Since by (31)
\[ \sum_{n \leq x} d(n)^6 \ll x (\log x)^{42} \quad (x \geq 3), \]
it follows that
\[ \sum_{\chi \in \mathcal{X}} \sum_{q \leq M} \frac{1}{\varphi(q)} \sum_{\chi} \left| \tau(\chi) \right|^2 \left| f(s_0 + it, \chi, \chi) \right|^2 \ll (\log M)^{62} < (\log MT)^{105}. \]

Now we introduce the integral function
\[ F(s) = \prod_{\chi \in \mathcal{X}} \prod_{q \leq M} \prod_{d | q} \prod_{\chi} \left( 1 - f(s, \chi, \chi') \right)^{\nu(\chi)} \]
where
\[ \nu(\chi) = (M/|q|) \left| \tau(\chi) \right|^2 \]
and consider that any zero of \( \zeta(s, \chi, \zeta, \zeta) \) is also a zero (of at least the same order) of the function \( 1 - f(s, \chi, \chi) \). Using (37) and (38) (which are the analogues of [1], Lemmas 8 and 9) and arguing as in the proof of [1], Theorem 5, we get (8).

References


Received on 15. 2. 1970

Diophantine approximation and certain sequences of lattices

by

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In memory of H. Davenport

1. Introduction. The present paper is a continuation of the joint work [2], [3] by Davenport and the author, but most of it can be read independently.

Let \( a_1, \ldots, a_n \) be real numbers. There are two forms of Dirichlet's theorem on simultaneous approximation.

(a) For any positive integer \( N \) there exist integers \( x_1, \ldots, x_n, y, \) not all zero, and satisfying
\[ |a_1 x_1 + \ldots + a_n x_n + y| < N^{-n}, \quad \max(|x_1|, \ldots, |x_n|) \leq N; \]

(b) for any positive integer \( N \) there are integers \( x_1, \ldots, x_n, y, \) not all zero, with
\[ \max(|a_1 y - x_1|, \ldots, |a_n y - x_n|) < N^{-1}, \quad |y| \leq N^n. \]

Now let \( A(a_1, \ldots, a_n; N) \) be the lattice in the space of dimension \( \ell = n + 1 \)

with basis vectors
\[ g_1 = (N^{-1}, 0, \ldots, 0, a_1 N^n), \]
\[ g_2 = (0, N^{-1}, \ldots, 0, a_2 N^n), \]
\[ \ldots \ldots \ldots \ldots \ldots \]
\[ g_n = (0, 0, \ldots, N^{-1}, a_n N^n), \]
\[ g_{n+1} = (0, 0, \ldots, 0, N^n). \]

Form (a) of Dirichlet's theorem says precisely that \( A(a_1, \ldots, a_n; N) \) has a nonzero point (namely \( x_1 g_1 + \ldots + x_n g_n + y g_{n+1} \)) in the cube defined by \( |x_1| \leq 1, \ldots, |x_n| \leq 1, |y| < 1. \) Dirichlet's theorem in form (a) can be improved for particular \( a_1, \ldots, a_n; N \) if the lattice \( A(a_1, \ldots, a_n; N) \)
has a nonzero point in some smaller cube \(|z| = 0, \ldots, |z| = 0\) where 0 < \( \varepsilon < 1\). Thus for given \(a_1, \ldots, a_n\), to study refinements of Dirichlet's theorem in form (b) it is natural to study the sequence of lattices \(A(a_1, \ldots, a_n; N)\) with \(N = 1, 2, \ldots\).

Form (b) of Dirichlet's theorem says that the lattice \(A^*(a_1, \ldots, a_n; N)\) with basis vectors
\[
\begin{align*}
g_1^* &= (N, 0, \ldots, 0, 0), \\
g_2^* &= (0, N, \ldots, 0, 0), \\
\vdots & \quad \vdots \\
g_n^* &= (0, 0, \ldots, N, 0), \\
g_*^* &= (-a_1N, -a_2N, \ldots, -a_nN, N^{-n})
\end{align*}
\]

has a nonzero point (namely \(x_1g_1^* + \cdots + x_ng_n^* + yg_*^*\)) in the cube \(|z| < 1, \ldots, |z| < 1, |z| < 1\). The lattice \(A^*(a_1, \ldots, a_n; N)\) is polar to \(A(a_1, \ldots, a_n; N)\). To study refinements of Dirichlet's theorem in form (b) for fixed \(a_1, \ldots, a_n\) one has to look at the sequence of lattices \(A^*(a_1, \ldots, a_n; N)\) with \(N = 1, 2, \ldots\).

Given a point \(x = (x_1, \ldots, x_l)\), write \(|x| = \max(|x_1|, \ldots, |x_l|)\). The \(l\times l\)-determinant \(A(x_1, \ldots, x_l)\) of \(l\) points \(x_1, \ldots, x_l\) in \(l\)-dimensional space \(E^l\) is defined as the \(l\times l\)-determinant with row vectors \(x_i = (x_{i1}, \ldots, x_{il}) (1 \leq i \leq l)\). We now recall that the lattices of determinant 1 in \(E^l\) form a topological space (see [1], § V.3.2). A sequence of lattices \(A_1, A_2, \ldots\) is everywhere dense in this space precisely if for every \(\varepsilon > 0\) and every \(l\)-tuple of points \(x_1, \ldots, x_l\) with determinant 1 there is a lattice \(A_k\) in the sequence with points \(h_1, \ldots, h_l\) in \(A_k\) such that
\[
|x_i - h_i| < \varepsilon \quad (i = 1, \ldots, l).
\]

It is easy to see that a sequence of lattices \(A_1, A_2, \ldots\) is everywhere dense if and only if the sequence of polar lattices \(A_1^*, A_2^*, \ldots\) is everywhere dense.

**Theorem 1.** The sequence of lattices \(A(a; N)\) with \(N = 1, 2, \ldots\) is everywhere dense in the space of lattices with determinant 1 in \(E^l\) if and only if every block of positive integers occurs infinitely often in the sequence of partial quotients of the expansion of \(x\) as a simple continued fraction.

Almost every \(x\) (in the sense of Lebesgue measure) has an expansion as a simple continued fraction with the property described in the theorem. We therefore have the following

**Corollary.** For almost every \(x\), the sequence \(A(a; N)\) with \(N = 1, 2, \ldots\) is everywhere dense.

When \(n > 1\) an appeal to continued fractions is not possible.

**Theorem 2.** Let \(n \geq 1\) and let \(N_1, N_2, \ldots\) be real numbers which increase to infinity. Then for almost every \(n\)-tuple \((a_1, \ldots, a_n)\), the sequence of lattices \(A(a_1, \ldots, a_n; N_k)\) with \(k = 1, 2, \ldots\) is everywhere dense in the space of lattices of determinant 1 in \(E^l\).

This result sharpens Theorem 3 of [3]. By a remark made above it remains true if the lattices \(A(a_1, \ldots, a_n; N_k)\) are replaced by the polar lattices \(A^*(a_1, \ldots, a_n; N_k)\).

Siegel [6] defined a measure on the space of lattices of determinant 1. Hence it is natural to ask whether a sequence of lattices is uniformly distributed in this space. It is easy to see that the lattices \(A(a_1, \ldots, a_n; N)\) change rather slowly as \(N\) varies, and hence \(A(a_1, \ldots, a_n; N)\) with \(N = 1, 2, \ldots\) is not uniformly distributed for any \(a_1, \ldots, a_n\). On the other hand it is likely that the lattices \(A(a_1, \ldots, a_n; N)\) are uniformly distributed for almost every \((a_1, \ldots, a_n)\). Nothing in this direction will be proved in the present paper.

**2. The necessity of the continued fraction condition.** We shall adopt the notation of [4], chapter X, for continued fractions. Thus \([a_0, a_1, \ldots, a_n]\) is the rational function of \(a_0, \ldots, a_n\) defined inductively by \([a_0] = a_0\) and by \([a_0, a_1, \ldots, a_k] = a_0 + [a_1, \ldots, a_k]^{-1} (k = 1, 2, \ldots)\). Every irrational number \(x\) has a unique expansion as an infinite continued fraction \(x = [a_0, a_1, a_2, \ldots] = \lim[a_0, a_1, \ldots, a_n]\) where \(a_n\) is an integer and \(a_1, a_2, \ldots\) are positive integers. The numbers \(a_0, a_1, a_2, \ldots\) are the partial quotients, and the rational \([a_0, a_1, \ldots, a_n]\) are the convergents of the continued fraction. One puts \([a_0, a_1, \ldots, a_n] = p_n/q_n\) where \(p_n, q_n\) are in their lowest terms, and \(a_n = [a_0, a_{n+1}, \ldots]\).

Suppose now that \(A(a_1, N)\) with \(N = 1, 2, \ldots\) is everywhere dense. Then \(x\) must be irrational. Let \(x_1, x_2\) be the points
\[
x_1 = (2\varepsilon, -1 + 2\varepsilon y), \quad x_2 = (1, y)
\]
where \(0 < y < \frac{1}{x}\) and where \(\varepsilon > 0\) is small. We have \(A(x_1, x_2) = 1\), and hence there are lattice points \(h_1, h_2\) in some lattice \(A(a; N)\) of the sequence with \(|h_i - x_i| < \varepsilon (i = 1, 2)\). We may write
\[
\begin{align*}
h_1 &= a_0g_1 + bg_2, \\
h_2 &= c_0g_1 + dg_2,
\end{align*}
\]
where \(g_1, g_2\) are given by (4) and where the coefficients \(a, b, c, d\) are integers. Now \(ad - bc = A(h_1, h_2)\), and this is close to \(A(x_1, x_2) = 1\) if \(\varepsilon\) is small. Hence
\[
ad - bc = 1
\]
if \(\varepsilon > 0\) is small. By virtue of (5) we have \(h_1 = (aN^{-1}, aN + bN)\), \(h_2 = (cN^{-1}, aN + dN)\), and hence the inequalities \(|h_i - x_i| < \varepsilon (i = 1, 2)\) imply that
In particular we have $a > 0$, $c > 0$ if $z > 0$ is small. We further have $a(aa + b) < 3\varepsilon N|aa + b| < 8\varepsilon (1 + e)$ by (7), and hence $a(aa + b) < \frac{1}{2}$ if $z$ is small. It follows from a well-known theorem (e.g. Theorem 13 of [4]) that $-b/c$ is a convergent to $a$, say $-b/a = p_k/q_k$. By (8) the numbers $a, b$ are coprime, and by (7) we have $aa + b < 0$. Hence $a = q_k$, $b = -p_k$, and $k$ is odd. Similarly from (8) we obtain that $a(aa + d) < (1 + e)N|aa + d| < (1 + e)(y + e) < \frac{1}{2}$ if $z$ is small, since $0 < y < \frac{1}{2}$. Thus also $-d/c$ is a convergent to $a$, say $-d/c = p_{k+1}/q_{k+1}$. Using (6) and (8) one sees that $c = q_k, d = -p_{k+1}$, and that $k$ is even. One sees from (7), (8) that
\[ |q_k a - p_k| = |a| \cdot |a + b| = |a - p_k|, \]
since $0 < y < \frac{1}{2}$ and since $z$ is small, and therefore one has $k > h$. Finally, $q_k p_k - q_{k+1} p_k = ad - be = 1$ implies that $h = k - 1$. For otherwise we would have $h < k - 1$, and
\[ (q_k, q_{k+1})^\varepsilon = (p_k/q_k) - (p_{k+1}/q_{k+1}) > (p_k/q_k) - (p_{k-1}/q_{k-1}) \]
\[ > (q_k q_{k-1})^\varepsilon > (q_k q_{k+1})^\varepsilon, \]
a contradiction. Altogether we have
\[ a = q_{k+1}, \quad b = -p_{k-1}, \quad c = q_k, \quad d = -p_k. \]

The inequalities (7), (8) imply that
\[ q_{k+1}/q_k = a/c < 3\varepsilon (1 - e)^{-1} < 4\varepsilon \]
if $z$ is small. We also have
\[ |q_k (aq_{k+1} - p_{k+1}) - y| = |c(aa + d) - y| < |N(aa + d) - y| + |N - c| |a + d| < e + N\varepsilon |y + e| < 3\varepsilon. \]
But by (4), § 10.9,
\[ q_k a(q_{k+1} - p_{k+1}) = q_k (-1)^{k}(a_{k+1} + q_{k+1} - q_{k+1}/q_k)^{-1} = (a_{k+1} + q_{k+1}/q_k)^{-1}. \]
Thus
\[ |a_{k+1} + q_{k+1}/q_k| - y| < 3\varepsilon, \quad \text{whence } |a_{k+1} + q_{k+1}/q_k| - y| < 4\varepsilon \]
if $z > 0$ is small, and using this together with (10) we obtain
\[ |a_{k+1} - y| < 10\varepsilon^2. \]
Since $e > 0$ was arbitrarily small, the sequence $a_1, a_2, \ldots$ comes arbitrarily close to $y$. Since $y$ was arbitrary in $0 < y < \frac{1}{2}$, the sequence is everywhere dense on the half line $x > 2$. Since $a_2 = a_0 + (a_{k+1})^{-1}$, the sequence $a_0', a_1', \ldots$ is in fact dense on $x > 1$. From this it follows easily that every block of positive integers occurs infinitely often among $a_1, a_2, \ldots$

3. The sufficiency of the continued fraction condition.

**Lemma 1.** Suppose every block of positive integers occurs infinitely often among $a_1, a_2, \ldots$. Then the points
\[ (a_n/q_n, a_{n+1}/q_{n+1}) \quad (n = 2, 4, 6, \ldots) \]
are everywhere dense in the quadrant $x > 1, y > 1$ of the plane. The same is true with $n = 3, 5, \ldots$.

Proof. Let $x > 1, y > 1$, and suppose $e > 0$ is small. There are integers $b_0, b_1, b_2, \ldots, b_0$ such that every number $x' = [b_0, b_1, b_2, \ldots, b_{n+1}, \ldots, b_1]$ with arbitrary $f$ and $b_2, b_3, \ldots, b_n$ satisfies $|x' - x| < e$. There are integers $c_0, c_1, c_2, \ldots$ such that every number $y' = [c_0, c_1, c_2, \ldots]$ with arbitrary $c_{n+1}, \ldots$ satisfies $|y' - y| < e$. Now suppose $n$ is large and such that
\[ a_n - 2 = b_2, \ldots, a_{n-1} = b_1, a_n = b_0, \]
\[ a_{n-1} = c_0, a_{n+2} = c_1, \ldots, a_{n+2} = c_n. \]
Since $q_n/q_{n-1} = [a_0, a_1, \ldots, a_n]$ ([5], § 11), we then have $|q_n/q_{n-1} - x| < e$, and similarly we have $|a_{n+1} - y| < e$. But (11) happens for infinitely many values of $n$. Since every block of integers occurs in $a_1, a_2, \ldots$, there are in fact infinitely many values of $n$ for which (11) holds both for $n$ and for $n' = n + 2r + 2s - 1$. But $n, n'$ have opposite parity, and hence there will in fact be infinitely many even as well as infinitely many odd $n$ with (11). This proves the lemma.

We now have to show that for any two points $x_1, x_2$ with $d(x_1, x_2) = 1$, there are lattice points $h_1, h_2$ in some lattice $A(a, N)$ with $h_1 - x_1$ and $h_2 - x_2$ both less than $32\varepsilon$ by replacing $h_1$ and $h_2$ by $-h_1$ and $-h_2$, respectively, if necessary. Let $y_1 = (x_1, y_1)$ be a minimal point in $A$, i.e. assume that $y_1 \neq 0$ and that there is no point $(x', y') \neq 0$ in $A$ with $|x'| < |x_1|$, $|y'| < |y_1|$. By replacing $y_1$ by $-y_1$, if necessary, we may assume that $x_1 > 0$. Let $y_1 = (x_1, y_1) \neq 0$ be a point with $|y_1| < x_1$, and with $|y_1| as small as possible. Then $y_1$ is again a minimal point, and in fact there is no point $(x, y) \neq 0$ with
\[ (x, y) < (x_1, y_1), \quad (y, y) < (y_1). \]
We may assume that $x_2 > 0$. The point $(x, y) = (x_1 - x_2, y_1 - y_2)$ has $0 < x < x_1$, and hence by the possibility of (12) it has $|y| = |y_1 - y_2| \geq |y_2|$. Since $|y_2| < |y_1|$, this implies that $y_2, y_1$ are of opposite sign.
Since there is no nonzero point in the region defined by (12), the triangle
\( \mathbf{0}, \mathbf{y}_1, \mathbf{y}_2 \) contains no lattice points but its vertices, and
\[
A(y_1, y_2) = x_1 y_2 - x_2 y_1 = \pm 1.
\]
One has
\[
|y_1| (x_1 + x_3) < |y_2| x_3 + |y_3| x_1 = |y_1 x_2 - y_2 x_3| = 1,
\]
and therefore
\[
\frac{1}{|x_1 y_1|} x_2 = 1 - \frac{x_3 |y_1|}{|x_1 y_1|} > 1.
\]

It will suffice to find points \( f_1, f_2 \) of \( A(a; N) \) which are close to
\( y_1, y_2 \), respectively. For since \( y_1, y_2 \) form a basis of \( A \), we have \( a = a_1 y_1 + a_2 y_2 \), and if \( f_1, f_2 \) are close to \( y_1, y_2 \), then \( h_i = a_1 f_i + a_2 f_2 \) is close to \( a_i \) \((i = 1, 2)\). From here on, \( y_1, y_2 \) will be fixed. Now choose \( n \)
even if \( y_1 > 0 \), and \( n \) odd if \( y_1 < 0 \), and such that
\[
(q_n/q_{n-1}) - (a y_2) < \delta,
\]
\[
a_{n+1} - \left( \frac{1}{|x_1 y_1|} - \frac{x_2}{x_1} \right) < \delta,
\]
where \( \delta \) is some small positive quantity. Let \( N \) be an integer with
\[|N x_2 - q_{n-1}| < |x_2|.
\]

(13)
\[
|q_{n-1}/N - x_2| < \delta
\]
if \( n \) and hence \( N \) is large. We also have
\[
|N x_2 - q_{n-1}| = \left| \frac{x_1}{x_2} (N x_2 - q_{n-1}) + q_{n-1} \left( \frac{x_1}{x_2} - \frac{q_{n-1}}{x_2} \right) \right| < x_2 + q_{n-1} \delta \ll N \delta,
\]
whence

(14)
\[
|q_{n-1}/N - x_2| \ll \delta.
\]
(The constants in \( \ll \) depend only on \( y_1, y_2 \).) We note that by a formula in
\[\{4\}, \S \text{10.5},\]
\[
|N (aq_n - p_n) - y_1| \ll \frac{q_n}{x_1} |aq_n - p_n| - y_1 + \frac{q_n}{x_1} N \ll \frac{q_n}{x_1} |aq_n - p_n| - y_1 + \frac{q_n}{x_1} N \ll \frac{q_n}{x_1} |aq_n - p_n| - y_1 + \frac{q_n}{x_1} N, \]
\[
|a_{n+1} + (q_{n-1}/q_n) - x_1 y_1| - \delta.
\]
But
\[
|a_{n+1} + (q_{n-1}/q_n) - x_1 y_1| - \delta.
\]
and therefore
\[
|N (aq_n - p_n) - y_1| \ll \delta.
\]
Putting
\[
f_1 = q_n g_1 - p_n g_2 = (q_n N^2, q_n x_1 N - p_n N) = (a_1, b_1), \text{ say},
\]
\[
f_2 = q_{n-1} g_1 - p_{n-1} g_2 = (q_{n-1} N^2, q_{n-1} x_1 N - p_{n-1} N) = (a_2, b_2), \text{ say},
\]
we have
\[
|a_1 - x_1| \ll \delta, \quad |a_2 - x_2| \ll \delta, \quad |b_1 - y_1| \ll \delta
\]
by (13), (14) and (15). Since
\[
a_1 b_2 - a_2 b_1 = q_{n-1} p_{n-1} g_2 = (-1)^{n-1} = x_1 y_2 - x_2 y_1,
\]

4. The method of proof of Theorem 2. We shall restrict ourselves
to the case when \( n = 2, l = 3 \). Throughout the proof, \( x_1, y_1, \ldots \) will denote
points in 3-dimensional space. We shall write \( A(a, \beta; N) \) instead of
\( A(a_1, a_2; N) \).

Let \( x_1, x_2, x_3 \) be points with \( A(x_1, x_2, x_3) = 1 \). Further let
\( T(N; x_1, x_2, x_3; \epsilon) \) consist of all pairs \((a, \beta)\) for which the lattice
\( A(a, \beta; N) \) contains points \( h_1, h_2, h_3 \) with \( |h_i - x_i| < \epsilon \) \((i = 1, 2, 3)\).

Proposition. There is a \( \theta = \theta(x_1, x_2, x_3; \epsilon) \) \(> 0 \) such that for every
square \( Q \) of the type
\[
(a - a, -a, \beta - \beta) < a, \quad |\beta - \beta| < a
\]
and every \( N > N_0(Q; x_1, x_2, x_3; \epsilon) \) the intersection of \( Q \) with \( T(N) \)
\(= T(N; x_1, x_2, x_3; \epsilon) \) has measure
\[
\mu(Q \cap T(N)) \geq \theta \mu(Q) \geq \theta a^2.
\]

Thus the complement of
\[
T(x_1, x_2, x_3; \epsilon) = \bigcup_{k=1}^{\infty} T(N_k; x_1, x_2, x_3; \epsilon)
\]
has density \( \ll 1 - \theta < 1 \) everywhere. Since a measurable set has density 1
at almost all of its points, the complement of \( T(x_1, x_2, x_3; \epsilon) \) has measure
zero, and almost every point \((a, \beta)\) belongs to \( T(x_1, x_2, x_3; \epsilon) \). Since
this is true for every \( \epsilon > 0 \) and every \( x_1, x_2, x_3 \) with determinant 1,
Theorem 2 follows. It remains to prove the proposition.

5. The set \( \Sigma(N) \). Write \( x_i = (\xi_{i1}, \xi_{i2}, \xi_{i3}) \) \((i = 1, 2, 3)\). We may
assume that \( x_1, x_2, x_3 \) satisfy only the equation \( A(x_1, x_2, x_3) = 1 \) and
equations implied by it, i.e., that \( x_1, x_2, x_3 \) is a generic point of the surface
in 9-dimensional space defined by \( A(x_1, x_2, \ldots, x_9) = 1 \). From now on, \( x_1, x_2, x_3 \) will be fixed. The constants in \( \leq \) may depend on \( x_1, x_2, x_3 \) and on \( \delta \), but they will be independent of \( N \) and of squares \( Q \).

Let \( \Sigma(N) = \Sigma(N; x_1, x_2, x_3; \delta) \) consist of all pairs \((\alpha, \beta)\) for which the lattice \( A(\alpha, \beta; N) \) contains points \( h_1, h_2, h_3 \) with
\[
A(h_1, h_2, h_3) = 1
\]
and with
\[
|h_1 - x_1| < \delta, \quad |h_2 - x_2| < \delta, \quad |h_3 - \xi_{x_1}| < \delta, \quad |h_3 - \xi_{x_2}| < \delta
\]
where \( h_3 = (h_{31}, h_{32}, h_{33}) \). Since \( A(h_1, h_2, h_3) = A(x_1, x_2, x_3) = 1 \), the eight inequalities implicit in \( (19) \) imply a ninth one, namely \( |h_3 - \xi_{x_3}| < \delta \).

Hence if \( \delta \) is sufficiently small in relation to \( \varepsilon \), then \( |h_i - x_i| < c \) \((i = 1, 2, 3)\), and \( \Sigma(N; x_1, x_2, x_3; \delta) \) is contained in \( T(N; x_1, x_2, x_3; \varepsilon) \).

Hence it will suffice to prove the proposition above with \( c \) replaced by \( \delta \) and \( T(N) \) replaced by \( \Sigma(N) \). It will suffice to prove the proposition for \( 0 < \delta < \delta_0 \), where \( \delta_0 = \delta(x_1, x_2, x_3) \) is arbitrarily small.

Recall that \( A(\alpha, \beta; N) \) has the basis
\[
(20) \quad g_1 = (N^{-1}, 0, \alpha N^2), \quad g_2 = (0, N^{-1}, \beta N^2), \quad g_3 = (0, 0, N^2).
\]
Any three points, \( h_1, h_2, h_3 \) of \( A(\alpha, \beta; N) \) may be written as
\[
h_1 = q_1 g_1 + q_2 g_2 + q_3 g_3,
\]
\[
h_2 = q_1 g_1 + q_2 g_2 + q_3 g_3,
\]
\[
h_3 = q_1 g_1 + q_2 g_2 + q_3 g_3
\]
with integer coefficients \( q_i \). For given integer points \( q_1, q_2, q_3 \) with \( q_i = (q_{i1}, q_{i2}, q_{i3}) \) \((i = 1, 2, 3)\), let \( F(N; q_1, q_2, q_3; \delta) \) be the set of pairs \((\alpha, \beta)\) for which \( h_1, h_2, h_3 \) as given by \((20)\) and \((21)\) satisfy \((18)\) and \((19)\). (\( F \) also depends on \( x_1, x_2, x_3 \), but these points are fixed.)

Now \( A(h_1, h_2, h_3) = 1 \) is equivalent with
\[
A(q_1, q_2, q_3) = 1
\]
and six of the eight inequalities implicit in \((19)\) are equivalent with
\[
|q_{i1} - x_{i1}| < N\delta, \quad |q_{i2} - N\xi_{x_i}| < N\delta \quad (i = 1, 2, 3).
\]
Thus \( F(N; q_1, q_2, q_3; \delta) \) is empty unless \((22)\) and \((23)\) hold. But if these inequalities hold, then \((\alpha, \beta)\) lies in \( F(N; q_1, q_2, q_3; \delta) \) precisely if
\[
|q_{i1} + q_{i2} + q_{i3} - \xi_{x_i} N^{-2}| < N\delta \quad (i = 1, 2, 3).
\]
(These are the remaining two inequalities of \((19)\).) Hence in this case \( P(N; q_1, q_2, q_3; \delta) \) is the parallelogram \( B(N; q_1, q_2, q_3; \delta) \) defined by \((24)\).

(Since \( x_1, x_2, x_3 \) are generic, and by \((23)\), we have \( q_{i1} q_{i2} - q_{i2} q_{i1} \neq 0 \) if \( \delta > 0 \) is sufficiently small, which we may assume.)

**Lemma 2.** Suppose \( \delta > 0 \) is sufficiently small, and the integer points \( q_1, q_2 \) satisfy \((23)\) for \( i = 1, 2 \). Then \( B(N; q_1, q_2; \delta) \) has area
\[
\mu(B(N; q_1, q_2; \delta)) = N^{-6}
\]
and diameter
\[
d(B(N; q_1, q_2; \delta)) = N^{-3}.
\]

**Proof.** This is Lemma 2 of \([3]\).

**Lemma 3.** Suppose \( N \) is large and suppose that integer points \( q_1, q_2 \) satisfy \((23)\) with \( i = 1, 2 \) and
\[
\left| \frac{q_{i1}}{q_{i2}} - \frac{q_{i1}}{q_{i3}} - a_0 \right| < \frac{\eta}{4},
\]
\[
\left| \frac{q_{i1}}{q_{i2}} - \frac{q_{i1}}{q_{i3}} + \beta_0 \right| < \frac{\eta}{4}.
\]

Then \( B(N; q_1, q_2; \delta) \) is contained in the square \( Q \) defined by \((16)\).

**Proof.** This is Lemma 3 of \([3]\).

Now let \( B'(N; q_1, q_2; \delta) \) be the parallelogram of points \((\alpha, \beta)\) which satisfy \((24)\) with \( \xi_{x_1}, \xi_{x_2} \) replaced by zero. Now if \((23)\) holds, then \( |q_{i1} q_{i2} - q_{i2} q_{i1}| > N^3 \), and \( B(N; q_1, q_2; \delta) \) is obtained from \( B'(N; q_1, q_2; \delta) \) by translation by a vector of length \( O(\delta^{-1}) \).

**Lemma 4.** Suppose \( q_1, q_2 \) are part of a basis and satisfy \((23)\) for \( i = 1, 2 \). Make the same assumptions on \( q_1, q_2 \). Then, if \( q_1, q_2 \neq (q_1, q_2) \), the parallelograms \( B'(N; q_1, q_2; \delta) \) and \( B'(N; q_1, q_2; \delta) \) are disjoint.

**Proof.** This is Lemma 4 of \([3]\).

**Lemma 5.** Suppose \( N \) is large. Then a point \((\alpha, \beta)\) lies in \( \leq 1 \) parallelograms \( B(N; q_1, q_2; \delta) \) with \( q_1, q_2 \) part of a basis and satisfying \((23)\) with \( i = 1, 2 \).

**Proof.** This is Lemma 5 of \([3]\).

6. The number of certain integer points. Let \( Z(N) \) be the set of triples of integer points \( q_1, q_2, q_3 \) with \((22)\), \((23)\) and \((25)\). Suppose that \( q_1, q_2, q_3 \) and \( q_1, q_2, q_3 \) lie in \( Z(N) \). Then by \((22)\) we have \( g_3 = g_3^* + v g_1 + v g_2 \) with integer coefficients \( u, v \). By \((23)\) we have
\[
|q_{i1} - N\xi_{x_i}| < N\delta \quad \text{and} \quad |q_{i1} - N\xi_{x_i}| < N\delta.
\]
whence $|q_1' - q_2| = |w_1 q_1 + w_2 q_2| < 2N \delta$. In the same manner one finds that $\max(|q_3' - q_2|, |q_3 - q_2|) < 2N \delta$. Now by (23) again one has
\[\max(|q_1|, |q_1 + q_3|, |q_2 + q_3|) \ll N^2\] and hence $u, v$ satisfy $|u|, |v| < \delta$. Hence if $\delta$ is sufficiently small we have $|u|, |v| < 1$, whence $u = v = 0$, whence $q_i = q_j$.

We have shown that if $q_1, q_2, q_3$ and $q_1', q_2', q_3'$ are distinct triples in $Z(N)$, then already the pairs $q_1, q_2$ and $q_1', q_2'$ are distinct.

Let $\mathcal{P}(N)$ denote the number of elements of $Z(N)$. By Lemma 3, the set $Q \cap \Sigma(N)$ contains at least $\mathcal{P}(N)$ parallelograms $B(N; q_1, q_3; \delta)$ where $q_1, q_2, q_3$ lie in $Z(N)$. But those parallelograms need not be disjoint. By what we just said, the pairs $q_1, q_2$ are all distinct here. Hence by Lemma 5, any given point $(a, b)$ lies in $\ll 1$ of these parallelograms. Since $B(N; q_1, q_3; \delta)$ has area $\mu(B) \gg N^{-1}$ by Lemma 2, we obtain
\[\mu(Q \cap \Sigma(N)) \gg N^{-1} \mathcal{P}(N).\]

Therefore to prove (17) and thus Theorem 2 it will suffice to show that (26)
\[\mathcal{P}(N) \gg N^{2/3} \delta.\]

7. Some further lemmas.

**Lemma 5.** Let $\mathcal{P}$ be a bounded Jordan measurable set in $6$-dimensional space. Then as $t \to \infty$, the number of integer points $X = (x_1, x_2)$ in $\mathcal{P}$ such that $x_1, x_2$ is part of a basis of the integer lattice in $3$-dimensional space is asymptotically $\mathcal{P}(N) \gg N^{-1}$.

**Proof.** This is the case if $m = 3$ and $l = 3$ of Theorem 4 in [3].

**Lemma 7.** Suppose $0 < \varepsilon < 1$ and $l = (l_1, l_2, l_3)$ are given. There is a basis $v_1, v_2, v_3$ of the integer lattice such that every point $x$ with
\[x = u_1 v_1 + u_2 v_2 + u_3 v_3\]
where
\[|u_i| < |u_1| + |u_2|\]
satisfies
\[|u_i v_i| < \varepsilon |x|\]
and
\[|v_i| |x| < \varepsilon |x|\]

**Proof.** We may assume that $l \neq 0$, and in fact we may assume that $|l_1| + |l_2| + |l_3| = 1$. The equation $lx = 0$ defines a plane $P$ in $\mathbb{R}^3$. Let $x_1, x_2, x_3$ be two nonzero orthogonal points on $P$, and let $\varepsilon > 0$ be small. Let $S$ be the set of points $x = (x_1, x_2)$ in $\mathbb{R}^2$ with
\[|x_1 - x_2| < \varepsilon, \quad |x_1 - x_2| < \varepsilon.\]

Lemma 6 tells us that for sufficiently large $t$ there will be points $(x_1, x_2)$ in $tS$ such that $x_1, x_2$ is part of a basis of the integer lattice. Let $(x_1, x_2)$ be such a point, and choose $x_3$ such that $x_1, x_2, x_3$ is a basis. Now let $v$ be a large integer and put
\[v_1 = x_1 v_1 + x_2, \quad v_2 = x_2 + x_3, \quad v_3 = x_3.\]

Then $v_1, v_2, v_3$ are again a basis of the integer lattice.

Now $x_1, x_2$ were orthogonal, and if $v$ is sufficiently small, the points $x_1, x_2$ will be "almost orthogonal", and if $v$ is sufficiently large, the points $v_1, v_2$ will be "almost orthogonal". To make this precise, we may ascertain that the angle between $v_1, v_2$ lies between $\pi/3$ and $2\pi/3$, say. Then
\[|u_1 v_1 + u_2 v_2| > c_0 (|u_1| + |u_2|) \min (|v_1|, |v_2|),\]
where $c_0 > 0$ is an absolute constant, and (29) implies that $|u_1 v_2| \ll (|u_1| + |u_2|) |x|$. Now $\min (|v_1|, |v_2|)$ becomes arbitrarily large for large $v$, while $|v_3|$ is independent of $v$. Thus for large $v$ we have
\[|u_2 v_3| \ll \frac{1}{\varepsilon} |u_1 v_2 + u_2 v_3|,\]
and hence the point $x$ given by (28) satisfies (30). Also
\[x = u_1 (v_1 v_2 + u_2 v_3 + u_3 v_1) + u_2 v_2 = \mathbb{P} + y\]
where
\[p = v_1 (u_1 v_2 + u_2 v_3),\]
and
\[y = v_1 (u_1 v_2 + u_2 v_3 + u_3 v_1) + u_2 v_2 + u_3 v_1 v_2 + u_3 v_1 v_3.\]

Here $p$ lies in the plane $P$, and $|p| \gg c_0 (|u_1| + |u_2|)$. On the other hand, $|x_1 - x_2| < \varepsilon$ ($l = 1, 2$), whence $|y| \ll c_0 (|u_1| + |u_2|) + \varepsilon (|u_1| + |u_2|)$. Thus if $\varepsilon$ is sufficiently small and if $v$ is sufficiently large, then $|y| \ll \varepsilon |p|$. But this yields (31), since
\[|x_1|x_2| \ll |y| \ll |x_1| \ll |x_1| \leq \varepsilon |x_1|,\]
and the conclusions of Lemma 7 are valid with (30) replaced by
\[|u_2 v_3| \ll \varepsilon (|u_1| + |u_2|),\]
and such that $r_1 r_2 r_3 - r_1 r_2 r_3 \neq 0$. 

**Lemma 8.** Suppose $s, l = (l_1, l_2, l_3)$ are as in Lemma 7, and assume that $l_3 \neq 0$. There is a basis of the integer lattice such that the conclusions of Lemma 7 are valid with (30) replaced by
\[|u_2 v_3| \ll \varepsilon (|u_1| + |u_2|)\]
Proof. Since \( l_3 \neq 0 \), one may choose \( z_1, z_2 \) in the proof of Lemma 7 such that \( \xi_{12} z_1 \pm \xi_{32} z_2 \neq 0 \). There is a constant \( c_0 > 0 \) such that for arbitrary \( u_1, u_3 \) one has

\[
|u_1 \xi_{11} + u_3 \xi_{31} + |u_1 \xi_{12} + u_3 \xi_{32}| > c_0 (|u_1| + |u_3|).
\]

Now if \( \epsilon \) is small and if \( \kappa \) is large, the points \( r_{12}/\epsilon \) and \( r_{32}/\epsilon \) will be arbitrarily close to \( z_1, z_2 \), respectively. Thus one will have \( r_{12} r_{32} - r_{21} r_{32} \neq 0 \) and

\[
|u_1 r_{12} + u_3 r_{32}| + |u_1 r_{21} + u_3 r_{32}| > \frac{c_0}{2} (|u_1| + |u_3|) \epsilon t.
\]

Hence

\[
|\xi_1| + |\xi_2| = |u_1 r_{12} + u_3 r_{32}| + |u_1 r_{21} + u_3 r_{32}| + |u_1 r_{12} + u_3 r_{32} + u_3 r_{32}|
\]

\[
> \frac{c_0}{2} (|u_1| + |u_3|) \epsilon t - 2 \epsilon \eta,
\]

by (30). Since \( |x| \) \( \leq c_2 \epsilon t (|u_1| + |u_3|) \), we obtain \( |\xi_1| + |\xi_2| \geq c_0 \eta \) if \( \epsilon > 0 \) is small. In conjunction with (30) this gives

\[
|u_3 r_{32}| \leq c_0^{-1} (|\xi_1| + |\xi_2|).
\]

Since \( \epsilon > 0 \) was arbitrary, the lemma follows.

8. A lower bound for \( \varepsilon(N) \). There are numbers \( l_1, l_2, l_3 \), not all zero, with

\[
l_1 \xi_{11} + l_2 \xi_{21} + l_3 \xi_{31} = 0, \quad l_1 \xi_{12} + l_2 \xi_{22} + l_3 \xi_{32} = 0.
\]

In fact, since \( \xi_1, \xi_2, \xi_3 \) were generic, the number \( l_3 \neq 0 \). The inequalities

\[
|q_{11} - N \xi_{11}| < N \delta/2, \quad |q_{12} - N \xi_{12}| < N \delta/2 \quad (i = 1, 2)
\]

are stronger than the cases \( i = 1, 2 \) of (23). There exists an \( \epsilon = \epsilon(\delta) > 0 \) such that (33) together with

\[
|l_1 q_{11} + l_2 q_{21} + l_3 q_{31}| < \epsilon \max(|q_{11}|, |q_{21}|, |q_{31}|),
\]

(34)

\[
|l_1 q_{12} + l_2 q_{22} + l_3 q_{32}| < \epsilon \max(|q_{12}|, |q_{22}|, |q_{32}|)
\]

implies (23) for \( i = 1, 2, 3 \).

Putting \( l = (l_1, l_2, l_3) \) and

\[
q_1 = (q_{11}, q_{21}, q_{31}), \quad q_2 = (q_{12}, q_{22}, q_{32}), \quad q_3 = (q_{13}, q_{23}, q_{33}),
\]

we may rewrite the inequalities (34) as

\[
|q_1| < \epsilon |q_1|, \quad |q_2| < \epsilon |q_2|.
\]

Let \( r_1, r_2, r_3 \) be the basis of Lemma 8. We may write

\[
q_1 = u_1 r_1 + u_2 r_2 + u_3 r_3,
\]

(37)

\[
q_2 = u_1 r_2 + u_2 r_2 + u_3 r_3,
\]

\[
q_3 = u_1 r_3 + u_2 r_3 + u_3 r_3,
\]

with integer coefficients \( u_i \). By (31) of Lemma 7 and 8, the inequalities (36) will be satisfied provided (29) holds, i.e. provided

\[
|u_i| \leq |u_{i1}| + |u_{31}| \quad (i = 1, 2, 3)
\]

holds for \( i = 1, 2, 3 \). Define points

\[
q'_1 = (q'_{11}, q'_{21}, q'_{31}), \quad q'_2 = (q'_{12}, q'_{22}, q'_{32}), \quad q'_3 = (q'_{13}, q'_{23}, q'_{33})
\]

by

\[
q'_1 = u_{11} r_1 + u_{21} r_2 + u_{31} r_3,
\]

(40)

\[
q'_2 = u_{12} r_1 + u_{22} r_2 + u_{32} r_3,
\]

\[
q'_3 = u_{13} r_1 + u_{23} r_2 + u_{33} r_3.
\]

By (32) of Lemma 8 we have

\[
|q_i - q'_i| < \epsilon (|q_{i1}| + |q_{i3}|) \quad (i = 1, 2, 3)
\]

provided (38) holds. Thus (38) implies that

\[
|q_i - q'_j| < \epsilon (|q_{i1}| + |q_{i3}|) \quad (i, j = 1, 2, 3).
\]

Thus if \( \epsilon > 0 \) is sufficiently small and if (38) holds, then

\[
|q'_{i1} - N q_{i1}| < N \delta/4, \quad |q'_{i2} - N q_{i2}| < N \delta/4 \quad (i = 1, 2)
\]

will imply (33). Similarly, (38), (41) together with

\[
|q'_{i1} - q_{i1}| \leq |q'_{i2} - q_{i2}| - \alpha_i < \eta/8,
\]

\[
\left| q'_{i3} q_{i1} - q_{i3} q_{i1} \right| - \beta_i < \eta/8
\]

will imply (23).

Thus \( \varepsilon(N) \geq \varepsilon'(N) \), where \( \varepsilon'(N) \) is the number of integer bases \( u_1, u_2, u_3 \) with (38) such that the quantities \( q'_i \) defined by (39) and (40) satisfy (41) and (42). The inequalities (41) and (42) with \( N = 1 \) define a bounded set in 6-dimensional space for \( (q_{11}, q_{12}, q_{13}, q_{21}, q_{22}, q_{23}) \). This
set has volume $\gg n^d$. Now \((q_{11}, q_{12}, q_{13}, q_{14}, q_{23}, q_{24})\) is related to \((u_1, u_2) = (u_{11}, u_{12}, u_{13}, u_{14}, u_{23}, u_{24})\) by the linear transformation \((40)\) of determinant \((r_{11} r_{22} - r_{12} r_{21})^2 \neq 0\). Hence \((41)\) and \((42)\) with \(N = 1\) together with \((39)\) and \((40)\) define a bounded set for \((u_1, u_2)\) in 6-dimensional space of volume \(\gg n^d\). For arbitrary \(N\) we obtain the same set but blown up by the factor \(N\). Hence by Lemma 6 there are \(\gg n^d N^k\) pairs of points \(u_1, u_2\) which are part of a basis such that \((41)\) and \((42)\) are satisfied. There still are \(\gg n^d N^k\) such pairs \(u_1, u_2\) all of whose components are different from zero.

It remains to be shown that for every such \(u_1, u_2\) one can find a third basis vector \(v_3\) such that \((38)\) holds. There certainly will be such a vector \(v_3\) of the type \(v_3 = \lambda_1 u_1 + \lambda_2 u_2 + u_0\), where \(\lambda_j \leq \frac{1}{2} (j = 1, 2)\) and where \(u_0\) is the point with \(d(u_1, u_2, u_0) = 1\) which is orthogonal to \(u_1\) and \(u_2\).

It is easy to see that the coordinates of \(u_0\) have absolute values at most 1, and hence
\[
|u_{0i}| \leq \frac{1}{2} |u_{1i}| + \frac{1}{2} |u_{2i}| + 1 \leq |u_{1i}| + |u_{2i}| \quad (i = 1, 2, 3),
\]
since we made sure that \(u_{1i} \neq 0, u_{2i} \neq 0\). Thus our \(u_0\) does satisfy \((38)\), and we have \(z(N) \gg z'(N) \gg n^d N^k\). This proves \((26)\) and hence the theorem.

**References**