Since by (21)

$$\sum_{n\leqslant x} d(n)^6 \ll x(\log x)^{63} \qquad (x\geqslant 3),$$

it follows that

$$(38)\qquad \sum_{\chi_{\mathfrak{K}}} \sum_{q\leqslant M} \frac{1}{\varphi(q)} \sum_{\chi_q}{}' |\tau(\chi_q)|^2\, |f(\sigma_0+it, \chi_q, \chi_{\mathfrak{K}})|^2 \ll (\log M_1)^{65} < (\log MT)^{195}.$$

Now we introduce the integral function

$$F(s) = \prod_{\chi_{\mathfrak{K}}} \prod_{\substack{q\leqslant M\\ d(q)\leqslant D}} \prod_{\substack{\chi_q\in\Gamma_1\\ \chi_q\neq\chi_q^0}} \big(1-f^2(s, \chi_q, \chi_{\mathfrak{K}})\big)^{e(\chi_q)}$$

where

$$e(\chi_q) = (M!/q)\,|\tau(\chi_q)|^2$$

and consider that any zero of $\zeta(s, \chi_q, \chi_{\mathfrak{K}})$ is also a zero (of at least the same order) of the function $1-f^2(s, \chi_q, \chi_{\mathfrak{K}})$. Using (37) and (38) (which are the analogues of [1], Lemmas 8 and 9) and arguing as in the proof of [1], Theorem 5, we get (3).

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Diophantine approximation and certain sequences of lattices

by

Wolfgang M. Schmidt (Boulder, Colo.)

In memory of H. Davenport

1. Introduction. The present paper is a continuation of the joint work [2], [3] by Davenport and the author, but most of it can be read independently.

Let $\alpha_1, \ldots, \alpha_n$ be real numbers. There are two forms of Dirichlet's theorem on simultaneous approximation.

(a) *For any positive integer N there exist integers $x_1, \ldots, x_n, y$, not all zero, and satisfying*

$$(1)\qquad |\alpha_1 x_1+\ldots+\alpha_n x_n+y| < N^{-n}, \qquad \max(|x_1|, \ldots, |x_n|)\leqslant N;$$

(b) *for any positive integer N there are integers $x_1, \ldots, x_n, y$, not all zero, with*

$$(2)\qquad \max(|\alpha_1 y-x_1|, \ldots, |\alpha_n y-x_n|) < N^{-1}, \qquad |y|\leqslant N^n.$$

Now let $\Lambda(\alpha_1, \ldots, \alpha_n; N)$ be the lattice in the space of dimension

$$(3)\qquad l = n+1$$

with basis vectors

$$(4)\qquad \begin{aligned} \boldsymbol{g}_1 &= (N^{-1}, 0, \ldots, 0, \alpha_1 N^n),\\ \boldsymbol{g}_2 &= (0, N^{-1}, \ldots, 0, \alpha_2 N^n),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ \boldsymbol{g}_n &= (0, 0, \ldots, N^{-1}, \alpha_n N^n),\\ \boldsymbol{g}_l &= (0, 0, \ldots, 0, N^n). \end{aligned}$$

Form (a) of Dirichlet's theorem says precisely that $\Lambda(\alpha_1, \ldots, \alpha_n; N)$ has a nonzero point (namely $x_1\boldsymbol{g}_1+\ldots+x_n\boldsymbol{g}_n+y\boldsymbol{g}_l$) in the cube defined by $|\xi_1|\leqslant 1, \ldots, |\xi_n|\leqslant 1, |\xi_l| < 1$. Dirichlet's theorem in form (a) can be improved for particular $\alpha_1, \ldots, \alpha_n; N$ if the lattice $\Lambda(\alpha_1, \ldots, \alpha_n; N)$

has a nonzero point in some smaller cube $|\xi_1| \leqslant c, \ldots, |\xi_l| \leqslant c$ where $0 < c < 1$. Thus for given $a_1, \ldots, a_n$, to study refinements of Dirichlet's theorem in form (a) it is natural to study the sequence of lattices $\Lambda(a_1, \ldots, a_n; N)$ with $N = 1, 2, \ldots$

Form (b) of Dirichlet's theorem says that the lattice $\Lambda^*(a_1, \ldots, a_n; N)$ with basis vectors

$$\begin{aligned} \boldsymbol{g}_1^* &= (N, 0, \ldots, 0, 0), \\ \boldsymbol{g}_2^* &= (0, N, \ldots, 0, 0), \\ &\cdots\cdots\cdots \\ \boldsymbol{g}_n^* &= (0, 0, \ldots, N, 0), \\ \boldsymbol{g}_l^* &= (-a_1N, -a_2N, \ldots, -a_nN, N^{-n}) \end{aligned}$$

has a nonzero point (namely $x_1\boldsymbol{g}_1^*+\ldots+x_n\boldsymbol{g}_n^*+y\boldsymbol{g}_l^*$) in the cube $|\xi_1| < 1, \ldots, |\xi_n| < 1, |\xi_l| \leqslant 1$. The lattice $\Lambda^*(a_1, \ldots, a_n; N)$ is polar to $\Lambda(a_1, \ldots, a_n; N)$. To study refinements of Dirichlet's theorem in form (b) for fixed $a_1, \ldots, a_n$, one has to look at the sequence of lattices $\Lambda^*(a_1, \ldots, a_n; N)$ with $N = 1, 2, \ldots$

Given a point $\boldsymbol{x} = (x_1, \ldots, x_l)$, write $|\boldsymbol{x}| = \max(|x_1|, \ldots, |x_l|)$. The *determinant* $\Delta(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l)$ of l points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l$ in l-dimensional space E^l is defined as the $l\times l$-determinant with row vectors $\boldsymbol{x}_i = (x_{i1}, \ldots, x_{il})$ $(1 \leqslant i \leqslant l)$. We now recall that the lattices of determinant 1 in E^l form a topological space (see [1], § V.3.2). A sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ is *everywhere dense* in this space precisely if for every $\varepsilon > 0$ and every l-tuple of points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_l$ with determinant 1 there is a lattice Λ_k in the sequence with points $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_l$ in Λ_k such that

$$|\boldsymbol{x}_i - \boldsymbol{h}_i| < \varepsilon \quad (i = 1, \ldots, l).$$

It is easy to see that a sequence of lattices $\Lambda_1, \Lambda_2, \ldots$ is everywhere dense if and only if the sequence of polar lattices $\Lambda_1^*, \Lambda_2^*, \ldots$ is everywhere dense.

THEOREM 1. *The sequence of lattices $\Lambda(a; N)$ with $N = 1, 2, \ldots$ is everywhere dense in the space of lattices with determinant 1 in E^2 if and only if every block of positive integers occurs infinitely often in the sequence of partial quotients of the expansion of a as a simple continued fraction.*

Almost every a (in the sense of Lebesgue measure) has an expansion as a simple continued fraction with the property described in the theorem. We therefore have the following

COROLLARY. *For almost every a, the sequence $\Lambda(a, N)$ with $N = 1, 2, \ldots$ is everywhere dense.*

When $n > 1$ an appeal to continued fractions is not possible.

THEOREM 2. *Let $n \geqslant 1$ and let $N_1, N_2, \ldots$ be real numbers which increase to infinity. Then for almost every n-tuple $(a_1, \ldots, a_n)$, the sequence of lattices $\Lambda_k = \Lambda(a_1, \ldots, a_n; N_k)$ with $k = 1, 2, \ldots$ is everywhere dense in the space of lattices of determinant 1 in E^l.*

This result sharpens Theorem 3 of [3]. By a remark made above it remains true if the lattices $\Lambda(a_1, \ldots, a_n; N_k)$ are replaced by the polar lattices $\Lambda^*(a_1, \ldots, a_n; N_k)$.

Siegel [6] defined a measure on the space of lattices of determinant 1. Hence it is natural to ask whether a sequence of lattices is *uniformly distributed* in this space. It is easy to see that the lattices $\Lambda(a_1, \ldots, a_n; N)$ change rather slowly as N varies, and hence $\Lambda(a_1, \ldots, a_n; N)$ with $N = 1, 2, \ldots$ is not uniformly distributed for any $a_1, \ldots, a_n$. On the other hand it is likely that the lattices $\Lambda(a_1, \ldots, a_n; 2^k)$ with $k = 1, 2, \ldots$ are uniformly distributed for almost every $(a_1, \ldots, a_n)$. Nothing in this direction will be proved in the present paper.

2. The necessity of the continued fraction condition. We shall adopt the notation of [4], chapter X, for continued fractions. Thus $[a_0, a_1, \ldots, a_n]$ is the rational function of $a_0, \ldots, a_n$ defined inductively by $[a_0] = a_0$ and by $[a_0, a_1, \ldots, a_k] = a_0 + [a_1, \ldots, a_k]^{-1}$ $(k = 1, 2, \ldots)$. Every irrational number a has a unique expansion as an infinite continued fraction $a = [a_0, a_1, a_2, \ldots] = \lim_{n\to\infty}[a_0, a_1, \ldots, a_n]$ where a_0 is an integer and $a_1, a_2, \ldots$ are positive integers. The numbers $a_0, a_1, a_2, \ldots$ are the *partial quotients*, and the rationals $[a_0, a_1, \ldots, a_n]$ are the *convergents* of the continued fraction. One puts $[a_0, a_1, \ldots, a_n] = p_n/q_n$ where p_n, q_n are in their lowest terms, and $a_n' = [a_n, a_{n+1}, \ldots]$.

Suppose now that $\Lambda(a; N)$ with $N = 1, 2, \ldots$ is everywhere dense. Then a must be irrational. Let $\boldsymbol{x}_1, \boldsymbol{x}_2$ be the points

$$\boldsymbol{x}_1 = (2\varepsilon, -1+2\varepsilon y), \quad \boldsymbol{x}_2 = (1, y)$$

where $0 < y < \frac{1}{2}$ and where $\varepsilon > 0$ is small. We have $\Delta(\boldsymbol{x}_1, \boldsymbol{x}_2) = 1$, and hence there are lattice points $\boldsymbol{h}_1, \boldsymbol{h}_2$ in some lattice $\Lambda(a; N)$ of the sequence with $|\boldsymbol{h}_i - \boldsymbol{x}_i| < \varepsilon$ $(i = 1, 2)$. We may write

$$\boldsymbol{h}_1 = a\boldsymbol{g}_1 + b\boldsymbol{g}_2, \quad \boldsymbol{h}_2 = c\boldsymbol{g}_1 + d\boldsymbol{g}_2, \tag{5}$$

where $\boldsymbol{g}_1, \boldsymbol{g}_2$ are given by (4) and where the coefficients a, b, c, d are integers. Now $ad - bc = \Delta(\boldsymbol{h}_1, \boldsymbol{h}_2)$, and this is close to $\Delta(\boldsymbol{x}_1, \boldsymbol{x}_2) = 1$ if ε is small. Hence

$$ad - bc = 1 \tag{6}$$

if $\varepsilon > 0$ is small. By virtue of (5) we have $\boldsymbol{h}_1 = (aN^{-1}, aaN + bN)$, $\boldsymbol{h}_2 = (cN^{-1}, caN + dN)$, and hence the inequalities $|\boldsymbol{h}_i - \boldsymbol{x}_i| < \varepsilon$ $(i = 1, 2)$ imply that

$$(7)\qquad |aN^{-1}-2\varepsilon| < \varepsilon, \quad |a\alpha N+bN+1-2\varepsilon y| < \varepsilon,$$

$$(8)\qquad |cN^{-1}-1| < \varepsilon, \quad |c\alpha N+dN-y| < \varepsilon.$$

In particular we have $a>0, c>0$ if $\varepsilon>0$ is small. We further have $a|a\alpha+b| < 3\varepsilon N|a\alpha+b| < 3\varepsilon(1+\varepsilon)$ by (7), and hence $a|a\alpha+b| < \frac{1}{2}$ if ε is small. It follows from a well known theorem (e.g. Theorem 184 of [4]) that $-b/a$ is a convergent to α, say $-b/a = p_h/q_h$. By (6), the numbers a, b are coprime, and by (7) we have $a\alpha+b<0$. Hence $a=q_h$, $b=-p_h$, and h is odd. Similarly from (8) we obtain that $c|c\alpha+d| < (1+\varepsilon)N|c\alpha+d| < (1+\varepsilon)(y+\varepsilon) < \frac{1}{2}$ if ε is small, since $0<y<\frac{1}{2}$. Thus also $-d/c$ is a convergent of α, say $-d/c = p_k/q_k$. Using (6) and (8) one sees that $c=q_k$, $d=-p_k$, and that k is even. One sees from (7), (8) that

$$|q_k\alpha-p_k| = |c\alpha+d| < |a\alpha+b| = |q_h\alpha-p_h|,$$

since $0<y<\frac{1}{2}$ and since ε is small, and therefore one has $k>h$. Finally, $q_kp_h-q_hp_k = ad-bc = 1$ implies that $h=k-1$. For otherwise we would have $h<k-1$ and

$$\begin{aligned}(q_hq_k)^{-1} &= (p_h/q_h)-(p_k/q_k) > (p_h/q_h)-(p_{k-1}/q_{k-1})\\ &\geqslant (q_hq_{k-1})^{-1} > (q_hq_k)^{-1},\end{aligned}$$

a contradiction. Altogether we have

$$(9)\qquad a=q_{k-1}, \quad b=-p_{k-1}, \quad c=q_k, \quad d=-p_k.$$

The inequalities (7), (8) imply that

$$(10)\qquad q_{k-1}/q_k = a/c < 3\varepsilon(1-\varepsilon)^{-1} < 4\varepsilon$$

if ε is small. We also have

$$\begin{aligned}|q_k(\alpha q_k-p_k)-y| &= |c(\alpha c+d)-y| \leqslant |N(\alpha c+d)-y|+|N-c||\alpha c+d|\\ &< \varepsilon+N\varepsilon(y+\varepsilon)N^{-1} < 3\varepsilon.\end{aligned}$$

But by [4], § 10.9,

$$q_k(\alpha q_k-p_k) = q_k(-1)^k(\alpha'_{k+1}q_k+q_{k-1})^{-1} = \big(\alpha'_{k+1}+(q_{k-1}/q_k)\big)^{-1}.$$

Thus $\left|\big(\alpha'_{k+1}+(q_{k-1}/q_k)\big)^{-1}-y\right| < 3\varepsilon$, whence $|\alpha'_{k+1}+(q_{k-1}/q_k)-y^{-1}| < 4y^{-2}\varepsilon$ if $\varepsilon>0$ is small, and using this together with (10) we obtain

$$|\alpha'_{k+1}-y^{-1}| < 10y^{-2}\varepsilon.$$

Since $\varepsilon>0$ was arbitrarily small, the sequence $\alpha'_1, \alpha'_2, \ldots$ comes arbitrarily close to y^{-1}. Since y was arbitrary in $0<y<\frac{1}{2}$, the sequence is everywhere dense on the half line $x>2$. Since $\alpha'_k = a_k+(\alpha'_{k+1})^{-1}$, the sequence $\alpha'_1, \alpha'_2, \ldots$ is in fact dense on $x>1$. From this it follows easily that every block of positive integers occurs infinitely often among $a_1, a_2, \ldots$

3. The sufficiency of the continued fraction condition.

LEMMA 1. *Suppose every block of positive integers occurs infinitely often among $a_1, a_2, \ldots$ Then the points*

$$(q_n/q_{n-1}, \alpha'_{n+1}) \qquad (n=2,4,6,\ldots)$$

are everywhere dense in the quadrant $x>1, y>1$ of the plane. The same is true with $n=3,5,7,\ldots$

Proof. Let $x>1$, $y>1$, and suppose $\varepsilon>0$ is small. There are integers $b_0, b_1, b_2, \ldots, b_s$ such that every number $x' = [b_0, b_1, \ldots, b_s, b_{s+1}, \ldots, b_t]$ with arbitrary t and $b_{s+1}, \ldots, b_t$ satisfies $|x'-x|<\varepsilon$. There are integers $c_0, c_1, \ldots, c_r$ such that every number $y' = [c_0, c_1, \ldots, c_r, c_{r+1}, \ldots]$ with arbitrary $c_{r+1}, \ldots$ satisfies $|y'-y|<\varepsilon$. Now suppose n is large and such that

$$(11)\qquad \begin{aligned}&a_{n-s}=b_s, \ \ldots, \ a_{n-1}=b_1, \ a_n=b_0,\\ &a_{n+1}=c_0, \ a_{n+2}=c_1, \ \ldots, \ a_{n+r+1}=c_r.\end{aligned}$$

Since $q_n/q_{n-1} = [a_n, a_{n-1}, \ldots, a_1]$ ([5], § 11), we then have $|(q_n/q_{n-1})-x| < \varepsilon$, and similarly we have $|\alpha'_{n+1}-y|<\varepsilon$. But (11) happens for infinitely many values of n. Since every block of integers occurs in $a_1, a_2, \ldots$, there are in fact infinitely many values of n for which (11) holds both for n and for $n' = n+2r+2s-1$. But n, n' have opposite parity, and hence there will in fact be infinitely many even as well as infinitely many odd n with (11). This proves the lemma.

We now have to show that for any two points $\boldsymbol{x}_1, \boldsymbol{x}_2$ with $\Delta(\boldsymbol{x}_1, \boldsymbol{x}_2) = 1$, there are lattice points $\boldsymbol{h}_1, \boldsymbol{h}_2$ in some lattice $\Lambda(\alpha; N)$ with $|\boldsymbol{h}_i-\boldsymbol{x}_i| < \varepsilon$ $(i=1,2)$. We lose no generality by restricting ourselves to points $\boldsymbol{x}_1, \boldsymbol{x}_2$ which span a lattice Λ which has no points on the coordinate axes except the origin. Let $\boldsymbol{y}_1 = (x_1, y_1)$ be a *minimal point* in Λ, i.e. assume that $\boldsymbol{y}_1 \neq \boldsymbol{0}$ and that there is no point $(x'_1, y'_1) \neq \boldsymbol{0}$ in Λ with $|x'_1|<|x_1|$, $|y'_1|<|y_1|$. By replacing $\boldsymbol{y}_1$ by $-\boldsymbol{y}_1$ if necessary, we may assume that $x_1>0$. Let $\boldsymbol{y}_2 = (x_2, y_2) \neq \boldsymbol{0}$ be a point with $|x_2|<x_1$ and with $|y_2|$ as small as possible. Then $\boldsymbol{y}_2$ is again a minimal point, and in fact there is no point $(x, y) \neq \boldsymbol{0}$ with

$$(12)\qquad |x|<|x_1|, \quad |y|<|y_2|.$$

We may assume that $x_2>0$. The point $(x, y) = (x_1-x_2, y_1-y_2)$ has $0<x<x_1$, and hence by the impossibility of (12) it has $|y| = |y_1-y_2| \geqslant |y_2|$. Since $|y_1|<|y_2|$, this implies that y_1, y_2 are of opposite sign.

Since there is no nonzero point in the region defined by (12), the triangle $\mathbf{0}, \mathbf{y}_1, \mathbf{y}_2$ contains no lattice points but its vertices, and

$$\Delta(\mathbf{y}_1, \mathbf{y}_2) = x_1 y_2 - x_2 y_1 = \pm 1.$$

One has

$$|y_1|(x_1+x_2) < |y_1|x_2 + |y_2|x_1 = |y_1 x_2 - y_2 x_1| = 1,$$

and therefore

$$\frac{1}{|x_1 y_1|} - \frac{x_2}{x_1} = \frac{1 - x_2|y_1|}{|x_1 y_1|} > 1.$$

It will suffice to find points $\mathbf{f}_1, \mathbf{f}_2$ of $\Lambda(\alpha; N)$ which are close to $\mathbf{y}_1, \mathbf{y}_2$, respectively. For since $\mathbf{y}_1, \mathbf{y}_2$ form a basis of Λ, we have $\mathbf{x}_i = c_{i1}\mathbf{y}_1 + c_{i2}\mathbf{y}_2$ $(i = 1, 2)$, and if $\mathbf{f}_1, \mathbf{f}_2$ are close to $\mathbf{y}_1, \mathbf{y}_2$, then $\mathbf{h}_i = c_{i1}\mathbf{f}_1 + c_{i2}\mathbf{f}_2$ is close to $\mathbf{x}_i$ $(i = 1, 2)$. From here on, $\mathbf{y}_1, \mathbf{y}_2$ will be fixed. Now choose n even if $y_1 > 0$, and n odd if $y_1 < 0$, and such that

$$|(q_n/q_{n-1}) - (x_1/x_2)| < \delta, \qquad \left|a'_{n+1} - \left(\frac{1}{|x_1 y_1|} - \frac{x_2}{x_1}\right)\right| < \delta,$$

where δ is some small positive quantity. Let N be an integer with $|Nx_2 - q_{n-1}| < |x_2|$. Then

$$|(q_{n-1}/N) - x_2| < \delta \tag{13}$$

if n and hence N is large. We also have

$$|Nx_1 - q_n| = \left|\frac{x_1}{x_2}(Nx_2 - q_{n-1}) + q_{n-1}\left(\frac{x_1}{x_2} - \frac{q_n}{q_{n-1}}\right)\right| < x_1 + q_{n-1}\delta \ll N\delta,$$

whence

$$|(q_n/N) - x_1| \ll \delta. \tag{14}$$

(The constants in $\ll$ depend only on $\mathbf{y}_1, \mathbf{y}_2$.) We note that by a formula in [4], § 10.9,

$$\begin{aligned}|N(\alpha q_n - p_n) - y_1| &\leqslant \left|\frac{q_n}{x_1}(\alpha q_n - p_n) - y_1\right| + \left|\frac{q_n}{x_1} - N\right| |\alpha q_n - p_n| \\ &\ll x_1^{-1}\left|\frac{(-1)^n q_n}{a'_{n+1} q_n + q_{n-1}} - x_1 y_1\right| + \delta N(a'_{n+1} q_n + q_{n-1})^{-1} \\ &\ll \left|(a'_{n+1} + (q_{n-1}/q_n))^{-1} - |x_1 y_1|\right| + \delta.\end{aligned}$$

But

$$\begin{aligned}&|a'_{n+1} + (q_{n-1}/q_n) - |x_1 y_1|^{-1}| \\ &\qquad \leqslant |a'_{n+1} - (|x_1 y_1|^{-1} - (x_2/x_1))| + |(q_{n-1}/q_n) - (x_2/x_1)| \ll \delta,\end{aligned}$$

and therefore

$$|N(\alpha q_n - p_n) - y_1| \ll \delta. \tag{15}$$

Putting

$$\mathbf{f}_1 = q_n \mathbf{g}_1 - p_n \mathbf{g}_2 = (q_n N^{-1}, q_n \alpha N - p_n N) = (a_1, b_1), \text{ say,}$$

$$\mathbf{f}_2 = q_{n-1} \mathbf{g}_1 - p_{n-1} \mathbf{g}_2 = (q_{n-1} N^{-1}, q_{n-1} \alpha N - p_{n-1} N) = (a_2, b_2), \text{ say,}$$

we have

$$|a_1 - x_1| \ll \delta, \quad |a_2 - x_2| \ll \delta, \quad |b_1 - y_1| \ll \delta$$

by (13), (14) and (15). Since

$$a_1 b_2 - a_2 b_1 = -(q_n p_{n-1} - p_n q_{n-1}) = (-1)^{n-1} = x_1 y_2 - x_2 y_1,$$

it follows that also $|b_2 - y_2| \ll \delta$. Hence we have $|\mathbf{f}_i - \mathbf{y}_i| < \varepsilon$ $(i = 1, 2)$ provided $0 < \delta < \delta(\varepsilon)$.

4. The method of proof of Theorem 2. We shall restrict ourselves to the case when $n = 2$, $l = 3$. Throughout the proof, $\mathbf{x}, \mathbf{y}, \ldots$ will denote points in 3-dimensional space. We shall write $\Lambda(\alpha, \beta; N)$ instead of $\Lambda(\alpha_1, \alpha_2; N)$.

Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be points with $\Delta(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = 1$. Further let $T(N; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \varepsilon)$ consist of all pairs (α, β) for which the lattice $\Lambda(\alpha, \beta; N)$ contains points $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ with $|\mathbf{h}_i - \mathbf{x}_i| < \varepsilon$ $(i = 1, 2, 3)$.

PROPOSITION. *There is a* $\theta = \theta(x_1, x_2, x_3; \varepsilon) > 0$ *such that for every square* Q *of the type*

$$|\alpha - \alpha_0| < \eta, \quad |\beta - \beta_0| < \eta \tag{16}$$

and every $N > N_0(Q; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \varepsilon)$ *the intersection of* Q *with* $T(N) = T(N; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \varepsilon)$ *has measure*

$$\mu(Q \cap T(N)) \geqslant \theta\mu(Q) = \theta 4\eta^2. \tag{17}$$

Thus the complement of

$$T(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \varepsilon) = \bigcup_{k=1}^{\infty} T(N_k; \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \varepsilon)$$

has density $\leqslant 1 - \theta < 1$ everywhere. Since a measurable set has density 1 at almost all of its points, the complement of $T(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \varepsilon)$ has measure zero, and almost every point (α, β) belongs to $T(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \varepsilon)$. Since this is true for every $\varepsilon > 0$ and every $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ with determinant 1, Theorem 2 follows. It remains to prove the proposition.

5. The set $\Sigma(N)$. Write $\mathbf{x}_i = (\xi_{i1}, \xi_{i2}, \xi_{i3})$ $(i = 1, 2, 3)$. We may assume that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ satisfy only the equation $\Delta(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = 1$ and equations implied by it, i.e. that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is a generic point of the surface in 9-dimensional space defined by $\Delta(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = 1$. From now on, $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ will be fixed. The constants in $\ll$ may depend on $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ and on δ, but they will be independent of N and of squares Q.

Let $\Sigma(N) = \Sigma(N; \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3; \delta)$ consist of all pairs (α, β) for which the lattice $\Lambda(\alpha, \beta; N)$ contains points $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ with

(18) $$\Delta(\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3) = 1$$

and with

(19) $$|\boldsymbol{h}_1-\boldsymbol{x}_1| < \delta, \quad |\boldsymbol{h}_2-\boldsymbol{x}_2| < \delta, \quad |h_{31}-\xi_{31}| < \delta, \quad |h_{32}-\xi_{32}| < \delta$$

where $\boldsymbol{h}_3 = (h_{31}, h_{32}, h_{33})$. Since $\Delta(\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3) = \Delta(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = 1$, the eight inequalities implicit in (19) imply a ninth one, namely $|h_{33}-\xi_{33}| \ll \delta$. Hence if δ is sufficiently small in relation to ε, then $|\boldsymbol{h}_i-\boldsymbol{x}_i| < \varepsilon$ $(i = 1, 2, 3)$, and $\Sigma(N; \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3; \delta)$ is contained in $T(N; \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3; \varepsilon)$. *Hence it will suffice to prove the proposition above with ε replaced by δ and $T(N)$ replaced by $\Sigma(N)$.* It will suffice to prove the proposition for $0 < \delta < \delta_0$, where $\delta_0 = \delta_0(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ is arbitrarily small.

Recall that $\Lambda(\alpha, \beta; N)$ has the basis

(20) $$\boldsymbol{g}_1 = (N^{-1}, 0, \alpha N^2), \quad \boldsymbol{g}_2 = (0, N^{-1}, \beta N^2), \quad \boldsymbol{g}_3 = (0, 0, N^2).$$

Any three points, $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ of $\Lambda(\alpha, \beta; N)$ may be written as

(21) $$\begin{aligned} \boldsymbol{h}_1 &= q_{11}\boldsymbol{g}_1+q_{12}\boldsymbol{g}_2+q_{13}\boldsymbol{g}_3, \\ \boldsymbol{h}_2 &= q_{21}\boldsymbol{g}_1+q_{22}\boldsymbol{g}_2+q_{23}\boldsymbol{g}_3, \\ \boldsymbol{h}_3 &= q_{31}\boldsymbol{g}_1+q_{32}\boldsymbol{g}_2+q_{33}\boldsymbol{g}_3 \end{aligned}$$

with integer coefficients q_{ij}. For given integer points $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ with $\boldsymbol{q}_i = (q_{i1}, q_{i2}, q_{i3})$ $(i = 1, 2, 3)$, let $F(N; \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3; \delta)$ be the set of pairs (α, β) for which $\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3$ as given by (20) and (21) satisfy (18) and (19). (F also depends on $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$, but these points are fixed.)

Now $\Delta(\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3) = 1$ is equivalent with

(22) $$\Delta(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3) = 1$$

and six of the eight inequalities implicit in (19) are equivalent with

(23) $$|q_{i1}-N\xi_{i1}| < N\delta, \quad |q_{i2}-N\xi_{i2}| < N\delta \quad (i = 1, 2, 3).$$

Thus $F(N; \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3; \delta)$ is empty unless (22) and (23) hold. But if these inequalities do hold, then (α, β) lies in $F(N; \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3; \delta)$ precisely if

(24) $$\begin{aligned} |q_{11}\alpha+q_{12}\beta+q_{13}-\xi_{13}N^{-2}| &< \delta N^{-2}, \\ |q_{21}\alpha+q_{22}\beta+q_{23}-\xi_{23}N^{-2}| &< \delta N^{-2}. \end{aligned}$$

(These are the remaining two inequalities of (19).) Hence in this case $F(N; \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3; \delta)$ is the parallelogram $E(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)$ defined by (24). (Since $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ are generic, and by (23), we have $q_{11}q_{22}-q_{12}q_{21} \neq 0$ if $\delta > 0$ is sufficiently small, which we may assume.)

Lemma 2. *Suppose $\delta > 0$ is sufficiently small, and the integer points $\boldsymbol{q}_1, \boldsymbol{q}_2$ satisfy* (23) *for $i = 1, 2$. Then $E(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)$ has area*

$$\mu\big(E(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)\big) \gg N^{-6}$$

and diameter

$$d\big(E(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)\big) \ll N^{-3}.$$

Proof. This is Lemma 2 of [3].

Lemma 3. *Suppose N is large and suppose that integer points $\boldsymbol{q}_1, \boldsymbol{q}_2$ satisfy* (23) *with $i = 1, 2$ and*

(25) $$\left|\begin{vmatrix} q_{12} & q_{13} \\ q_{22} & q_{23} \end{vmatrix} \Big/ \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} - \alpha_0\right| < \eta/4, \qquad \left|\begin{vmatrix} q_{13} & q_{11} \\ q_{23} & q_{21} \end{vmatrix} \Big/ \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} - \beta_0\right| < \eta/4.$$

Then $E(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)$ is contained in the square Q defined by (16).

Proof. This is Lemma 3 of [3].

Now let $E^*(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)$ be the parallelogram of points (α, β) which satisfy (24) with ξ_{13}, ξ_{23} replaced by zero. Now if (23) holds, then $|q_{11}q_{22}-q_{12}q_{21}| \gg N^2$, and $E(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)$ is obtained from $E^*(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)$ by translation by a vector of length $O(N^{-3})$.

Lemma 4. *Suppose $\boldsymbol{q}_1, \boldsymbol{q}_2$ are part of a basis and satisfy* (23) *for $i = 1, 2$. Make the same assumptions on $\boldsymbol{q}_1', \boldsymbol{q}_2'$. Then if $(\boldsymbol{q}_1, \boldsymbol{q}_2) \neq (\boldsymbol{q}_1', \boldsymbol{q}_2')$, the parallelograms $E^*(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)$ and $E^*(N; \boldsymbol{q}_1', \boldsymbol{q}_2'; \delta)$ are disjoint.*

Proof. This is Lemma 4 of [3].

Lemma 5. *Suppose N is large. Then a point (α, β) lies in $\ll 1$ parallelograms $E(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)$ with $\boldsymbol{q}_1, \boldsymbol{q}_2$ part of a basis and satisfying* (23) *with $i = 1, 2$.*

Proof. This is Lemma 5 of [3].

6. The number of certain integer points. Let $Z(N)$ be the set of triples of integer points $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ with (22), (23) and (25). Suppose that $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ and $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3'$ lie in $Z(N)$. Then by (22) we have $\boldsymbol{q}_3' = \boldsymbol{q}_3 + u\boldsymbol{q}_1 + v\boldsymbol{q}_2$, with integer coefficients u, v. By (23) we have

$$|q_{31}-N\xi_{31}| < N\delta \quad \text{and} \quad |q_{31}'-N\xi_{31}| < N\delta,$$

whence $|q'_{31}-q_{31}| = |uq_{11}+vq_{21}| < 2N\delta$. In the same manner one finds that $|uq_{21}+vq_{22}| < 2N\delta$. Now by (23) again one has

$$\max(|q_{11}|, |q_{12}|, |q_{21}|, |q_{22}|) \ll N \quad \text{and} \quad |q_{11}q_{22}-q_{12}q_{21}| \gg N^2,$$

and hence u, v satisfy $|u| \ll \delta$, $|v| \ll \delta$. Hence if δ is sufficiently small we have $|u| < 1$, $|v| < 1$, whence $u = v = 0$, whence $\boldsymbol{q}'_3 = \boldsymbol{q}_3$.

We have shown that if $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ and $\boldsymbol{q}'_1, \boldsymbol{q}'_2, \boldsymbol{q}'_3$ are distinct triples in $Z(N)$, then already the pairs $\boldsymbol{q}_1, \boldsymbol{q}_2$ and $\boldsymbol{q}'_1, \boldsymbol{q}'_2$ are distinct.

Let $z(N)$ denote the number of elements of $Z(N)$. By Lemma 3, the set $Q \cap \Sigma(N)$ contains at least $z(N)$ parallelograms $E(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)$ where $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ lie in $Z(N)$. But these parallelograms need not be disjoint. By what we just said, the pairs $\boldsymbol{q}_1, \boldsymbol{q}_2$ are all distinct here. Hence by Lemma 5, any given point (α, β) lies in $\ll 1$ of these parallelograms. Since $E(N; \boldsymbol{q}_1, \boldsymbol{q}_2; \delta)$ has area $\mu(E) \gg N^{-6}$ by Lemma 2, we obtain

$$\mu\big(Q \cap \Sigma(N)\big) \gg N^{-6} z(N).$$

Therefore to prove (17) and thus Theorem 2 it will suffice to show that

$$z(N) \gg \eta^2 N^6. \tag{26}$$

7. Some further lemmas.

LEMMA 6. *Let S be a bounded Jordan measurable set in 6-dimensional space. Then as $t \to \infty$, the number of integer points $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ in tS such that $\boldsymbol{x}_1, \boldsymbol{x}_2$ is part of a basis of the integer lattice in 3-dimensional space is asymptotically equal to*

$$t^6 V(S)\big(\zeta(3)\zeta(2)\big)^{-1}. \tag{27}$$

Proof. This is the case $m = 2$, $l = 3$ of Theorem 4 in [3].

LEMMA 7. *Suppose $0 < \varepsilon < 1$ and $\boldsymbol{l} = (l_1, l_2, l_3)$ are given. There is a basis $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$ of the integer lattice such that every point $\boldsymbol{x}$ with*

$$\boldsymbol{x} = u_1\boldsymbol{r}_1 + u_2\boldsymbol{r}_2 + u_3\boldsymbol{r}_3 \tag{28}$$

where

$$|u_3| \leqslant |u_1| + |u_2| \tag{29}$$

satisfies

$$|u_3\boldsymbol{r}_3| \leqslant \varepsilon|\boldsymbol{x}| \tag{30}$$

and

$$|\boldsymbol{l}\boldsymbol{x}| = |l_1x_1 + l_2x_2 + l_3x_3| \leqslant \varepsilon|\boldsymbol{x}|. \tag{31}$$

Proof. We may assume that $\boldsymbol{l} \neq \boldsymbol{0}$, and in fact we may assume that $|l_1| + |l_2| + |l_3| = 1$. The equation $\boldsymbol{l}\boldsymbol{x} = \boldsymbol{0}$ defines a plane P in E^3. Let

$\boldsymbol{z}_1, \boldsymbol{z}_2$ be two nonzero orthogonal points on P, and let $\varrho > 0$ be small. Let S be the set of points $\boldsymbol{X} = (\boldsymbol{x}_1, \boldsymbol{x}_2)$ in E^6 with

$$|\boldsymbol{x}_1 - \boldsymbol{z}_1| < \varrho, \quad |\boldsymbol{x}_2 - \boldsymbol{z}_2| < \varrho.$$

Lemma 6 tells us that for sufficiently large t there will be points $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ in tS such that $\boldsymbol{x}_1, \boldsymbol{x}_2$ is part of a basis of the integer lattice. Let $(\boldsymbol{x}_1, \boldsymbol{x}_2)$ be such a point, and choose $\boldsymbol{x}_3$ such that $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ is a basis. Now let v be a large integer and put

$$\boldsymbol{r}_1 = v\boldsymbol{x}_1 + \boldsymbol{x}_2, \quad \boldsymbol{r}_2 = v\boldsymbol{x}_2 + \boldsymbol{x}_3, \quad \boldsymbol{r}_3 = \boldsymbol{x}_1.$$

Then $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$ are again a basis of the integer lattice.

Now $\boldsymbol{z}_1, \boldsymbol{z}_2$ were orthogonal, and if ϱ is sufficiently small, the points $\boldsymbol{x}_1, \boldsymbol{x}_2$ will be "almost orthogonal", and if v is sufficiently large, the points $\boldsymbol{r}_1, \boldsymbol{r}_2$ will be "almost orthogonal". To make this precise, we may ascertain that the angle between $\boldsymbol{r}_1, \boldsymbol{r}_2$ lies between $\pi/3$ and $2\pi/3$, say. Then

$$|u_1\boldsymbol{r}_1 + u_2\boldsymbol{r}_2| \geqslant c_0(|u_1| + |u_2|)\min(|\boldsymbol{r}_1|, |\boldsymbol{r}_2|),$$

where $c_0 > 0$ is an absolute constant, and (29) implies that $|u_3\boldsymbol{r}_3| \leqslant (|u_1| + |u_2|)|\boldsymbol{r}_3|$. Now $\min(|\boldsymbol{r}_1|, |\boldsymbol{r}_2|)$ becomes arbitrarily large for large v, while $|\boldsymbol{r}_3|$ is independent of v. Thus for large v we have

$$|u_3\boldsymbol{r}_3| \leqslant \tfrac{1}{2}\varepsilon|u_1\boldsymbol{r}_1 + u_2\boldsymbol{r}_2|,$$

and hence the point $\boldsymbol{x}$ given by (28) satisfies (30). Also

$$\boldsymbol{x} = u_1(v\boldsymbol{x}_1 + \boldsymbol{x}_2) + u_2(v\boldsymbol{x}_2 + \boldsymbol{x}_3) + u_3\boldsymbol{x}_1 = \boldsymbol{p} + \boldsymbol{y}$$

where

$$\boldsymbol{p} = vt(u_1\boldsymbol{z}_1 + u_2\boldsymbol{z}_2),$$

$$\boldsymbol{y} = vu_1(\boldsymbol{x}_1 - t\boldsymbol{z}_1) + vu_2(\boldsymbol{x}_2 - t\boldsymbol{z}_2) + u_1\boldsymbol{x}_2 + u_2\boldsymbol{x}_3 + u_3\boldsymbol{x}_1.$$

Here $\boldsymbol{p}$ lies in the plane P, and $|\boldsymbol{p}| \geqslant vtc_0(|u_1| + |u_2|)$. On the other hand $|\boldsymbol{x}_i - t\boldsymbol{z}_i| < t\varrho$ $(i = 1, 2)$, whence $|\boldsymbol{y}| \ll vt\varrho(|u_1| + |u_2|) + t(|u_1| + |u_2|)$. Thus if ϱ is sufficiently small and if v is sufficiently large, then $|\boldsymbol{y}| \leqslant \frac{1}{2}\varepsilon|\boldsymbol{p}|$. But this yields (31), since

$$|\boldsymbol{l}\boldsymbol{x}| = |\boldsymbol{l}\boldsymbol{y}| \leqslant |\boldsymbol{y}| \leqslant \varepsilon|\boldsymbol{p}| - |\boldsymbol{y}| \leqslant \varepsilon(|\boldsymbol{p}| - |\boldsymbol{y}|) \leqslant \varepsilon|\boldsymbol{x}|.$$

LEMMA 8. *Suppose $\varepsilon, \boldsymbol{l} = (l_1, l_2, l_3)$ are as in Lemma 7, and assume that $l_3 \neq 0$. There is a basis of the integer lattice such that the conclusions of Lemma 7 are valid with (30) replaced by*

$$|u_3\boldsymbol{r}_3| \leqslant \varepsilon(|x_1| + |x_2|), \tag{32}$$

and such that $r_{11}r_{22} - r_{12}r_{21} \neq 0$.

Proof. Since $l_3 \neq 0$, one may choose z_1, z_2 in the proof of Lemma 7 such that $z_{11}z_{22}-z_{12}z_{21} \neq 0$. There is a constant $c_1 > 0$ such that for arbitrary u_1, u_2 one has

$$|u_1 z_{11}+u_2 z_{21}|+|u_1 z_{12}+u_2 z_{22}| \geqslant c_1(|u_1|+|u_2|).$$

Now if ϱ is small and if v is large, the points r_1/vt and r_2/vt will be arbitrarily close to z_1, z_2, respectively. Thus one will have $r_{11}r_{22}-r_{12}r_{21} \neq 0$ and

$$|u_1 r_{11}+u_2 r_{21}|+|u_1 r_{12}+u_2 r_{22}| \geqslant \frac{c_1}{2}(|u_1|+|u_2|)vt.$$

Hence

$$|x_1|+|x_2| = |u_1 r_{11}+u_2 r_{21}+u_3 r_{31}|+|u_1 r_{12}+u_2 r_{22}+u_3 r_{32}|$$
$$\geqslant \frac{c_1}{2}(|u_1|+|u_2|)vt-2\varepsilon|\boldsymbol{x}|$$

by (30). Since $|\boldsymbol{x}| \leqslant c_2 vt(|u_1|+|u_2|)$, we obtain $|x_1|+|x_2| \geqslant c_3|\boldsymbol{x}|$ if $\varepsilon > 0$ is small. In conjunction with (30) this gives

$$|u_3 \boldsymbol{r}_3| \leqslant \varepsilon c_3^{-1}(|x_1|+|x_2|).$$

Since $\varepsilon > 0$ was arbitrary, the lemma follows.

8. A lower bound for $z(N)$. There are numbers l_1, l_2, l_3, not all zero, with

$$l_1\xi_{11}+l_2\xi_{21}+l_3\xi_{31} = 0, \quad l_1\xi_{12}+l_2\xi_{22}+l_3\xi_{32} = 0.$$

In fact, since $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ were generic, the number $l_3 \neq 0$. The inequalities

$$(33) \qquad |q_{i1}-N\xi_{i1}| < N\delta/2, \quad |q_{i2}-N\xi_{i2}| < N\delta/2 \quad (i = 1, 2)$$

are stronger than the cases $i = 1, 2$ of (23). There exists an $\varepsilon = \varepsilon(\delta) > 0$ such that (33) together with

$$(34) \qquad \begin{aligned} |l_1 q_{11}+l_2 q_{21}+l_3 q_{31}| &< \varepsilon\max(|q_{11}|, |q_{21}|, |q_{31}|), \\ |l_1 q_{12}+l_2 q_{22}+l_3 q_{32}| &< \varepsilon\max(|q_{12}|, |q_{22}|, |q_{32}|) \end{aligned}$$

implies (23) for $i = 1, 2, 3$.

Putting $\boldsymbol{l} = (l_1, l_2, l_3)$ and

$$(35) \qquad \boldsymbol{q}_1 = (q_{11}, q_{21}, q_{31}), \quad \boldsymbol{q}_2 = (q_{12}, q_{22}, q_{32}), \quad \boldsymbol{q}_3 = (q_{13}, q_{23}, q_{33}),$$

we may rewrite the inequalities (34) as

$$(36) \qquad |\boldsymbol{l}\boldsymbol{q}_1| < \varepsilon|\boldsymbol{q}_1|, \quad |\boldsymbol{l}\boldsymbol{q}_2| < \varepsilon|\boldsymbol{q}_2|.$$

Let $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$ be the basis of Lemma 8. We may write

$$(37) \qquad \begin{aligned} \boldsymbol{q}_1 &= u_{11}\boldsymbol{r}_1+u_{21}\boldsymbol{r}_2+u_{31}\boldsymbol{r}_3, \\ \boldsymbol{q}_2 &= u_{12}\boldsymbol{r}_1+u_{22}\boldsymbol{r}_2+u_{32}\boldsymbol{r}_3, \\ \boldsymbol{q}_3 &= u_{13}\boldsymbol{r}_1+u_{23}\boldsymbol{r}_2+u_{33}\boldsymbol{r}_3, \end{aligned}$$

with integer coefficients u_{ij}. By (31) of Lemma 7 and 8, the inequalities (36) will be satisfied provided (29) holds, i.e. provided

$$(38) \qquad |u_{3i}| \leqslant |u_{1i}|+|u_{2i}| \quad (i = 1, 2, 3)$$

holds for $i = 1, 2$. Define points

$$(39) \qquad \boldsymbol{q}_1' = (q_{11}', q_{21}', q_{31}'), \quad \boldsymbol{q}_2' = (q_{12}', q_{22}', q_{32}'), \quad \boldsymbol{q}_3' = (q_{13}', q_{23}', q_{33}')$$

by

$$(40) \qquad \begin{aligned} \boldsymbol{q}_1' &= u_{11}\boldsymbol{r}_1+u_{21}\boldsymbol{r}_2, \\ \boldsymbol{q}_2' &= u_{12}\boldsymbol{r}_1+u_{22}\boldsymbol{r}_2, \\ \boldsymbol{q}_3' &= u_{13}\boldsymbol{r}_1+u_{23}\boldsymbol{r}_2. \end{aligned}$$

By (32) of Lemma 8 we have

$$|\boldsymbol{q}_i-\boldsymbol{q}_i'| \leqslant \varepsilon(|q_{1i}|+|q_{2i}|) \quad (i = 1, 2, 3)$$

provided (38) holds. Thus (38) implies that

$$|q_{ji}-q_{ji}'| \leqslant \varepsilon(|q_{1i}|+|q_{2i}|) \quad (i, j = 1, 2, 3).$$

Thus if $\varepsilon > 0$ is sufficiently small and if (38) holds, then

$$(41) \qquad |q_{i1}'-N\xi_{i1}| < N\delta/4, \quad |q_{i2}'-N\xi_{i2}| < N\delta/4 \quad (i = 1, 2)$$

will imply (33). Similarly, (38), (41) together with

$$(42) \qquad \begin{aligned} &\left|\begin{vmatrix} q_{12}' & q_{13}' \\ q_{22}' & q_{23}' \end{vmatrix} \Big/ \begin{vmatrix} q_{11}' & q_{12}' \\ q_{21}' & q_{22}' \end{vmatrix} - \alpha_0\right| < \eta/8, \\ &\left|\begin{vmatrix} q_{13}' & q_{11}' \\ q_{23}' & q_{21}' \end{vmatrix} \Big/ \begin{vmatrix} q_{11}' & q_{12}' \\ q_{21}' & q_{22}' \end{vmatrix} - \beta_0\right| < \eta/8 \end{aligned}$$

will imply (25).

Thus $z(N) \geqslant z'(N)$, where $z'(N)$ is the number of integer bases $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ with (38) such that the quantities q_{ij}' defined by (39) and (40) satisfy (41) and (42). The inequalities (41) and (42) with $N = 1$ define a bounded set in 6-dimensional space for $(q_{11}', q_{12}', q_{13}', q_{21}', q_{22}', q_{23}')$. This set has volume $\gg \eta^2$. Now $(q'_{11}, q'_{12}, q'_{13}, q'_{21}, q'_{22}, q'_{23})$ is related to $(\boldsymbol{u}_1, \boldsymbol{u}_2) = (u_{11}, u_{12}, u_{13}, u_{21}, u_{22}, u_{23})$ be the linear transformation (40) of determinant $(r_{11}r_{22}-r_{12}r_{21})^3 \neq 0$. Hence (41) and (42) with $N = 1$ together with (39) and (40) define a bounded set for $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ in 6-dimensional space of volume $\gg \eta^2$. For arbitrary N we obtain the same set but blown up by the factor N. Hence by Lemma 6 there are $\gg \eta^2 N^6$ pairs of points $\boldsymbol{u}_1, \boldsymbol{u}_2$ which are part of a basis such that (41) and (42) are satisfied. There still are $\gg \eta^2 N^6$ such pairs $\boldsymbol{u}_1, \boldsymbol{u}_2$ all of whose components are different from zero.

It remains to be shown that for every such $\boldsymbol{u}_1, \boldsymbol{u}_2$ one can find a third basis vector $\boldsymbol{u}_3$ such that (38) holds. There certainly will be such a vector $\boldsymbol{u}_3$ of the type $\boldsymbol{u}_3 = \lambda_1\boldsymbol{u}_1+\lambda_2\boldsymbol{u}_2+\boldsymbol{u}_0$, where $|\lambda_j| \leqslant \frac{1}{2}$ $(j = 1, 2)$ and where $\boldsymbol{u}_0$ is the point with $\Delta(\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_0) = 1$ which is orthogonal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. It is easy to see that the coordinates of $\boldsymbol{u}_0$ have absolute values at most 1, and hence

$$|u_{0i}| \leqslant \tfrac{1}{2}|u_{1i}|+\tfrac{1}{2}|u_{i2}|+1 \leqslant |u_{1i}|+|u_{2i}| \qquad (i = 1, 2, 3),$$

since we made sure that $u_{1i} \neq 0$, $u_{2i} \neq 0$. Thus our $\boldsymbol{u}_3$ does satisfy (38), and we have $z(N) \geqslant z'(N) \gg \eta^2 N^6$. This proves (26) and hence the theorem.

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UNIVERSITY OF COLORADO
Boulder, Colorado

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Bounds for solutions of diagonal inequalities

by

JANE PITMAN (Adelaide, South Australia)

In memory of H. Davenport

1. Introduction. In 1958 the following theorem was proved by Birch and Davenport [1]:

If $\lambda_1, \lambda_2, \ldots, \lambda_5$ are real numbers, not all of the same sign, such that $|\lambda_i| \geqslant 1$ for all i, then for any $\theta > 0$ the Diophantine inequality

$$|\lambda_1 x_1^2+\ldots+\lambda_5 x_5^2| < 1$$

has a solution in integers $x_1, \ldots, x_5$, not all zero, such that

$$|\lambda_1 x_1^2|+\ldots+|\lambda_5 x_5^2| < K_\theta |\lambda_1\lambda_2 \ldots \lambda_5|^{1+\theta}.$$

A corresponding theorem on solutions of the diagonal cubic inequality

$$|\lambda_1 x_1^3+\ldots+\lambda_9 x_9^3| < 1$$

such that

$$|\lambda_1 x_1^3|+\ldots+|\lambda_9 x_9^3| < K'_\theta |\lambda_1 \ldots \lambda_9|^{(3/2)+\theta}$$

was proved in Pitman and Ridout [7]. In this paper I obtain a similar theorem for the diagonal inequality

$$|\lambda_1 x_1^k+\ldots+\lambda_n x_n^k| < 1, \tag{1}$$

where k is an integer, $k \geqslant 4$, and $\lambda_1, \ldots, \lambda_n$ are not all of the same sign if k is even. By a *solution* of a Diophantine equation or inequality I shall always mean a solution in integers $x_1, \ldots, x_n$, not all zero.

For the case when the λ_i/λ_j are not all rational, Davenport and Heilbronn [4] found that the condition $n \geqslant 2^k+1$ is sufficient for the existence of infinitely many solutions of (1); later Davenport and Roth [5] showed that $n > ck\log k$ is sufficient if $k \geqslant 12$, and Danicic [2] showed that $n \geqslant 14$ is sufficient if $k = 4$.

In order to find bounds for solutions of (1) by analytic methods similar to those of [1] and [7], we must first deal independently with the