set has volume $\gg \eta^2$. Now $(q'_{11}, q'_{12}, q'_{13}, q'_{21}, q'_{22}, q'_{23})$ is related to $(\boldsymbol{u}_1, \boldsymbol{u}_2) = (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23})$ be the linear transformation (40) of determinant $(r_{11}r_{22}-r_{12}r_{21})^3 \neq 0$. Hence (41) and (42) with $N = 1$ together with (39) and (40) define a bounded set for $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ in 6-dimensional space of volume $\gg \eta^2$. For arbitrary N we obtain the same set but blown up by the factor N. Hence by Lemma 6 there are $\gg \eta^2 N^6$ pairs of points $\boldsymbol{u}_1, \boldsymbol{u}_2$ which are part of a basis such that (41) and (42) are satisfied. There still are $\gg \eta^2 N^6$ such pairs $\boldsymbol{u}_1, \boldsymbol{u}_2$ all of whose components are different from zero.

It remains to be shown that for every such $\boldsymbol{u}_1, \boldsymbol{u}_2$ one can find a third basis vector $\boldsymbol{u}_3$ such that (38) holds. There certainly will be such a vector $\boldsymbol{u}_3$ of the type $\boldsymbol{u}_3 = \lambda_1\boldsymbol{u}_1 + \lambda_2\boldsymbol{u}_2 + \boldsymbol{u}_0$, where $|\lambda_j| \leqslant \frac{1}{2}$ $(j = 1, 2)$ and where $\boldsymbol{u}_0$ is the point with $\Delta(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_0) = 1$ which is orthogonal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. It is easy to see that the coordinates of $\boldsymbol{u}_0$ have absolute values at most 1, and hence

$$|u_{0i}| \leqslant \tfrac{1}{2}|u_{1i}| + \tfrac{1}{2}|u_{i2}| + 1 \leqslant |u_{1i}| + |u_{2i}| \qquad (i = 1, 2, 3),$$

since we made sure that $u_{1i} \neq 0$, $u_{2i} \neq 0$. Thus our $\boldsymbol{u}_3$ does satisfy (38), and we have $z(N) \geqslant z'(N) \gg \eta^2 N^6$. This proves (26) and hence the theorem.

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UNIVERSITY OF COLORADO
Boulder, Colorado

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Bounds for solutions of diagonal inequalities

by

JANE PITMAN (Adelaide, South Australia)

In memory of H. Davenport

1. Introduction. In 1958 the following theorem was proved by Birch and Davenport [1]:

If $\lambda_1, \lambda_2, \ldots, \lambda_5$ are real numbers, not all of the same sign, such that $|\lambda_i| \geqslant 1$ for all i, then for any $\theta > 0$ the Diophantine inequality

$$|\lambda_1 x_1^2 + \ldots + \lambda_5 x_5^2| < 1$$

has a solution in integers $x_1, \ldots, x_5$, not all zero, such that

$$|\lambda_1 x_1^2| + \ldots + |\lambda_5 x_5^2| < K_\theta |\lambda_1 \lambda_2 \ldots \lambda_5|^{1+\theta}.$$

A corresponding theorem on solutions of the diagonal cubic inequality

$$|\lambda_1 x_1^3 + \ldots + \lambda_9 x_9^3| < 1$$

such that

$$|\lambda_1 x_1^3| + \ldots + |\lambda_9 x_9^3| < K'_\theta |\lambda_1 \ldots \lambda_9|^{(3/2)+\theta}$$

was proved in Pitman and Ridout [7]. In this paper I obtain a similar theorem for the diagonal inequality

$$|\lambda_1 x_1^k + \ldots + \lambda_n x_n^k| < 1, \tag{1}$$

where k is an integer, $k \geqslant 4$, and $\lambda_1, \ldots, \lambda_n$ are not all of the same sign if k is even. By a *solution* of a Diophantine equation or inequality I shall always mean a solution in integers $x_1, \ldots, x_n$, not all zero.

For the case when the λ_i/λ_j are not all rational, Davenport and Heilbronn [4] found that the condition $n \geqslant 2^k+1$ is sufficient for the existence of infinitely many solutions of (1); later Davenport and Roth [5] showed that $n > ck \log k$ is sufficient if $k \geqslant 12$, and Danicic [2] showed that $n \geqslant 14$ is sufficient if $k = 4$.

In order to find bounds for solutions of (1) by analytic methods similar to those of [1] and [7], we must first deal independently with the case when the λ_i/λ_j are all rational, that is, with the diagonal Diophantine equation

$$\mu_1 x_1^k + \ldots + \mu_n x_n^k = 0, \tag{2}$$

where $\mu_1, \ldots, \mu_n$ are non-zero integers; I do this in another paper [6], which I shall call DE (diagonal equations). In the present paper I use the main result of DE (Lemma 1 below) to prove the following theorem (of which Lemma 1 is a special case).

THEOREM. *Let k be an integer, $k \geq 4$, and let n be the integer defined by*

$$\begin{cases} n = 2^k+1 & \text{if} \quad 4 \leqslant k \leqslant 11, \\ n \geqslant 2k^2(2\log k + \log\log k + 3) + 1 > n-1 & \text{if} \quad k \geqslant 12. \end{cases} \tag{3}$$

Then for any $\theta > 0$ there exists a constant K_θ, depending only on θ and k, with the following property. If $\lambda_1, \ldots, \lambda_n$ are real numbers which satisfy $|\lambda_i| \geqslant 1$ for all i and which are not all of the same sign if k is even, then the inequality (1) *has a solution in non-zero integers such that*

$$|\lambda_1 x_1^k| + \ldots + |\lambda_n x_n^k| < K_\theta |\lambda_1 \ldots \lambda_n|^{k\psi+\theta},$$

where

$$\begin{cases} \psi = \tfrac{1}{2} & \text{if} \quad 4 \leqslant k \leqslant 11, \\ \psi = 1 & \text{if} \quad k \geqslant 12. \end{cases} \tag{4}$$

The method of proof is similar to that of Theorem 2 of [7], except that for $k \geqslant 12$ we use Vinogradov's estimates and that we avoid certain complications because the cases $k = 2$, $k = 3$ are excluded. In DE I discuss the possibility of reducing the number of variables by modifying this method.

This paper depends heavily on ideas developed by Professor Davenport and owes much to his advice. I am deeply grateful for all his generous help and encouragement.

2. Notation and preliminaries. Let k be an integer, $k \geqslant 4$; let n be the integer defined by (3); and let θ be given such that $0 < \theta < 1$ (this involves no loss of generality). Let $\lambda_1, \ldots, \lambda_n$ be n real numbers which satisfy

$$|\lambda_i| \geqslant 1 \quad (i = 1, \ldots, n)$$

and which are not all of the same sign if k is even.

We write

$$\nu = \frac{1}{k}, \quad \Pi = \prod_{i=1}^{n} |\lambda_i|, \quad \Lambda = \max_i |\lambda_i|,$$

and take P to be a large positive integer such that

$$P \geqslant |\lambda_i|^\nu \quad (i = 1, \ldots, n). \tag{5}$$

We define

$$S_i(\alpha) = \sum_{x_i} e(\lambda_i \alpha x_i^k),$$

where x_i runs through all integral values in the range

$$P \leqslant |\lambda_i|^\nu x_i \leqslant 3P, \tag{6}$$

and we write

$$V(\alpha) = \prod_{i=1}^{n} S_i(\alpha). \tag{7}$$

We also define

$$I(\beta) = \sum_m \nu m^{-1+\nu} e(\beta m), \tag{8}$$

where m runs through all integral values in the closed interval

$$[P^k, (3P)^k]. \tag{9}$$

We use the notational conventions of DE, § 2 (which correspond exactly to those of [7], § 2); in particular, ε denotes an arbitrarily small positive number, and the constants implied by $O, \ll, \gg$ are always independent of P and of the λ_i. In addition to some standard general lemmas on exponential sums which are collected together in DE, § 2, we shall use the following preliminary results.

LEMMA 1. *Let k and n be as above. Then for any $\delta > 0$ there exists a constant C_δ, depending only on δ and k, with the following property. If $\mu_1, \ldots, \mu_n$ are non-zero integers which are not all of the same sign if k is even, then the equation* (2) *has a solution in non-zero integers such that*

$$|\mu_1 x_1^k| + \ldots + |\mu_n x_n^k| < C_\delta |\mu_1 \ldots \mu_n|^{k\psi+\delta},$$

where ψ is defined by (4).

Proof. See DE, Theorem 1.

LEMMA 2. *For any positive integer r, there exists a real valued function of a real variable, f, such that*

$$|f(\alpha)| < C(r)\min(1, \alpha^{-r-1}) \tag{10}$$

for $\alpha > 0$ and the following conditions are satisfied. If

$$g(\eta) = \mathscr{R}\int_0^\infty e(\eta\alpha) f(\alpha)\, d\alpha,$$

then

$$\begin{aligned} 0 \leqslant g(\eta) &\leqslant 1 && \text{for all real } \eta, \\ g(\eta) &= 0 && \text{for } |\eta| \geqslant 1, \\ g(\eta) &= 1 && \text{for } |\eta| \leqslant \tfrac{1}{3}. \end{aligned}$$

Proof. See Davenport [3], Lemma 1.

Let r be a positive integer (whose value will be decided later) and let f be the corresponding function given by the lemma. Let $\mathcal{N}(P)$ denote the number of solutions of (1) such that (6) holds for all i. Let

(11) $$\mathcal{J}(P) = \mathcal{R}\int_J V(a)f(a)\,da,$$

where $V(a)$ is defined by (7) and $J = [0, \infty)$. It follows from Lemma 2 that

(12) $$\mathcal{N}(P) \geqslant \mathcal{J}(P).$$

We therefore set out to show that $\mathcal{J}(P) > 0$ whenever P is somewhat larger than Π^{ν}, and when this approach fails we shall fall back on Lemma 1.

Since the main term in our estimate of $\mathcal{J}(P)$ will be $\gg \Pi^{-\nu}P^{n-k}$, an error will be "permissible" if it is substantially smaller than $\Pi^{-\nu}P^{n-k}$.

3. Dissection of the interval J. In order to estimate the $S_i(a)$ (and hence $\mathcal{J}(P)$), we must consider rational approximations to the $\lambda_i a$. Our estimates involve a fixed number δ such that $0<\delta<1$ whose value will be decided later.

For each $a \in [0, P^{\delta}]$, by (5) and Dirichlet's theorem on Diophantine approximations, there exist rationals a_i/q_i such that

(13) $$\begin{cases}(a_i, q_i) = 1, \quad \lambda_i a = (a_i/q_i)+\beta_i,\\ 0<q_i\leqslant (|\lambda_i|^{-\nu}P)^{k-1+\delta}, \quad |\beta_i| \leqslant q_i^{-1}(|\lambda_i|^{-\nu}P)^{-k+1-\delta}.\end{cases}$$

We shall distinguish between the cases

(14) $$q_i^{\nu} \leqslant |\lambda_i|^{-1/(n-1)}P^{\sigma},$$

(15) $$q_i^{\nu} > |\lambda_i|^{-1/(n-1)}P^{\sigma},$$

where

(16) $$\sigma = \begin{cases}\dfrac{1}{2^{k-1}} & \text{if} \quad 4\leqslant k\leqslant 11,\\[2ex] \dfrac{1}{2k^2(2\log k+\log\log k+3)} & \text{if} \quad k\geqslant 12\end{cases}$$

(i.e., σ is as in DE, Lemma 4). Fortunately the bound in (14) and (15), which is the smallest that will work in Lemma 8 below, is small enough to give the following lemma.

LEMMA 3. *Let $a\in[0, P^{\delta}]$.*

(i) *If $P > 2|\lambda_i|^{1/4}$ then there is at most one approximation a_i/q_i to $\lambda_i a$ such that* (13) *and* (14) *hold.*

(ii) *Suppose that $P > 4|\lambda_i\lambda_j|^{1/2}$, $i\neq j$, and the approximations a_i/q_i, a_j/q_j satisfy* (13) *and* (14); *then*

(17) $$\left|\frac{a_i}{\lambda_i q_i} - \frac{a_j}{\lambda_j q_j}\right| < \frac{1}{4|\lambda_i\lambda_j|(q_iq_j)^2P^{\delta}}.$$

If, further, $a_i\neq 0$, then $a_j \neq 0$ and a_jq_i/a_iq_j is a convergent in the continued fraction expansion of λ_j/λ_i.

Proof. (i) This is similar to Lemma 8 (i) of [7]; see also DE, Lemma 9.

(ii) By (13) and (14), we have

$$\left|\frac{a_i}{\lambda_i q_i} - \frac{a_j}{\lambda_j q_j}\right| = \left|\frac{\beta_i}{\lambda_i} - \frac{\beta_j}{\lambda_j}\right| \leqslant \frac{2P^{-k+1+\sigma k}}{q_iq_jP^{\delta}}.$$

By (14) again, it follows that (17) holds if

$$8|\lambda_i\lambda_j|^{1-k/(n-1)} < P^{k-1-3\sigma k},$$

which is easily verified under our assumptions. (It is false for $k = 2, 3$.) Now suppose that $a_i\neq 0$. Since a_i is integral, (17) implies that $a_j \neq 0$. We know that a_jq_i/a_iq_j is a convergent to λ_j/λ_i if

$$\left|\frac{\lambda_j}{\lambda_i} - \frac{a_jq_i}{a_iq_j}\right| < \frac{1}{2|a_iq_j|^2};$$

and this inequality follows from (17), since, by (13),

$$0<|a_i|<2|\lambda_i|q_iP^{\delta}.$$

In order to use Lemma 3, we assume from now on that

(18) $$P > 4\,\Pi^{1/2}$$

(without loss of generality, since $\Pi^{\nu}\geqslant \Pi^{1/2}$). We dissect the interval $J = [0, \infty)$ as follows. We write

(19) $$Q = \Lambda^{-\nu(1-\delta)}P^{-k+1-\delta},$$

and define G, K as the intervals

$$G = [0, Q], \quad K = [P^{\delta}, \infty);$$

we define H as the set of all a in (Q, P^{δ}) such that for each i there is a rational a_i/q_i which satisfies (13) and (14). The main term in our estimate of the integral $\mathcal{J}(P)$ defined by (11) comes from G, which is simply the set of a for which $a_i/q_i = 0/1$ satisfies (13) and (14) for all i. The essential difficulty arises from the contribution to $\mathcal{J}(P)$ from H, and we therefore deal with H first.

4. Contribution from H. Let $a\in H$ and for each i let a_i/q_i be an approximation to $\lambda_i a$ which satisfies (13) and (14). By (18) and Lemma 3, the a_i/q_i are unique, $a_i\neq 0$ for all i, and a_iq_1/a_1q_i is a convergent to

λ_i/λ_1 for $i = 2, \ldots, n$. For $i = 2, \ldots, n$, we define A_i, B_i to be the integers such that $(A_i, B_i) = 1$, $B_i > 0$, and $A_i/B_i = a_i q_1/a_1 q_i$, that is,

$$\frac{a_i}{q_i} = \frac{a_1}{q_1} \cdot \frac{A_i}{B_i}. \tag{20}$$

The A_i, B_i are bounded above by fixed powers of P, since this is true of a_1, a_i, q_1, q_i. Hence the number of possible sets of convergents $A_2/B_2, \ldots, A_n/B_n$ is $O(P^\varepsilon)$.

We define

$$a = \frac{a_1}{(a_1, B_2 B_3 \ldots B_n)}, \quad q = \frac{q_1}{(q_1, A_2 A_3 \ldots A_n)},$$

so that, by (20), a_i is divisible by a and q_i is divisible by q, for all i. We can therefore write

$$a_i = a a_i', \quad q_i = q q_i' \quad (i = 1, \ldots, n). \tag{21}$$

Here, $a_1' = (a_1, B_2 \ldots B_n)$, $q_1' = (q_1, A_2 \ldots A_n)$, so that a_1', q_1' are both divisors of $A_2 \ldots A_n B_2 \ldots B_n$, which is bounded by a fixed power of P. Hence the number of possible pairs a_1', q_1' is $O(P^\varepsilon)$.

Now a_1, q_1 are uniquely determined by a, q, a_1', q_1'; and, by (20), $a_2, \ldots, a_n, q_2, \ldots, q_n$ are uniquely determined by a_1, q_1 and the set of convergents $A_2/B_2, \ldots, A_n/B_n$. Hence, by the concluding remarks of the last two paragraphs, the number of sets of approximations $a_1/q_1, \ldots, a_n/q_n$ which can correspond to a given pair a, q in the manner described above is $O(P^\varepsilon)$.

We shall find that the error term contributed by H is permissible provided that $a_1' \ldots a_n'(q_1' \ldots q_n')^{k-1}$ is reasonably large for all α in H. Therefore we start by applying Lemma 1 to do what we can with the case when this product is small.

LEMMA 4. *Suppose that $a > 0$ and that for all i*

$$\lambda_i \alpha = \frac{a_i}{q_i} + \beta_i, \quad |\beta_i| < \tfrac{1}{2} q_i^{-1}, \quad a_i = a a_i', \quad q_i = q q_i',$$

where a_i, q_i, a, a_i', etc., are integers, $a_i \neq 0$, $q_i > 0$, $a > 0$, $q > 0$. Let

$$B = \max_i |\beta_i/\lambda_i|,$$

and let C_δ be as in Lemma 1. Then (1) has a solution in non-zero integers such that

$$\sum_i |\lambda_i x_i^k| \leqslant (3P)^k, \tag{22}$$

provided that

$$C_\delta |a_1' \ldots a_n'(q_1' \ldots q_n')^{k-1}|^{k\psi+\delta} < \tfrac{1}{2} a\alpha^{-1} q \min\left((3P)^k, \alpha B^{-1}\right). \tag{23}$$

Proof. We have

$$\alpha\Big(\sum_i \lambda_i x_i^k\Big) = \frac{a}{q}\Big(\sum_i \frac{a_i'}{q_i'} x_i^k\Big) + \sum_i \beta_i x_i^k.$$

Writing $x_i = q_i' y_i$, we obtain

$$\alpha\Big(\sum_i \lambda_i x_i^k\Big) = \frac{a}{q}\Big(\sum_i a_i'(q_i')^{k-1} y_i^k\Big) + \sum_i \beta_i (q_i')^k y_i^k.$$

By Lemma 1, there exist non-zero integers $y_1, \ldots, y_n$ such that

$$\sum_i a_i'(q_i')^{k-1} y_i^k = 0$$

and

$$\sum_i |a_i'(q_i')^{k-1} y_i^k| < C_\delta |a_1' \ldots a_n'(q_1' \ldots q_n')^{k-1}|^{k\psi+\delta}.$$

Suppose now that (23) holds. Then

$$2\alpha^{-1} a q^{-1} \sum_i |a_i'(q_i')^{k-1} y_i^k| < \min\left((3P)^k, \alpha B^{-1}\right). \tag{24}$$

It is easily deduced from our hypotheses that $\beta_i q_i' = \beta_i q_i q^{-1}$ satisfies the inequalities

$$|\beta_i q_i'| \leqslant 2\alpha^{-1} B a q^{-1} |a_i'|, \quad |\beta_i q_i'| \leqslant a q^{-1} |a_i'|.$$

Hence for the non-zero integers $x_1, \ldots, x_n$ corresponding to $y_1, \ldots, y_n$ we obtain

$$\Big|\sum_i \lambda_i x_i^k\Big| = \alpha^{-1}\Big|\sum_i \beta_i (q_i')^k y_i^k\Big| \leqslant 2\alpha^{-2} B a q^{-1} \sum_i |a_i'(q_i')^{k-1} y_i^k|,$$

$$\sum_i |\lambda_i x_i^k| \leqslant 2\alpha^{-1} a q^{-1} \sum_i |a_i'(q_i')^{k-1} y_i^k|.$$

It now follows from (24) that $x_1, \ldots, x_n$ satisfy (1) and (22).

We now consider the contribution from H to $\mathscr{J}(P)$ in the cases which are not covered by the above lemma. We resume the notation introduced at the beginning of § 4.

LEMMA 5. *Suppose that $\delta < 1/n$, and let C_δ be as in Lemma 1. For $\alpha \in H$ let*

$$B = \max_i |\beta_i/\lambda_i| = |\beta_j/\lambda_j|,$$

say, where $j = j(\alpha)$. Suppose that for all α in H

$$C_\delta |a_1' \ldots a_n'(q_1' \ldots q_n')^{k-1}|^{k\psi+\delta} \geqslant \tfrac{1}{2} a\alpha^{-1} q \min\left((3P)^k, \alpha B^{-1}\right). \tag{25}$$

Then

$$\int_H |V(\alpha) f(\alpha)|\, d\alpha \ll \Pi^{-\nu} P^{n-k} \cdot \Pi^{\nu^2} P^{2s-\mu(\delta)},$$

where

(26) $$\mu(\delta) = \frac{1-n\delta}{k(k\psi+\delta)}.$$

Proof. Suppose that $\alpha \in H$. Since $|f(\alpha)| \ll 1$, it follows from (14) and Lemma 3 of DE that

(27) $$|V(\alpha)f(\alpha)| \ll \Pi^{-\nu}(q_1 \dots q_n)^{-\nu} P^n \prod_{i=1}^{n} \min(1, P^{-k}|\lambda_i/\beta_i|).$$

We now obtain an upper bound for the right-hand side of (27). By (13) and (21) we have

$$a_1' \dots a_n'(q_1' \dots q_n')^{k-1} \ll \Pi a^{-n}(q_1 \dots q_n)\, a^n \cdot \{q^{-n}(q_1 \dots q_n)\}^{k-1}.$$

Also $q \geqslant 1$, $a < P^{\delta}$, $P^{\delta} \geqslant 1$, and

$$P^{\delta(1-nk\psi-n\delta)} \geqslant P^{-nk\delta}$$

(as $\psi \leqslant 1$ and $\delta < 1/n$). Hence we deduce from (25) that

$$(\Pi a^{-n} q^{-n(k-1)} q_1^k \dots q_n^k)^{k\psi+\delta} \gg a^{-1} P^{k-nk\delta} \min(1, P^{-k}B^{-1}).$$

Since $\nu^2/(k\psi+\delta) \leqslant 2\nu^3$ and $k-1 > 1$, it then follows from (27) that

(28) $$|V(\alpha)f(\alpha)| \ll \Pi^{-\nu+\nu^2} a^{2\nu^3-n\nu^2} q^{-n\nu^2} P^{n-\mu(\delta)} m(\alpha),$$

where

$$m(\alpha) = \prod_{i \neq j} \min(1, P^{-k}|\lambda_i/\beta_i|) \leqslant \sum_i \min(1, P^{-k}|\lambda_i/\beta_i|).$$

Now for all α corresponding to a fixed set of approximations a_i/q_i we have $|\beta_i/\lambda_i| = |\alpha - (a_i/\lambda_i q_i)| < 1$. Also

$$\int_{-1}^{1} \min(1, P^{-k}|\beta|^{-1})\, d\beta \ll P^{-k}\log P \ll P^{-k+\varepsilon}.$$

Therefore, by integrating (28) with respect to α, we see that the contribution to $\mathscr{J}(P)$ from all α in H corresponding to a particular set of approximations is

$$\ll \Pi^{-\nu+\nu^2} a^{2\nu^3-n\nu^2} q^{-n\nu^2} P^{n-\mu(\delta)-k+\varepsilon}.$$

By (3), we have $n > n-2\nu > k^2$, since $k \geqslant 4$, and therefore the series $\sum a^{2\nu^3-n\nu^2}$, $\sum q^{-n\nu^2}$ are absolutely convergent. Hence, summing over the $O(P^{\varepsilon})$ sets of approximations corresponding to a given pair a, q, and then summing over all a, q we obtain the required result.

5. The main term. We now give the main term of our estimate of $\mathscr{J}(P)$ that is, the contribution from G.

LEMMA 6. *We have*

$$\mathscr{R}\int_G V(\alpha)f(\alpha)\,d\alpha = c\Pi^{-\nu}P^{n-k} + O(P^{n-1-k}) + O\{\Pi^{-\nu}P^{n-k}(\Lambda^{-\nu}P)^{-(n-1)(1-\delta)}\},$$

where $c = c(P) \geqslant$ a positive constant depending only on n and k.

Proof. Suppose $\alpha \in G$. Then for all i the pair $a_i = 0$, $q_i = 1$ satisfies (13), and also, by (5), satisfies

$$q_i \leqslant (|\lambda_i|^{-\nu}P)^{1-\delta}.$$

Hence, by Lemma 3 of DE, we have for all i

$$S_i(\alpha) = |\lambda_i|^{-\nu} I(\pm\alpha) + O(1),$$

where $\pm$ is the sign of λ_i and each term of the right-hand side is

$$\ll |\lambda_i|^{-\nu} P \min(1, P^{-k}\alpha^{-1}).$$

Therefore we have

$$V(\alpha) = \Pi^{-\nu}\prod_{i=1}^{n} I(\pm\alpha) + E,$$

where

$$E \ll \{P\min(1, P^{-k}\alpha^{-1})\}^{n-1} \ll P^{n-1}\min(1, P^{-2k}\alpha^{-2}).$$

Since $|f(\alpha)| \ll 1$ and $G \subset [0, 1]$ and

$$\int_0^1 \min(1, P^{-2k}\alpha^{-2})\,d\alpha \ll P^{-k},$$

it follows that

(29) $$\mathscr{R}\int_G V(\alpha)f(\alpha)\,d\alpha = \Pi^{-\nu}\mathscr{R}\int_G \prod_{i=1}^{n} I(\pm\alpha)\cdot f(\alpha)\,d\alpha + O(P^{n-1-k}).$$

The error in (29) caused by replacing G on the right-hand side by $J = [0, \infty)$ is

(30) $$\ll \int_Q^{\infty} \Pi^{-\nu}P^n\{\min(1, P^{-k}\alpha^{-1})\}^n\,d\alpha,$$

where Q is defined by (19). Now

$$\int_Q^{\infty} \alpha^{-n}\,d\alpha \ll Q^{-n+1} = \Lambda^{\nu(1-\delta)(n-1)}P^{(k-1+\delta)(n-1)}.$$

Thus by (30) the error is

$$\ll \Pi^{-\nu}P^{n-k}\cdot(\Lambda^{-\nu}P)^{-(n-1)(1-\delta)},$$

and so we have

(31) $$\mathscr{R}\int_G V(\alpha)f(\alpha)\,d\alpha = \Pi^{-\nu}\mathscr{R}\int_0^{\infty}\prod_{i=1}^{n} I(\pm\alpha)\cdot f(\alpha)\,d\alpha + O(P^{n-1-k}) + O\{\Pi^{-\nu}P^{n-k}(\Lambda^{-\nu}P)^{-(n-1)(1-\delta)}\}.$$

It is easily deduced from Lemma 2 and the definition (8) of $I(\beta)$ that

$$(32)\qquad \mathscr{R}\int_0^\infty \prod_{i=1}^n I(\pm\alpha)\cdot f(\alpha)\,d\alpha = \nu^n \sum_{m_1,\ldots,m_n} (m_1\ldots m_n)^{-1+\nu} = z,$$

say, where the summation is over all integral $m_1, \ldots, m_n$ such that the m_i are in the interval (9) and

$$\pm m_1 \pm m_2 \pm \ldots \pm m_n = 0.$$

Since we may assume without loss of generality that $\lambda_1 > 0$ and $\lambda_2 < 0$, we may take this equation to be of the form

$$m_1 - m_2 \pm m_3 \pm \ldots \pm m_n = 0.$$

By Lemma 6 of DE, we now have $z \gg P^{n-k}$, and the lemma follows from (31) and (32).

6. Contributions from K and $J-G-H-K$. First we consider the "tail" of the integral $\mathscr{J}(P)$, that is, the contribution from K.

LEMMA 7. *We have*

$$\int_K |V(\alpha)f(\alpha)|\,d\alpha \ll \Pi^{-\nu}P^{n-r\delta}.$$

Proof. This follows from (10) in Lemma 2 and the trivial inequality $|V(\alpha)| \leqslant \Pi^{-\nu}P^n$.

Finally, we estimate the contribution from the remaining subset $J-G-H-K$, which consists essentially of the "minor arcs". The limitations on this estimate determine both the number of variables and the bound in the theorem; since the argument is exactly similar to that in Lemma 8 of DE, we omit some of the details.

LEMMA 8. *We have*

$$\int_{J-G-H-K} |V(\alpha)f(\alpha)|\,d\alpha \ll \Pi^{-\nu}P^{n-k}\cdot \Pi^{1/(n-1)}P^{-(\sigma-2\delta-\varepsilon)}.$$

Proof. Let $\alpha \epsilon J-G-H-K$; note that $\alpha < P^\delta$. Since (15) holds for some i and $\nu\sigma < 1/(n-1)$ (by (3)), it follows from (16) and Lemmas 3 and 4 of DE that for some i

$$(33)\qquad S_i(\alpha) \ll \max\{(|\lambda_i|^{-\nu}P)^{1-\sigma+\delta},\ |\lambda_i|^{-\nu}q_i^{-\nu}P\} \ll |\lambda_i|^{-\nu+1/(n-1)}P^{1-\sigma+\delta}.$$

We now use (10), Hölder's inequality and Lemma 5 of DE (with $X = P^\delta$), and deduce that the contribution to $\mathscr{J}(P)$ from all α in $[0, P^\delta]$ such that (33) holds for a particular i is

$$\ll |\lambda_i|^{-\nu+1/(n-1)}P^{1-\sigma+\delta}\Big\{\prod_{j\neq i}\big((|\lambda_j|^{-\nu}P)^{n-1-k+\varepsilon}P^\delta\big)\Big\}^{1/(n-1)}.$$

The required result follows immediately.

7. Completion of the proof. Suppose for the moment that (25) holds for all α in H and that $\delta < 1/n$. Then by Lemmas 5, 6, 7 and 8 we have

$$(34)\qquad \mathscr{J}(P) = c\Pi^{-\nu}P^{n-k} + \Pi^{-\nu}P^{n-k}E,$$

where $c \geqslant$ a positive constant which depends only on n and k, and

$$(35)\qquad E \ll \Pi^{\nu^2}P^{2\varepsilon-\mu(\delta)} + \Pi^{\nu}P^{-1} + (\Lambda^{-\nu}P)^{-(n-1)(1-\delta)} + P^{k-r\delta} + \Pi^{1/(n-1)}P^{-\sigma+2\delta+\varepsilon},$$

where $\mu(\delta)$ and σ are defined by (26) and (16).

We choose $\varepsilon > 0$, $\delta > 0$ and then a positive integer r in such a way that $\delta < 1/n$ and that the right-hand side of (35) is bounded by a constant multiple of

$$\{\Pi P^{-(1-\theta)/\psi}\}^{\nu^2}P^{-\nu^2\theta/2} + \{\Pi^\nu P^{-(1-\theta)}\}P^{-\theta} + \{\Pi^{1/(n-1)}P^{-(1-\theta)\sigma}\}P^{-\theta\sigma/2}.$$

(This is possible because $\mu(\delta)\uparrow \nu^2/\psi$ as $\delta\downarrow 0$ and $\Pi \geqslant \Lambda \geqslant 1$.) Each of the expressions $\{\ldots\}$ is at most 1 if $P^{1-\theta} \geqslant \Pi^\psi$, since $1-\theta > 0$, $\psi > \nu$ and $\psi \geqslant 1/\{(n-1)\sigma\}$. Hence there exists a constant $L_\theta \geqslant 4$ such that if $P^{1-\theta} > L_\theta \Pi^\psi$ and (34) holds, then $\mathscr{J}(P) > 0$. We now choose the positive integer P so that

$$L_\theta \Pi^\psi < P^{1-\theta} < 2L_\theta \Pi^\psi;$$

this implies that (5) and (18) also hold (as we have assumed throughout).

If (25) holds for all α in H, then, by the preceding discussion together with (12), we have $\mathscr{N}(P) \geqslant \mathscr{J}(P) > 0$, and therefore there is a solution of (1) in non-zero integers such that

$$\sum_i |\lambda_i x_i^k| \leqslant n(3P)^k.$$

On the other hand, if (25) fails for some α in H, then by Lemma 4 there is a solution of (1) in non-zero integers such that

$$\sum_i |\lambda_i x_i^k| \leqslant (3P)^k.$$

Hence in either case, by our choice of P, we have

$$\sum_i |\lambda_i x_i^k| < L'_\theta \Pi^{\psi k/(1-\theta)},$$

say, and since $\psi k/(1-\theta)\downarrow \psi k$ as $\theta\downarrow 0$, this completes the proof of the theorem.

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THE UNIVERSITY OF ADELAIDE
Adelaide, South Australia

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On Bombieri's estimate for exponential sums

by

J. H. H. Chalk and R. A. Smith (Toronto, Canada)

In memory of H. Davenpor

1. Introduction. In the course of an article [2] devoted mainly to the structure and interpretation of multiple exponential sums over finite fields, Bombieri included an estimate for the magnitude of certain special exponential sums "along a curve" (and, incidentally, generalized Weil's method [20] for similar sums "along a line"). For comparison purposes, it will be useful to have an abridged statement of this result (cf. [2], Theorem 6, p. 97). Thus, let $k = [q]$ denote the finite field of $q = p^a$ elements ($a \geqslant 1$) and characteristic p, σ denote the absolute trace from $[q^m]$ to $[p]$, $e(x)$ denote $\exp(2\pi ix/p)$, X a projective curve of degree d_1 defined over k and embedded in projective n-space $\boldsymbol{P}^n$ over k, X_m the set of points of X defined over $[q^m]$, $R(X_0, X_1, \ldots, X_n)$ a homogeneous rational function in $\boldsymbol{P}^n$ defined over k (d_2 being the degree of its numerator) and

(1) $$\mathscr{S}_m(R, X) = \sum_{x\in X_m}{}' e\big[\sigma\big(R(x)\big)\big],$$

where "′" indicates that the poles of R are omitted. Then

(2) $$|\mathscr{S}_m(R, X)| \leqslant (d_1^2 - 3d_1 + 2d_1d_2)q^{m/2} + d_1^2,$$

provided that

(A) for every homogeneous rational $h \in \bar{k}(X_0, \ldots, X_n)$, the function

(3) $$R - (h^p - h)$$

does not vanish identically on any absolutely irreducible component of X.

This condition (A), which restricts the choice of R not only in its behaviour over k but also over the algebraic closure $\bar{k}$ of k, is stated (without details) to hold if(1)

(B) $p > d_1 d_2$ and R is not constant on any such component of X.

(1) It seems likely that (2) continues to hold even if the restriction on p in (B) is ignored.