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The multiplicity of partial coverings of space

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1. Let K be a convex body in n-dimensional space. Consider a system of translates of K such that no point of space belongs to more than $h-1$ of the translates. This system is an $(h-1)$-fold packing. Let the proportion of space belonging to at least one of the bodies be δ, and let

$$k = -\log(1-\delta). \tag{1}$$

We prove that, provided n is sufficiently large, and

$$n4^{-n} < \delta < 1-e^{-n/6}, \tag{2}$$

there is such a system with $h-1 = [l]$, where

$$l = \frac{n\log 4(n+1)-2ke-\log\delta-\frac{1}{2}\log n+n}{\log n-\log 2ke}, \tag{3}$$

and we also prove that the density of the system is greater than $2k$ and $\sim 2k$.

These results are illustrated by examples in § 7.

This paper uses methods of Erdös and Rogers [1], and the notation of that paper is used where convenient.

2. In this section we take K to be a Lebesgue measurable set with finite positive measure V. Let Λ be the lattice of all points with integral coordinates, and suppose that all the distinct translates of K by the vectors of Λ are disjoint.

Let the N points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N$ be chosen at random in the cube C of points $\boldsymbol{x}$ with

$$0 \leqslant x_i \leqslant 1 \quad (i = 1, 2, \ldots, n).$$

Consider the system of sets

$$K+\boldsymbol{x}_i+\boldsymbol{g} \quad (1 \leqslant i \leqslant N,\ \boldsymbol{g}\in\Lambda) \tag{4}$$

and, for $0 \leqslant h \leqslant N$, the set E_h of points belonging to just h of the sets (4). Then, given K and h, the density $\delta(E_h)$ of the set E_h is a function of

$x_1, \ldots, x_N$, and it has been proved by Erdös and Rogers [1] that the mean value, $\mathscr{M}(\delta(E_h))$, of this density over all choices of the points $x_1, \ldots, x_N$ in C is

$$\mathscr{M}(\delta(E_h)) = \frac{N!}{h!(N-h)!} V^h (1-V)^{N-h}. \tag{5}$$

3. Now take K to be a convex body with volume V. By a result of Rogers and Shephard [3] there is a lattice Λ_1 of determinant $4^n V$ such that the distinct translates of K by the vectors of Λ_1 are disjoint. Thus, after applying a suitable linear transformation to K, we may suppose that the volume V of K is 4^{-n} and that the distinct translates of K, by the vectors of the lattice Λ of all points with integral coordinates, are disjoint.

Let F_h be the set of points covered by at least h bodies of the system

$$K + x_i + g \quad (1 \leqslant i \leqslant N,\ g \in \Lambda),$$

and let E_0 be the set of points belonging to no body of the system

$$(1-\eta)K + x_i + g \quad (1 \leqslant i \leqslant N,\ g \in \Lambda),$$

where $0 < \eta < 1$. Then it follows from (5) that

$$\mathscr{M}(\delta(F_h)) = \sum_{t=h}^{N} \frac{N!}{t!(N-t)!} V^t (1-V)^{N-t} \tag{6}$$

$$= \frac{N!}{h!(N-h)!} V^h (1-V)^{N-h} \sum_{t=0}^{N-h} \frac{h!(N-h)!}{(h+t)!(N-h-t)!} \left(\frac{V}{1-V}\right)^t,$$

and that

$$\mathscr{M}(\delta(E_0)) = \left(1-(1-\eta)^n V\right)^N. \tag{7}$$

4. It follows from (1) and (2) that

$$k < \tfrac{1}{6} n \tag{8}$$

so that

$$\log n - \log 2ke > \log 3 - 1 = a > 0. \tag{9}$$

Let

$$V = 4^{-n}, \quad N^* = 2ke4^n, \quad N = [N^*]+1. \tag{10}$$

We have, by (1) and (2),

$$k > \delta > n4^{-n}, \tag{11}$$

so that, by (10),

$$N > n. \tag{12}$$

By (3), (10) and (2)

$$\frac{l}{N^*} < \frac{n \log 16e(n+1)}{k4^n(\log n - \log 2ke)}. \tag{13}$$

Let

$$f(k) = k(\log n - \log 2ke),$$

so that

$$f'(k) = \log n - \log 2ke^2.$$

Thus, if $n4^{-n} \leqslant k \leqslant n/2e^2$, we have from (13)

$$\frac{l}{N^*} < \frac{n \log 16e(n+1)}{n(n\log 4 - \log 2e)} = o(1), \tag{14}$$

and, if $n/2e^2 \leqslant k \leqslant \frac{1}{6}n$, we have from (13),

$$\frac{l}{N^*} < \frac{6n \log 16e(n+1)}{an4^n} = o(1) \tag{15}$$

where a is defined in (9). Thus by (2), (8), (11), (14) and (15)

$$\frac{l}{N^*} = o(1). \tag{16}$$

Hence

$$\frac{h}{N} \leqslant \frac{l+1}{N^*} = o(1) \tag{17}$$

by (16) and (12). Hence, by (17) and (12),

$$N - h = N\left(1 - \frac{h}{N}\right) \to \infty \quad \text{as} \quad n \to \infty. \tag{18}$$

By (3), (2), (8) and (11),

$$h > \frac{n \log n - \frac{1}{3}ne - \frac{1}{2}\log n}{n \log 4 - \log 2e}$$

so that

$$h \to \infty \quad \text{as} \quad n \to \infty. \tag{19}$$

By (3), (10) and (2)

$$lV < \frac{n \log 16e(n+1)}{a4^n} = o(1)$$

so that

$$hV < (l+1)V = o(1). \tag{20}$$

By (10), (3) and (8)

$$\frac{N^* V}{h} < \frac{2ke(\log n - \log 2ke)}{n \log n - \frac{1}{3}ne - \frac{1}{2}\log n}.$$

The right-hand side of this inequality, treated as a function of k with n fixed, is maximum when $k = n/(2e^2)$. Hence

$$\frac{N^* V}{h} < n/\{e(n\log n - \tfrac{1}{3}ne - \tfrac{1}{2}\log n)\} = o(1)$$

so that, using (10) and (19),

$$\frac{(N+1)V}{h+1} < \frac{(N^*+2)V}{h} = \frac{N^*V}{h} + \frac{2V}{h} = o(1), \tag{21}$$

and, similarly,

$$\frac{NV}{h} = o(1). \tag{22}$$

Also, since $1-V > \frac{1}{2}$ by (10), we have by (21),

$$\frac{(N-h)V}{(h+1)(1-V)} < \frac{2(N+1)V}{h+1} < 1 \tag{23}$$

for n sufficiently large.

5. In the sum in (6) the ratio of the $(t+1)$st term to the tth term is

$$\frac{(N-h-t)V}{(h+t+1)(1-V)} \leqslant \frac{(N-h)V}{(h+1)(1-V)} < 1$$

by (23). Hence

$$\Delta = \mathscr{M}\big(\delta(F_h)\big) \leqslant \frac{N!}{h!(N-h)!} V^h (1-V)^{N-h} \sum_{t=0}^{\infty} \left\{\frac{(N-h)V}{(h+1)(1-V)}\right\}^t$$

$$= \frac{N!}{h!(N-h)!} V^h(1-V)^{N-h} \left\{\frac{(h+1)(1-V)}{(h+1)-(N+1)V}\right\}.$$

Using Stirling's formula, which we may by (12), (18) and (19)

$$\log\Delta \leqslant (N-h)\log\left(1+\frac{h}{N-h}\right) - h\log\frac{h}{NV} + (N-h)\log(1-V) -$$

$$-\frac{1}{2}\log\left(1-\frac{h}{N}\right) - \log\left(1-\frac{(N+1)V}{h+1}\right) - \frac{1}{2}\log h - \frac{1}{2}\log 2\pi + o(1).$$

Hence, by (20), (17) and (21),

$$\log\Delta < F(h,N) - \tfrac{1}{2}\log 2\pi + o(1) \tag{24}$$

where

$$F(h,N) = h - h\log h + h\log NV - NV - \tfrac{1}{2}\log h. \tag{25}$$

Now,

$$\frac{\partial F}{\partial h} = \log\frac{NV}{h} - \frac{1}{2h} < 0$$

by (22) and (19). Hence

$$F(h,N) \leqslant F(l,N).$$

Also

$$\frac{\partial F(l,N)}{\partial N} = \frac{l}{N} - V$$

so that the error in replacing N by N^* in $F(l,N)$ is at most $\dfrac{l}{N^*} + V = o(1)$ by (16) and (10). Hence, from (24), (25) and (10),

$$\log\Delta \leqslant l - l\log l + l\log 2ke - 2ke - \tfrac{1}{2}\log l - \tfrac{1}{2}\log 2\pi + o(1).$$

Hence, substituting for $n\log 4(n+1) - \log\delta$ from (3),

$$\begin{aligned} \log\Delta - \log\delta + n\log 4(n+1) &\leqslant l(1-\log l+\log n) + \tfrac{1}{2}(\log n - \log l - \log 2\pi - 2n) + o(1) \\ &< l\left(1+\frac{1}{2n} - \log l + \log n\right) + \frac{1}{2}(\log n - \log l - 2n - 1) = g(l) \end{aligned} \tag{26}$$

for n sufficiently large. Now

$$\frac{dg}{dl} = \frac{1}{2n} - \log l + \log n - \frac{1}{2l},$$

$$\frac{d^2g}{dl^2} = -\frac{1}{l} + \frac{1}{2l^2} < 0$$

by (19). When $l = n$, $dg/dl = 0$, so that $g(l) \leqslant g(n)$, and, by (26),

$$\log\Delta - \log\delta + n\log 4(n+1) < g(n) = 0.$$

Hence

$$\mathscr{M}\big(\delta(F_h)\big) < \delta\eta^n V \quad \text{where} \quad \eta = 1/(n+1). \tag{27}$$

With this choice of η we have, from (7), (10) and (1),

$$\log\mathscr{M}\big(\delta(E_0)\big) < -NV\left(1-\frac{1}{n+1}\right)^n < -\frac{NV}{e}$$

$$< -\frac{N^*V}{e} = -2k = \log(1-\delta)^2.$$

Hence

$$\mathscr{M}\big(\delta(E_0)\big) < (1-\delta)^2, \tag{28}$$

and, from (27) and (28)

$$(1-\delta)\mathscr{M}\big(\delta(F_h)\big)+\eta^n V\mathscr{M}\big(\delta(E_0)\big) < \{\delta(1-\delta)+(1-\delta)^2\}\eta^n V = (1-\delta)\eta^n V.$$

Hence we can choose points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ so that

$$(1-\delta)\,\delta(F_h)+\eta^n V\delta(E_0) < (1-\delta)\eta^n V.$$

Thus, with this choice of $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$

(29) $$\delta(F_h) < \eta^n V$$

and

(30) $$\delta(E_0) < 1-\delta.$$

6. We prove that the system of sets

(31) $$(1-\eta)K+\boldsymbol{x}_i+\boldsymbol{g} \quad (1 \leqslant i \leqslant N,\ \boldsymbol{g} \in \Lambda),$$

where $\eta = 1/(n+1)$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ are chosen as in § 5, has the properties stated in § 1.

Since, by (30), $\delta(E_0) < 1-\delta$, it follows that the proportion of space belonging to at least one set of the system is at least δ. The density of the system (31) is $NV(1-\eta)^n \sim 2k$ by (10). Also $NV(1-\eta)^n > 2k$.

Suppose that a point $\boldsymbol{x}$ of space is covered h or more times by sets of the system (31). Then each point of the set

$$\eta K+\boldsymbol{x}$$

is covered at least h times by sets of the system

$$K+\boldsymbol{x}_i+\boldsymbol{g} \quad (1 \leqslant i \leqslant N,\ \boldsymbol{g} \in \Lambda).$$

Hence F_h contains the union

$$\bigcup_{\boldsymbol{g}\in\Lambda}\{\eta K+\boldsymbol{x}+\boldsymbol{g}\}.$$

No two distinct sets of this union have any common point and the density of the union is $\eta^n V$. Hence $\delta(F_h) \geqslant \eta^n V$ which contradicts (29). This completes the proof that the system (31) has the required properties.

7. We illustrate the results stated in § 1. If $\delta < 1-\exp(-1/8e^2)$ it is easily proved that $h-1 < n$. If β is a constant and $\delta = n^{-\beta}$ then $h \sim n/(1+\beta)$, and if $\delta = \beta^{-n}$ then $h \sim (\log n)/(\log\beta)$.

It follows from (2) and (3) that

(32) $$2k > (n/e)(16ne)^{-n/(h-1)}.$$

By a result of Few [2] there are h-fold packings of equal spheres with density at least

(33) $$\delta_1\left(\frac{2h}{h+1}\right)^{n/2}$$

where δ_1 is the density of the closest packing of equal spheres. Since it is only known that $\delta_1 > Cn2^{-n}$, where C is a constant, the result (33) only ensures that there is an h-fold packing whose density δ_h satisfies

(34) $$\delta_h > Cn\left\{\frac{h}{2(h+1)}\right\}^{n/2}.$$

The density of the $(h-1)$-fold packing (31) is at least $2k$, so that there are $(h-1)$-fold packings whose density is greater than the right-hand side of (32). For large n this lower bound is better than that given by (34), with h replaced by $h-1$ provided

$$h-1 > \frac{2\log 16ne}{\log 2}.$$

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