = 2\sum_{j=0}^{k-1} c(n, j) + c(n, k) + C_k,$$

where C_k is independent of n. Making use of (5.4) we can show that $C_k = 0$, thus completing the induction.

6. Exactly as in § 3, we can show that

$$g_0(n, a) = \sum_{j=0}^{k} A'_{kj} g_{k-j}(n-k, a) \quad (0 \leqslant k \leqslant a) \tag{6.1}$$

and

$$g_k(n, a) = \sum_{j=0}^{k} B'_{kj} g_0(n+k-j, a) \quad (0 \leqslant k \leqslant a), \tag{6.2}$$

where

$$A'_{kj} = c(k, k+j) - c(k, k+j+2) \quad (0 \leqslant j \leqslant k) \tag{6.3}$$

and

$$B'_{kj} = c(-k+j-1, j) \quad (0 \leqslant j \leqslant k), \tag{6.4}$$

while $c(n, j)$ is defined by

$$(1+x+x^2)^n = \sum_{j=0}^{\infty} c(n, j) x^j.$$

We have also

$$g_k(n, a) = g_{a-k}(n, a) \quad (0 \leqslant k \leqslant a). \tag{6.5}$$

Thus for $k = a$, (6.2) becomes

$$g_0(n, a) = \sum_{j=0}^{a} B'_{aj} g_0(n+a-j, a); \tag{6.6}$$

thus $g_0(n, a)$ satisfies a recurrence of order $\leqslant a$. As in § 3 recurrences of lower order can usually be obtained.

As for initial values, we have

$$g_0(n, a) = g_0(n) \quad (1 \leqslant n \leqslant a+1). \tag{6.7}$$

Hence $g_0(n, a)$ is uniquely determined by means of (6.6) and (6.7).

We give a few special results. For $a = 2$ we have

$$g_0(n,2) = 2g_0(n-1,2)+g_0(n-2,2),$$
$$g_0(1,2) = 1, \quad g_0(2,2) = 2, \quad g_0(3,2) = 5.$$

It follows that

$$g_0(n,2) = \frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}.$$

For $a = 3$ we have

$$g_0(n,3)-3g_0(n-1,3)+g_0(n-2,3) = 0,$$
$$g_0(1,3) = 1, \quad g_0(2,3) = 2, \quad g_0(3,3) = 5.$$

It follows that

$$g_0(n,3) = F_{2n-1}, \quad g_1(n,3) = F_{2n},$$

where F_n is the nth Fibonacci number.

Finally we have

$$g(n,a) = \sum_{k=0}^{a} g_k(n,a)$$
$$= \sum_{k=0}^{a} [g_{k-1}(n-1,a)+g_k(n-1,a)+g_{k+1}(n-1,a)]$$
$$= 3g(n-1,a)-2g_0(n-1,a).$$

It follows that

$$g(n,a) = 3^{n-1}(a+1)-2\sum_{k=1}^{n-1} 3^{n-k-1}g_0(k,a). \tag{6.8}$$

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Zur Definition der Diskrepanz

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Es sei I^s der s-dimensionale Einheitswürfel $0 \leqslant \xi_1 < 1, \ldots, 0 \leqslant \xi_s < 1$ und $x_1, \ldots, x_\nu$ Punkte in I^s. Um die Verteilung der Folge $\omega_\nu = (x_1, \ldots, x_\nu)$ zu studieren betrachtet man die sogenannte Diskrepanz $D(\omega_\nu)$. Sie ist so definiert: Ist J ein beliebiges Intervall, so sei $\nu^*(J)$ die Anzahl der Punkte von ω_ν in J, $V(J)$ das Volumen von J, dann ist

$$D(\omega_\nu) = \sup_J \left| \frac{\nu^*(J)}{\nu} - V(J) \right|.$$

Es ist stets $\nu^{-1} \leqslant D \leqslant 1$. Es ist D umso kleiner, je "gleichmäßiger" ω_ν in $I = I^s$ verteilt ist. Bekanntlich ist der Begriff der Diskrepanz aus der Theorie der Gleichverteilung entstanden. Hier werden unendliche Folgen $\omega = (x_i)_{i\in N}$ betrachtet und eine solche Folge ist gleichverteilt, wenn $\lim_{\nu\to\infty} D(\omega_\nu) = 0$, wo ω_ν die Folge der ersten ν Glieder aus ω ist. Genauer gesagt heißt ω gleichverteilt, wenn $\lim_{\nu\to\infty} \frac{\nu^*(J)}{\nu} = V(J)$ für alle J. Man kann dann zeigen, daß daraus $\lim D(\omega_\nu) = 0$ folgt. Die Umkehrung ist trivial. Man kann fragen, ob man nicht die Familie der Intervalle durch andere Familien von Teilmengen von I ersetzen kann. Wir werden zunächst zeigen, daß die Familie $\mathscr{C}$ aller konvexen Körper C das Gewünschte leistet:

(1) Ist $D_C(\omega_\nu) = \sup_{C\in\mathscr{C}} \left| \frac{\nu^*(C)}{\nu} - V(C) \right|$ so ist $D_C(\omega) \leqslant 72^s D^{1/s}(\omega)$.

Trivialer Weise ist $D(\omega) \leqslant D_C(\omega)$. Aus einem Satz von S. K. Zaremba (1) folgt, daß man den Exponenten $1/s$ in (1) nicht weglassen kann. Es liegt nahe, statt Intervalle Kugeln K zu nehmen, die zugehörige Diskrepanz wollen wir D_K nennen. Dann gilt für $D_K(\omega)$ trivialerweise (1). Schwieriger ist jetzt die Frage einer Abschätzung von D durch D_K wenn $s > 1$. (Im Falle $s = 1$ ist natürlich $D = D_K$.) Wir werden zeigen: Es gibt

(1) Monatshefte für Mathematik 72 (1968), S. 264–269.