Fortunately many of his lectures were either published or mimeographed and will be a source of pleasure for many years to come.

His early death is a great loss to mathematics and to all who knew him. I count myself very fortunate to have known him for some 45 years. He is assured of a permanent place among those great mathematicians who have advanced the theory of numbers, called by Gauss, the Queen of Mathematics.

ACTA ARITHMETICA
XVIII (1971)

Some aspects of Davenport's work

by

L. J. MORDELL (Cambridge)

It is well known what an important part has been played by problems, even of the simplest character, in furthering research, discovery and the advancement of mathematics. Hilbert's famous address on problems is a classic illustration. The solution of a problem frequently requires new ideas and new methods. The generalization it suggests, its consideration from a different point of view or its rephrasing may lead to a new problem of far greater significance than the original one which may turn out to be only a very special case of a general theorem. Sometimes it seems almost incredible what striking and far-reaching fundamental developments have arisen in directions which seem very remote indeed from the problem from which they arose.

Problems are the life-blood of mathematics. Davenport wrote nearly two hundred papers and in them he studied a great variety of problems on number theory and related topics. These stimulated a great deal of research and often proved the starting point for much work by his colleagues and students. Two problems in particular led him and others to investigations of the greatest importance which have greatly enriched mathematics, opening up entirely new and unexpected fields of research. Even at the present time, their possibilities have not been exhausted.

The first problem was tackled in his very first paper published in 1931. It is really wonderful what this led to and it makes a fascinating story to follow its consequences and to discuss the influence it had in shaping some of the best mathematical research for many years. I propose to do this in some detail. The problem, proposed to him by Littlewood, was to estimate the sum

$$S = \sum_{x=0}^{p-1}\left(\frac{(x+a)(x+b)(x+c)}{p}\right) = \sum_{x=0}^{p-1}\left(\frac{f(x)}{p}\right), \tag{1}$$

say, where p is a large prime and the bracket denotes the Legendre quadratic character $\bmod p$. The case when $f(x)$ is a quadratic polynomial is really very simple and had been investigated by Jacobsthal as long ago as 1906. It is very surprising that the only results known for the cubic case were nearly trivial ones. Davenport found the estimate

(2) $$S = O(p^{3/4}),$$

where the constant involved in O is independent of $f(x)$. This was a really significant result. The problem was a most suggestive and interesting one, and I noticed that it could be considered from a different angle. Let N be the number of solutions of the congruence

(3) $$y^2 \equiv (x+a)(x+b)(x+c) \pmod{p}.$$

Clearly

(4) $$N = \sum_{x=0}^{p-1}\left(1+\left(\frac{(x+a)(x+b)(x+c)}{p}\right)\right) = p+S,$$

where the bracket denotes the Legendre character. One immediately thinks of the general problem of estimating the number N of solutions of a polynomial congruence

(5) $$f(x, y) \equiv 0 \pmod{p}$$

in the form

(6) $$N = p+O(p^{\lambda}),$$

where λ is a constant, and in particular, of the best possible value of λ. Isolated results had been known for many years. Gauss' work on cyclotomy had led him to some cases, e.g.,

$$f(x, y) = ax^3+by^3+c,$$

and Fermat's last theorem had led to the case

$$f(x, y) = ax^n+by^n+c.$$

The consideration of the congruence (5) seemed to attract little attention or proved too difficult. I made a start which enabled me to deal with some new instances of (5), by applying a method of discrete averaging. Let $f(x, y)$ involve linear parameters $a_1, a_2, \ldots, a_n$. Then it is well known and easy to see that N is given by, say,

(7) $$N_{a_1,a_2,\ldots,a_n} = N_a = \frac{1}{p}\sum e\left(\frac{2\pi i}{p}\, t_1 f(x_1, y_1)\right),$$

where the summation is extended over 0 to $p-1$ for t_1, x_1, y_1, since the sum in t_1 vanishes unless $f(x_1, y_1) \equiv 0$. Then

$$p(N_a - p) = \sum_{t_1=1}^{p-1}\sum_{x_1,y_1} e\left(\frac{2\pi i}{p}\, t_1 f(x_1, y_1)\right).$$

Hence

$$p^r(N_a-p)^r = \sum_{t_1,\ldots,t_r=1}^{p-1}\sum_{x,y} e\left(\frac{2\pi i}{p}\, E\right),$$

where $E = t_1 f(x_1, y_1)+\ldots+t_r f(x_r, y_r)$, and r is an arbitrary even number. Sum now for the parameters a from 0 to $p-1$. The sums in the a vanish unless their coefficients are congruent to zero, and this leads to a number of congruences for the variables x, y. The number M of their solutions can sometimes be found, and so we have a result of the form

$$p^r \sum_a (N_a-p)^r = p^n M.$$

This simplifies on noting the solutions when all the a are zero. If some of the N_a are equal, say for p^{μ} values of the parameters a, we have an estimate

(8) $$p^r(N_a-p)^r = O(p^{n-\mu}M).$$

In this way with $r = 6$, I showed that Davenport's exponent $\frac{3}{4}$ in (2) could be replaced by $\frac{2}{3}$; and also attacked several other cases. Some years before, Artin had produced a conjecture that the exponent should be $\frac{1}{2}$. In problems involving orders of magnitude, it often takes a long time to sharpen estimates, but not for the present one. Davenport was staying with Hasse at Marburg in the early thirties and challenged him to find a concrete illustration of abstract algebra. This led Hasse to his theory of the elliptic function fields, a study initiated by Artin, and he proved Artin's conjecture, namely:

The number N of solutions of the congruence

(9) $$y^2 \equiv x^3+Ax+B$$

satisfies the inequality

(10) $$|N-p-1| < 2\sqrt{p}.$$

Then Weil took up the question and developed a general theory for the congruences (5) and found the best possible result for the number of solutions in the form

$$|N-p| < k\sqrt{p},$$

where k is an explicitly given absolute constant.

In recent years, attention has been paid to the more general case when $f(x, y)$ is replaced by a function of several variables. One could never have foreseen what applications of congruence theory would have been made to diophantine equations and to zeta functions associated with the manifolds defined by polynomial congruences.

The study of congruences was facilitated by the use of exponential sums. I noticed that the method of discrete averaging proved useful also for them. Write

$$S = \sum_{x=0}^{p-1} e\left(\frac{2\pi i}{p} f(x)\right), \tag{11}$$

where p is a large prime and $f(x)$ is a polynomial of degree n with integer coefficients. Then I found that

$$S = O(p^{1-1/n}), \tag{12}$$

where the constant in O is independent of p and the coefficients of $f(x)$. This estimate was improved for some n by Davenport in [5]. Here again the problem arises of finding the best possible value. It has been shown by Weil that

$$|S| < (n-1)p^{1/2}. \tag{13}$$

While the proof for (12) is really elementary, that of (13) is of a very advanced nature.

Obvious extensions arise when $f(x)$ is replaced by a function of several variables.

Linnik told me that the method of discrete averaging led Vinogradoff to his important estimation of exponential sums by continuous averaging.

Davenport was very much interested throughout his life in congruences and exponential sums as can be seen from the large number of papers on these subjects mentioned in his list of papers. One might mention in particular two of his early papers. Extensions and generalizations of many classical results are contained in [8], a joint paper with H. Hasse. In [27], he finds estimates for character sums by elementary means, results which could be improved only by Weil's deep methods. He also proved a conjecture of Hasse on some functional equations for L functions.

What a wonderful cornucopia his first paper proved to be!

Davenport did not play a vital part in the greatest developments arising from his first problem. In his second one, however, he was the pioneer who was able to find a path over untrodden ground. Let $L_1, L_2, \ldots, L_n$ be n linear homogeneous forms in n variables (x) with determinant D. It is easily shown that integer values of the x not all zero exist such that

$$|L_1 L_2 \ldots L_n| \leqslant k|D|, \tag{14}$$

where k is a constant independent of the coefficients of the L. Values of k are easily found from Minkowski's convex region theorem applied to the region

$$|L_1|+|L_2|+\ldots+|L_n| \leqslant l\sqrt[n]{|D|}; \tag{15}$$

e.g. if the coefficients of the L are all real, then k can be taken as $n!/n^n$. The problem is to find the best possible value of k. When $n = 2$, the problem is easy. When $n = 3$, the known results follow from results for (15). I suggested to Davenport that he should try to improve them. Two cases must be considered. The first is when the coefficients of the L are all real, and the second when the coefficients of L_1 are real and those of L_2 and L_3 are conjugate imaginary. He first sharpened the old estimates. Then in paper [25], he found the best possible value, $k = 1/7$ in the first case. This was the first time that an exact result had been found in the Geometry of Numbers for non-convex regions apart from those arising for $n = 2$. Then in paper [26], he settled the second case and found that $k = 1/\sqrt{23}$. These were great accomplishments indeed and gave him an international reputation. The proofs, however, as might be expected from such difficult problems, were rather complicated and so it was not easy to extract from them a simple idea which permitted of further applications or generalization.

In thinking about his work, I hit upon a method of reducing the three dimensional problem to a two dimensional one. Here, this led to the consideration of an inequality

$$|f(x, y)| = |(a_1x+b_1y)(a_2x+b_2y)(a_3x+b_3y)| \leqslant \sqrt[4]{|D|/j}$$

where D is the discriminant of the cubic $f(x, y)$. If the best possible value of j could be found, the value for l in (15) when $n = 3$ follows easily. In this way, the question arose of finding the best possible estimate for the minimum of a binary cubic form

$$f(x, y) = ax^3+bx^2y+cxy^2+dy^3 \tag{16}$$

of discriminant D for integer values of x, y not both zero, in the shape

$$|f(x, y)| \leqslant \sqrt[4]{|D|/j}. \tag{17}$$

Partial results had been found around the middle of the last century by Eisenstein, Arndt and Hermite but real progress seemed so slow and difficult that the problem was put aside until I was led to it by the consideration of Davenport's work. Fortunately, after much hard work, the problem was solved by reduction to a problem in the geometry of numbers for non-convex regions. Thus when $D > 0$, a linear substitution, with real coefficients led to the region

$$|x^3-xy^2-y^3| \leqslant 1,$$

a type which had never been previously considered. When $D > 0$, I showed that $j = 49$ and that equality was required only when

$$af(x, y) \sim x^3 + x^2y - 2xy^2 - y^3. \tag{18}$$

When $D < 0$, then $j = 23$ and equality was required only when

$$af(x, y) \sim x^3 - xy^2 - y^3. \tag{19}$$

The proof was very complicated but before long Davenport found two proofs of which the second was a characteristically beautiful and simple one. For the case $D > 0$, in [45] he followed Hermite in calling the cubic reduced if its quadratic covariant

$$Ax^2 + Bxy + Cy^2 = (bx + cy)^2 - (3ax + by)(cx + 3dy)$$

is reduced, i.e. if $C \geqslant A \geqslant 2B$. He then showed that if $f(x, y)$ is reduced and has determinant 49, at least one of

$$f(1, 0), \quad f(0, 1), \quad f(1, 1), \quad f(1, -1)$$

does not exceed 1 numerically. One of them is numerically less than 1 except when

$$\pm f(x, y) = x^3 + x^2y - 2xy^2 - y^3.$$

In [46], he considered the case $D < 0$. Now write

$$f(x, y) = (x + \theta y)(Px^2 + Qxy + Ry^2),$$

where θ, P, Q, R are real and $f(x, y)$ is called reduced if $|Q| \leqslant P \leqslant R$ and $\theta > 0$. Then if $f(x, y)$ is a reduced binary cubic of discriminant -23, at least one of

$$f(1, 0), \quad f(0, 1), \quad f(1, -1), \quad f(1, -2)$$

does not exceed 1 numerically. One of them is numerically less than 1 except when

$$f(x, y) = x^3 + x^2y + 2xy^2 + y^3 \sim x^3 - xy^2 - y^3,$$

and then all four values are ± 1.

The new method employed for the binary cubic led to great developments in the Geometry of Numbers. Previously only convex regions had been studied, but now the road was open to the study of non-convex regions. Important contributions were made by myself, Davenport, Mahler, Rogers and others.

Davenport's results for $n = 3$ in equation (14) were extended in a most unexpected and remarkable way. It is well known that when $n = 2$, there is a succession of isolated minima, the so-called Markoff chain. I suggested the possibility of an analogous result for $n = 3$. Before long, Davenport found a second minimum $k = 1/9$. He also showed that $k < 1/9.1$ except for two special cases. Recently by applying Davenport's methods and a computer, Swinnerton-Dyer has found some additional twenty minima! The proof appears in this volume.

Davenport maintained his interest in the general problem of the minimum of the product of n linear forms. However, even the case $n = 4$ has not been settled. But shortly before Davenport's death, he and Swinnerton-Dyer were engaged upon joint work upon this case. Swinnerton-Dyer thinks there is now a possibility of finding the best possible result.

Both Davenport and I were interested in the similar problem for the product of n inhomogeneous linear forms. Here the problem requires an estimate

$$|L_1 + c_1|\,|L_2 + c_2| \ldots |L_n + c_n| \leqslant |D|k,$$

where the c's are real constants.

A conjecture attributed to Minkowski states that the best possible value of k is $1/2^n$. He gave a proof for $n = 2$, and several others are known. Remak found a lengthy and very involved one for $n = 3$, but in [29], Davenport produced a short and elegant proof for this. The result for $n = 4$ is due to Dyson. For general n, an estimate $k \leqslant 2^{-n/2} + \varepsilon$ had been found by Tchebotareff. I and then Davenport gave sharpened estimates in several papers. The general problem had been in our minds for many years and we spent much time on it. Our efforts, however, have not been successful. The problem still remains unsolved.