ACTA ARITHMETICA
XVIII (1971)

On the zeros of a class of L-functions

by

E. Fogels (Riga)

In memory of H. Davenport

Introduction

1. Let K be any quadratic field of the discriminant $\Delta \ll 1$, $\mathfrak{K}$ — classes of ideals defined either by (4) or by (4) and (5) (see below), $\chi_{\mathfrak{K}}$ — characters of classes $\mathfrak{K}$, q — any natural number >1, χ_q — Dirichlet characters $\bmod q$. Then there is a group G_1 of reduced classes of residues $l(\bmod q)$ formed by the residues of the ideal norms $N\mathfrak{a}$ with $(\mathfrak{a}, [q]) = 1$ and $\mathfrak{a}$ belonging to the principal class $\mathfrak{K}_1$. The group of characters of any Abelian group being isomorphic with the group itself (see [7], § 10), there is a group Γ_1 of characters χ_q corresponding to G_1. Let us introduce the function

$$\zeta(s, \chi_q, \chi_{\mathfrak{K}}) = \sum_{\mathfrak{a}} \frac{\chi_q(N\mathfrak{a})\chi_{\mathfrak{K}}(\mathfrak{a})}{N\mathfrak{a}^s} \quad (s = \sigma + it,\ \sigma > 1). \tag{1}$$

In § 2 we shall prove that (1) is a Hecke L-function with a character $\chi(\mathfrak{a}) \bmod [q]$. We denote by $N(a, T, \chi_q, \chi_{\mathfrak{K}})$ the number of zeros of the function (1) in the rectangle $(a \leqslant \sigma \leqslant 1,\ |t| \leqslant T)$, by $d(n)$ and $\varphi(q)$ the number of natural numbers dividing n and the number of reduced classes $\bmod q$, respectively. Let

$$\tau(\chi_q) = \sum_{1 \leqslant m \leqslant q} \chi_q(m) e^{2\pi i m/q} \tag{2}$$

and let $\sum_{\chi_q}'$ denote a sum over all characters of the group Γ_1 excluding the principal character χ_q^0. The aim of the present paper is the proof of the following

Theorem. *For all $a \in [1/2, 1]$ we have uniformly in $D \geqslant 2$, $M \geqslant 2$, $T \geqslant 2$*

$$\sum_{\chi_{\mathfrak{K}}} \sum_{\substack{q \leqslant M \\ d(q) \leqslant D}} \frac{1}{\varphi(q)} \sum_{\chi_q}' |\tau(\chi_q)|^2 N(a, T, \chi_q, \chi_{\mathfrak{K}}) \ll D^4 T (M^2 T)^{\frac{4(1-a)}{3-2a}} \log^{196} MT. \tag{3}$$

(3) is an analogous of a theorem due to Bombieri about the number of zeros of Dirichlet L-functions ([1], Theorem 5). With respect to M the right-hand side of (3) is in essential as small as in Bombieri's work, but it is larger with respect to T. Nevertheless it can be applied (see [6])(1) for proving a mean value theorem of Bombieri's type ([1], Theorem 4) but for primes which are ideal norms for a given class $\mathfrak{K}$ in any quadratic field.

Preliminaries

2. Lemma 1. *In any algebraic number field K (of degree $n \geqslant 2$) (1) is a Hecke L-function with character $\chi(\mathfrak{a}) \bmod [q]$.*

This has been stated in [4], § 4, yet the proof given there being insufficient, we are going to complete it here.

The integer ideals $\mathfrak{a}$ and $\mathfrak{b}$ are of the same class $\mathfrak{K}$ if there are integers $\alpha, \beta \in K$ such that

(4) $$\mathfrak{a}[\alpha] = \mathfrak{b}[\beta]$$

and

(5) $$\alpha \succ 0, \qquad \beta \succ 0$$

where $\xi \succ 0$ means that all the real conjugates (if any) of ξ are positive. Let further $\mathfrak{f} = [q]$ and $(\mathfrak{a}, \mathfrak{f}) = (\mathfrak{b}, \mathfrak{f}) = 1$. If besides (4) and (5)

(6) $$\alpha \equiv \beta \equiv 1 (\bmod \mathfrak{f}),$$

then $\mathfrak{a}$ and $\mathfrak{b}$ are in the same class $\mathfrak{H} \bmod \mathfrak{f}$ (see [8], Definition VIII). Since $\mathfrak{f} = [q]$, by (6) we have $\alpha = 1+q\gamma$ with a suitable integer $\gamma \in K$. Multiplying by the conjugate numbers $\alpha' = 1+q\gamma', \alpha'' = 1+q\gamma'', \ldots$ and considering that the elementary symmetric functions of $\gamma, \gamma', \gamma'', \ldots$ are rational integers, we deduce that $N\alpha \equiv 1 (\bmod q)$; similarly $N\beta \equiv 1 (\bmod q)$. By (4) $N\mathfrak{a} \cdot N[\alpha] = N\mathfrak{b} \cdot N[\beta]$, whence

(7) $$N\mathfrak{a} \equiv N\mathfrak{b} (\bmod q),$$

since by (5) $N\alpha > 0$ and $N\beta > 0$ and thus $N[\alpha] = N\alpha$, $N[\beta] = N\beta$ (cf. [9], Satz 812). Thus all ideals of the same class $\mathfrak{H}$ have the same norm residue $l (\bmod q)$. By [4], § 4, for any class $\mathfrak{K}$ there is the same number $\nu = \varphi_1(q)$ (say) of mod q incongruent numbers l with $(l, q) = 1$ and such that $N\mathfrak{a} \equiv l (\bmod q)$ for appropriate ideal $\mathfrak{a} \in \mathfrak{K}$(2). Let $\mathfrak{H}_1^0, \mathfrak{H}_2^0, \ldots, \mathfrak{H}_g^0$

(1) The restriction imposed on q in [6], footnote on p. 8, is unnecessary.

(2) ν is the order of groups G_1 and Γ_1 introduced in § 1.

be all the classes $\mathfrak{H} \subset \mathfrak{K}_1$ of ideals $\mathfrak{a}$ with $N\mathfrak{a} \equiv 1 (\bmod q)$(3). Then there is a group

(8) $$H_1 = (\mathfrak{H}_1^0) + (\mathfrak{H}_2^0) + \ldots + (\mathfrak{H}_g^0)$$

whose elements are the classes $\mathfrak{H}_i^0$ $(1 \leqslant i \leqslant g)$. Since in the group representation $\mathfrak{K}_1 = \mathfrak{H}_1 H_1 + \mathfrak{H}_2 H_1 + \ldots$ any two cosets are either identical or they have no common element (see [7], § 6), taking merely the different cosets we get a representation

(9) $$\mathfrak{K}_1 = \mathfrak{H}_1 H_1 + \mathfrak{H}_2 H_1 + \ldots + \mathfrak{H}_\nu H_1$$

where all the classes $\mathfrak{H}$ of the same coset $\mathfrak{H}_j H_1$ contain exclusively ideals with the same norm residue $l (\bmod q) \in G_1$. The characters of the group $\mathfrak{K}_1$ (of elements $\mathfrak{H} = \mathfrak{H}_j \cdot \mathfrak{H}_i^0$) can be represented by the products

(10) $$\chi_q(\mathfrak{H}_j) \cdot \varkappa(\mathfrak{H}_i^0)$$

where $\chi_q(\mathfrak{H}_j) = \chi_q(N\mathfrak{a})$ (if $\mathfrak{a} \in \mathfrak{H}_j$) runs through the characters $\chi_q \in \Gamma_1$ (see § 1)(4) and $\varkappa(\mathfrak{H}_i^0)$ runs through all characters of the group (8). In a similar manner we can represent the characters of all the classes $\mathfrak{H}$. Let $\mathfrak{K}_1, \ldots, \mathfrak{K}_h$ be all the classes $\mathfrak{K}$. We denote by $\mathfrak{H}_1', \ldots, \mathfrak{H}_h'$ a fixed set of classes $\mathfrak{H}$ such that $\mathfrak{H}_1' \subset \mathfrak{K}_1, \ldots, \mathfrak{H}_h' \subset \mathfrak{K}_h$. Defining $\chi_{\mathfrak{K}}(\mathfrak{H}_l') = \chi_{\mathfrak{K}}(\mathfrak{K}_l)$ if $\mathfrak{H}_l' \subset \mathfrak{K}_l$, all the characters $\chi(\mathfrak{H})$ of the classes $\mathfrak{H} = \mathfrak{H}_j \mathfrak{H}_i^0 \mathfrak{H}_l'$ can be represented by

(11) $$\chi(\mathfrak{H}) = \chi_q(\mathfrak{H}_j) \cdot \varkappa(\mathfrak{H}_i^0) \chi_{\mathfrak{K}}(\mathfrak{H}_l').$$

Agreeing that $\chi(\mathfrak{a}) = \chi(\mathfrak{H})$ if $\mathfrak{a} \in \mathfrak{H}$, (11) represents all characters $\chi(\mathfrak{a}) \bmod [q]$. If in particular $\varkappa$ is the principal character, then $\chi(\mathfrak{a}) = \chi_q(N\mathfrak{a}) \chi_{\mathfrak{K}}(\mathfrak{a})$ and the lemma follows.

Lemma 2. *If $\chi_q \in \Gamma_1$ and if not both χ_q and $\chi_{\mathfrak{K}}$ are principal characters, then (1) is an integral function*(5).

Proof. By Lemma 1

(12) $$\sum_{\mathfrak{a}} \frac{\chi_q(N\mathfrak{a}) \chi_{\mathfrak{K}}(\mathfrak{a})}{N\mathfrak{a}^s} = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}$$

where $\chi(\mathfrak{a})$ is a character mod $[q]$. If the right-hand side is not an integral

(3) There is at least one such class $\mathfrak{H}$, viz. the class containing $\mathfrak{a} = [1]$.

(4) Consider that the factor-group $\mathfrak{K}_1/H_1$ is isomorphic with the group $(\mathfrak{H}_1) + (\mathfrak{H}_2) + \ldots + (\mathfrak{H}_\nu)$ which is isomorphic with G_1.

(5) The following example shows that we cannot dispense with the condition $\chi_q \in \Gamma_1$. If $q = 4$, then there are two characters $\chi_q(n)$: the principal one and the character $(-1)^{(n-1)/2}$. Since odd sums of two squares are $\equiv 1 (\bmod 4)$, in the field generated by $\sqrt{-1}$ the corresponding functions (1) are both identical and have a pole at $s = 1$.

function, then $\chi(\mathfrak{a})$ is the principal character (cf. [8], Satz LXIII) whence the coefficient a_n in the Dirichlet-expansion $\sum_n a_n n^{-s}$ of (12) is

$$a_n = \sum_{\substack{\mathfrak{a} \\ N\mathfrak{a}=n}} 1, \tag{13}$$

provided that $(n, q) = 1$. Let us suppose that on the left-hand side in (12) $\chi_{\mathfrak{K}}(\mathfrak{a})$ is the principal character but χ_q non-principal (whence $\varphi_1(q) \geqslant 2$). Then there are numbers $n = N\mathfrak{a}$ with $(N\mathfrak{a}, q) = 1$ and $\operatorname{re}\chi_q(N\mathfrak{a}) < 1$, whence the sum

$$\sum_{\substack{\mathfrak{a} \\ N\mathfrak{a}=n}} \chi_q(N\mathfrak{a})$$

disagrees with (13); a contradiction to the uniqueness theorem for Dirichlet series.

If $\chi_{\mathfrak{K}}(\mathfrak{a})$ is not the principal character yet the function (1) is not regular at $s = 1$, then we use some fixed class $\mathfrak{K}_l$ with $\operatorname{re}\chi_{\mathfrak{K}}(\mathfrak{K}_l) < 1$ containing an ideal $\mathfrak{a}_0$ with the norm $N\mathfrak{a}_0 \equiv 1 \pmod{q}$ (or congruent to some other suitable number)(6) and arrive at the same contradiction.

3. Lemma 3. *Let*

$$\zeta(s, \chi) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) N\mathfrak{a}^{-s} \tag{14}$$

be the Hecke L-function of the quadratic field K (of a discriminant Δ) with a primitive character mod $\mathfrak{f}$. *Let further $s_0 = \frac{1}{2}+it_0$, $s_1 = \frac{1}{2}+it_1$ $(t_1 \leqslant 1)$, $l \geqslant l_0 > 3$,*

$$d = \frac{1}{\pi}\sqrt{|\Delta|\cdot N\mathfrak{f}}, \qquad x = \begin{cases} d^4(1+|t_0|)^4 l^4 & \text{if} \quad N\mathfrak{f} > 1, \\ d^4(1+|t_0|)^8 l^4 & \text{if} \quad N\mathfrak{f} = 1. \end{cases} \tag{15}$$

Then we have uniformly in d and t_0

$$\zeta(s_0, \chi) = \sum_{\substack{\mathfrak{a} \\ N\mathfrak{a}\leqslant X}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s_0}} \eta_1\left(\frac{N\mathfrak{a}}{d}, t_0\right) + \sum_{\substack{\mathfrak{a} \\ N\mathfrak{a}\leqslant X}} \frac{\overline{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s_0}} \eta_2\left(\frac{N\mathfrak{a}}{d}, t_0\right) + \sum_{\substack{\mathfrak{a} \\ N\mathfrak{a}\leqslant X}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^{s_1}} \eta_3\left(\frac{N\mathfrak{a}}{d}, t_1\right) + \sum_{\substack{\mathfrak{a} \\ N\mathfrak{a}\leqslant X}} \frac{\overline{\chi}(\mathfrak{a})}{N\mathfrak{a}^{1-s_1}} \eta_4\left(\frac{N\mathfrak{a}}{d}, t_1\right) + O(l^{-1}) \tag{16}$$

(6) We consider separately the cases (i) $\varphi_1(q) = \varphi(q)$; (ii) $\varphi_1(q) = 1$; (iii) $1 < \varphi_1(q) < \varphi(q)$. In the case (i) in any class $\mathfrak{K}$ there is an $\mathfrak{a}$ with $N\mathfrak{a} \equiv 1 \pmod{q}$. In case (ii) there is no other $\chi_q \in \Gamma_1$ than the principal one, which makes the condition $N\mathfrak{a} \equiv 1 \pmod{q}$ superfluous. In case (iii) there is in Γ_1 a character χ_q which takes at least two different values at the normresidues $\pmod{q}$ of $\mathfrak{K}_1$ and as many at those of $\mathfrak{K}_l$ (since by (9) the residues of $\mathfrak{K}_l$ are those of $\mathfrak{K}_1$ multiplied by a suitable number $a_l \equiv N\mathfrak{a} \pmod{q}$ with $\mathfrak{a} \in \mathfrak{H}_l'$). Hence there is at least one $\mathfrak{a} \in \mathfrak{K}_l$ such that $\chi_q(N\mathfrak{a})\chi_{\mathfrak{K}}(\mathfrak{K}_l) \neq 1$, etc.

where the η_j are appropriate functions such that for $j = 1, 2, 3, 4$

$$\eta_j(x, t) \ll \begin{cases} 1 & \text{in any case,} \\ x^{-1/2} & \text{if} \quad x > l(1+\frac{2}{3}|t|)\log^2 d \end{cases} \tag{17}$$

with an absolute constant in the notation(7).

For the proof see [5], Corollary.

Lemma 4. *Let $\tau(\chi_q)$ be defined by* (2) *and let a_n be any complex numbers. Then*

$$\sum_{q\leqslant X} \frac{1}{\varphi(q)} \sum_{\chi_q} |\tau(\chi_q)|^2 \Big|\sum_{Y+1\leqslant n\leqslant Y+U} \chi_q(n) a_n\Big|^2 < 2\cdot 3(\log X)\max(X^2, U) \sum_{Y+1\leqslant n\leqslant Y+U} |a_n|^2, \tag{18}$$

provided that X is greater than some numerical constant.

This is Theorem 3 of [3].

Lemma 5. *Let a_n $(-N \leqslant n \leqslant N)$ be any complex numbers and let*

$$S(x) = \sum_{-N\leqslant n\leqslant N} a_n e^{2\pi i n x}.$$

Further let $x_1, x_2, \ldots, x_R$ $(R \geqslant 2)$ be any real numbers and define

$$\delta = \min_{j\neq k} \|x_j - x_k\|$$

where $\|x\|$ stands for the distance from x to the nearest integer. Then

$$\sum_{1\leqslant r\leqslant R} |S(x_r)|^2 \leqslant 2\cdot 2\max(\delta^{-1}, 2N) \sum_{-N\leqslant n\leqslant N} |a_n|^2.$$

This is Theorem 1 of [2].

Lemma 6. *Let $c_{n,q}$ be any complex numbers, the other notation as before. Then for $X \geqslant 2$*

$$\sum_{q\leqslant X} \frac{1}{q} \sum_{\chi_q} |\tau(\chi_q)|^2 \Big|\sum_{Y+1\leqslant n\leqslant Y+U} \chi_q(n) a_n c_{n,q}\Big|^2 \ll \max(X^2, XU) \sum_{Y+1\leqslant n\leqslant Y+U} d(n)|a_n|^2 \max_q |c_{n,q}|^2 \tag{19}$$

with an absolute constant in the notation.

(7) I have been informed by A. I. Vinogradov and A. F. Lavrik that a suitable approximate equation follows also from Lavrik's paper in Izv. Akad. Nauk SSSR, Ser. math. 32 (1968), pp. 134–185.

Proof. Arguing as in the proof of Theorem 3 of [2] (but with $a_n c_{n,q}$ instead of a_n) we deduce that

$$\sum_{\chi_q} |\tau(\chi_q)|^2 \Big| \sum_{Y+1\leqslant n\leqslant Y+U} \chi_q(n) a_n c_{n,q}\Big|^2 = \varphi(q) \sum_{\substack{1\leqslant m\leqslant q\\(m,q)=1}} \Big|S_q\Big(\frac{m}{q}\Big)\Big|^2$$

and

$$\varphi(q) \sum_{\substack{1\leqslant m\leqslant q\\(m,q)=1}} \Big|S_q\Big(\frac{m}{q}\Big)\Big|^2 \leqslant q \sum_{d|q} d \sum_{\substack{1\leqslant m\leqslant q\\(m,q)=1}} \Big| \sum_{\substack{Y+1\leqslant n\leqslant Y+U\\ d|n}} a_n c_{n,q} e^{2\pi i n m/q}\Big|^2$$

where

$$S_q(x) = \sum_{Y+1\leqslant n\leqslant Y+U} a_n c_{n,q} e^{2\pi i n x}.$$

Hence, by S denoting the left-hand side of (19), we have

$$S \leqslant \sum_{d\leqslant X} d \sum_{\substack{q\leqslant X\\ d|q}} \sum_{\substack{1\leqslant m\leqslant q\\(m,q)=1}} \Big| \sum_{\substack{Y+1\leqslant n\leqslant Y+U\\ d|n}} a_n c_{n,q} e^{2\pi i n m/q}\Big|^2. \tag{20}$$

For any fixed d let us write $q = dq'$, $n = dn'$. By the previous lemma (with $x_r = m/q'$, $\delta = 1/q'$)

$$\sum_{\substack{1\leqslant m\leqslant q\\(m,q)=1}} \Big| \sum_{\substack{Y+1\leqslant n\leqslant Y+U\\ d|n}} a_n c_{n,q} e^{2\pi i n m/q}\Big|^2$$
$$\leqslant d \sum_{\substack{1\leqslant m\leqslant q'\\(m,q')=1}} \Big| \sum_{(Y+1)/d\leqslant n'\leqslant (Y+U)/d} a_{dn'} c_{dn',dq'} e^{2\pi i n'(m/q')}\Big|^2$$
$$\ll d\cdot \max\Big(q', 1+\frac{U}{d}\Big) \sum_{(Y+1)/d\leqslant n'\leqslant (Y+U)/d} |a_{dn'} c_{dn',dq'}|^2.$$

Summing over all $q' \leqslant X/d$ we obtain the inequality

$$\sum_{\substack{q\leqslant X\\ d|q}} \sum_{\substack{1\leqslant m\leqslant q\\(m,q)=1}} \Big| \sum_{\substack{Y+1\leqslant n\leqslant Y+U\\ d|n}} a_n c_{n,q} e^{2\pi i n m/q}\Big|^2$$
$$\ll \Big\{\frac{X^2}{d} + X\Big(1+\frac{U}{d}\Big)\Big\} \sum_{(Y+1)/d\leqslant n'\leqslant (Y+U)/d} |a_{dn'}|^2 \max_{q'} |c_{dn',dq'}|^2,$$

whence, after multiplying by d and summing over all $d \leqslant X$ (cf. (20)), (19) follows.

Lemma 7. *Let $\tau_k(n)$ be the number of solutions of the equation $n = x_1 x_2 \ldots x_k$ in natural numbers. Then for any natural numbers k, l and for $x \geqslant 3$ we have*

$$\sum_{n\leqslant x} \tau_k(n)^l \ll x(\log x)^{k^l-1} \tag{21}$$

with the constant in the notation depending on k and l.

For the proof see [10].

Proof of the inequality (33)

4. Let M be an arbitrarily large natural number and q be a natural number such that

$$1 < q \leqslant M.$$

Further let K be a given quadratic field of the discriminant $\Delta \ll 1$ and let $\zeta(s, \chi_q, \chi_{\mathfrak{K}}) = \zeta(s, \chi)$ ($\chi \bmod [q]$) be the function (1) (see Lemma 1). The character $\chi \bmod [q]$ may be imprimitive one. Then there is an ideal $\mathfrak{f}_0 | [q]$ ($\mathfrak{f}_0 = \mathfrak{f}_0(\chi_q)$) and a primitive character $\chi'(\mathfrak{a}) \bmod \mathfrak{f}_0$ such that

$$\zeta(s, \chi) = \zeta(s, \chi') \prod_{\mathfrak{p}|[q], \mathfrak{p}\nmid \mathfrak{f}_0} \big(1-\chi'(\mathfrak{p}) N\mathfrak{p}^{-s}\big) \tag{22}$$

where $\mathfrak{p}$ denotes prime ideals (cf. [8], p. 102). In the case of a primitive $\chi = \chi'$ the product is empty and it has the value 1. In the present paragraph we shall use (16) with

$$s_0 = s = \tfrac{1}{2}+it, \quad s_1 = \tfrac{1}{2}, \tag{23}$$
$$d = \frac{1}{\pi}\sqrt{|\Delta|\cdot N\mathfrak{f}_0}, \quad X = cM^4(1+|t|)^8$$

where c stands for some constant $> 3^4\Delta^2$. The character χ' being a primitive one we may replace $\zeta(s, \chi')$ in (22) by the four sums (16) (with χ' instead of χ) and a remaining term $O(1)$. If $\mathfrak{a}$ is not divisible by any of the prime ideals $\mathfrak{p}$ under the product in (22), then by (1) and (12)

$$\chi'(\mathfrak{a}) = \chi_q(N\mathfrak{a})\chi_{\mathfrak{K}}(\mathfrak{a}), \quad \bar{\chi}'(\mathfrak{a}) = \bar{\chi}_q(N\mathfrak{a})\bar{\chi}_{\mathfrak{K}}(\mathfrak{a}). \tag{24}$$

Let us partition the remaining ideals $\mathfrak{a}$ into sets in such a manner that all ideals of the same set have in common the same powers of prime ideals dividing the square-free ideal

$$\mathfrak{f}_1 = \prod \mathfrak{p} \quad (\mathfrak{p}|[q], \mathfrak{p}\nmid \mathfrak{f}_0).$$

This common divisor denoting by $\mathfrak{c}$ in case of the first sum we take before the parenthesis $\chi'(\mathfrak{c})/N\mathfrak{c}^s$; for the other sums the corresponding factors are $\bar{\chi}'(\mathfrak{c})/N\mathfrak{c}^{1-s}$, $\chi'(\mathfrak{c})/N\mathfrak{c}^{s_1}$, $\bar{\chi}'(\mathfrak{c})/N\mathfrak{c}^{1-s_1}$, respectively. Considering that the product in (22) is equivalent to the sum

$$\sum_{\mathfrak{b}|\mathfrak{f}_1}\frac{\mu(\mathfrak{b})\chi'(\mathfrak{b})}{N\mathfrak{b}^s}$$

($\mu(\mathfrak{b})$ — the Möbius function of ideals), we get the representation

$$\zeta(s,\chi) = \sum_{\mathfrak{b}|\mathfrak{f}_1}\frac{\mu(\mathfrak{b})\chi'(\mathfrak{b})}{N\mathfrak{b}^s}\{\Sigma_1+\Sigma_2+\Sigma_3+\Sigma_4+O(1)\} \tag{25}$$

where

$$\Sigma_1 = \sum_{\mathfrak{c}}\frac{\chi'(\mathfrak{c})}{N\mathfrak{c}^s}\sum_{\substack{\mathfrak{a}_1\\ N\mathfrak{a}_1\leqslant X/N\mathfrak{c}}}\frac{\chi'(\mathfrak{a}_1)}{N\mathfrak{a}_1^s}\eta_1\left(\frac{N\mathfrak{a}_1\mathfrak{c}}{d},t\right)$$

and Σ_2, Σ_3, Σ_4 stand for analogous sums with χ' replaced by $\bar{\chi}'$, s replaced by $1-s$ or $\frac{1}{2}$ and η_1 replaced by η_2, η_3 or η_4 (with t in η_3 and η_4 replaced by 0). Hence by Cauchy's inequality and by (24)

$$|\Sigma_1|^2 \leqslant \sum_{\mathfrak{c}}\frac{1}{N\mathfrak{c}}\Bigg|\sum_{\substack{\mathfrak{a}_1\\ N\mathfrak{a}_1\leqslant X/N\mathfrak{c}}}\frac{\chi_q(N\mathfrak{a}_1)\chi_{\mathfrak{K}}(\mathfrak{a}_1)}{N\mathfrak{a}_1^{1/2+it}}\eta_1\left(\frac{N\mathfrak{a}_1\mathfrak{c}}{d},t\right)\Bigg|^2.$$

Let a_j $(1\leqslant j\leqslant h\ll 1)$ run through the possible values of $\chi_{\mathfrak{K}}(\mathfrak{a}_1)$. Then

$$|\Sigma_1|^2 \ll \sum_{\mathfrak{c}}\frac{1}{N\mathfrak{c}}\Bigg|\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant X/N\mathfrak{c}\\ \chi_{\mathfrak{K}}(\mathfrak{a})=a_j}}\frac{\chi_q(N\mathfrak{a})}{N\mathfrak{a}^{1/2+it}}\eta_1\left(\frac{N\mathfrak{a}\mathfrak{c}}{d},t\right)\Bigg|^2 \tag{26}$$

$$\ll \sum_{\mathfrak{c}}\frac{1}{N\mathfrak{c}}(S'_{j\mathfrak{c}}+S''_{j\mathfrak{c}})$$

where

$$S'_{j\mathfrak{c}} = \Bigg|\sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}\leqslant x/N\mathfrak{c}\\ \chi_{\mathfrak{K}}(\mathfrak{a})=a_j}}\frac{\chi_q(N\mathfrak{a})}{N\mathfrak{a}^{1/2+it}}\eta_1\left(\frac{N\mathfrak{a}\mathfrak{c}}{d},t\right)\Bigg|^2,$$

(27)

$$S''_{j\mathfrak{c}} = \Bigg|\sum_{\substack{\mathfrak{a}\\ x/N\mathfrak{c}\leqslant N\mathfrak{a}\leqslant X/N\mathfrak{c}\\ \chi_{\mathfrak{K}}(\mathfrak{a})=a_j}}\frac{\chi_q(N\mathfrak{a})}{N\mathfrak{a}^{1/2+it}}\eta_1\left(\frac{N\mathfrak{a}\mathfrak{c}}{d},t\right)\Bigg|^2,$$

$$x = cM(1+|t|)\log^2 M.$$

In what follows we choose a fixed number $D>2$ and we consider merely numbers $q\in(1,M]$ with the restriction

$$d(q)\leqslant D.$$

Then there are at most D^2 different values of $N\mathfrak{f}_0$ and, by (23), as many values of d. Now let us partition the characters χ_q into classes K_ν $(\nu = 1, 2, \ldots \ll D^2)$ in such a manner that for all χ_q of the same class K_ν the number d in (27) is the same one-valued function of q. Hence in estimating the sum

$$V_1 = \sum_{\substack{q\leqslant M\\ d(q)\leqslant D}}\frac{1}{\varphi(q)}\sum_{\chi_q}|\tau(\chi_q)|^2 S'_{j\mathfrak{c}} \tag{28}$$

we may use (19) for each of the classes K_ν separately and add the results. Since $\eta_1(y,t)\ll 1$, by (17), and since $\varphi(q)\gg q/\log\log q$ if $q>3$ (cf. [11], p. 24), we deduce that

$$V_1 \ll D^2(\log\log M)\max\left(M^2, M\frac{x}{N\mathfrak{c}}\right)\sum_{n\leqslant x/N\mathfrak{c}}\frac{d(n)^3}{n}$$

(consider that there are no more than $d(n)$ ideals $\mathfrak{a}$ with $N\mathfrak{a} = n$; cf. [9], Satz 882). Hence, by (27), (26) and partial summation

$$V_1 \ll D^2M^2(1+|t|)\log^{10}M(1+|t|)\log\log M. \tag{29}$$

In order to estimate the sum

$$V_2 = \sum_{\substack{q\leqslant M\\ d(q)\leqslant D}}\frac{1}{\varphi(q)}\sum_{\chi_q}|\tau(\chi_q)|^2 S''_{j\mathfrak{c}} \tag{30}$$

we split the interval $x/N\mathfrak{c}<N\mathfrak{a}\leqslant X/N\mathfrak{c}$ into $\ll\log M(1+|t|)$ parts $(U, 2U)$ where $U\geqslant x/M$ and get

$$V_2 \ll D^2\log M(1+|t|)\log\log M\max_{\frac{x}{N\mathfrak{c}}\leqslant U\leqslant\frac{X}{N\mathfrak{c}}}\sum_{\substack{q\leqslant M\\ d(q)\leqslant D}}\frac{1}{q}\max_{K_\nu}\sum_{\chi_q\in K_\nu}|\tau(\chi_q)|^2\times$$

$$\times\Bigg|\sum_{U\leqslant n<2U}\chi_q(n)\frac{c_{nj}}{n^{1/2+it}}\eta_1\left(\frac{nN\mathfrak{c}}{d},t\right)\Bigg|^2$$

where

$$c_{nj} = \sum_{\substack{\mathfrak{a}\\ N\mathfrak{a}=n\\ \chi_{\mathfrak{K}}(\mathfrak{a})=a_j}}1.$$

Using Cauchy's inequality, (19) and the estimate $\eta_1(y,t)\ll y^{-1/2}$ for $y\geqslant x/M$ (cf. (17) and (27)) we prove that

(31) $$V_2 \ll D^2 \log^2 M(1+|t|) \max_{\frac{x}{N\mathfrak{c}} \leqslant U \leqslant \frac{X}{N\mathfrak{c}}} \max(M^2, MU) \sum_{U \leqslant n < 2U} \frac{d(n)^3}{n} \frac{M}{nN\mathfrak{c}}$$
$$\ll D^2 \log^2 M(1+|t|) \max_{\frac{x}{N\mathfrak{c}} \leqslant U \leqslant \frac{X}{N\mathfrak{c}}} \max(M, U) \frac{M^2}{N\mathfrak{c}} \frac{(\log U)^7}{U}$$
$$\ll D^2 M^2 \log^9 M(1+|t|).$$

Since

$$\sum_{\mathfrak{c}} \frac{1}{N\mathfrak{c}} \leqslant \prod_{\mathfrak{p}|[q]} \frac{1}{1-1/N\mathfrak{p}} \ll \Big(\prod_{p|q} \frac{1}{1-1/p}\Big)^2 \ll \exp\Big\{2 \sum_{p \ll \log q} \frac{1}{p}\Big\} \ll (\log\log M)^2$$

(p denoting primes), from (26) and (28)–(31) we deduce that

(32) $$\sum_{\substack{q \leqslant M \\ d(q) \leqslant D}} \frac{1}{\varphi(q)} \sum_{\chi_q} |\tau(\chi_q)|^2 |\Sigma_1|^2$$
$$\ll D^2 M^2 (1+|t|) \log^{10} M(1+|t|)(\log\log M)^3.$$

The same estimate holds if Σ_1 in (32) is replaced by Σ_2, Σ_3, Σ_4 or by $O(1)$.

Considering that by Cauchy's inequality

$$\Big(\sum_{\mathfrak{b}|\mathfrak{f}_1} \frac{1}{N\mathfrak{b}^{1/2}}\Big)^2 \ll \sum_{\mathfrak{b}|\mathfrak{f}_1} 1 \sum_{\mathfrak{b}|\mathfrak{f}_1} \frac{1}{N\mathfrak{b}} \ll D^2 \sum_{n \leqslant M^2} \frac{d(n)}{n} \ll D^2 \log^2 M,$$

from (25), (1), (12), (32) (and the analogous inequalities with $\Sigma_2, \ldots, O(1)$ instead of Σ_1) we deduce that

(33) $$\sum_{\substack{q \leqslant M \\ d(q) \leqslant D}} \frac{1}{\varphi(q)} \sum_{\chi_q} |\tau(\chi_q)|^2 |\zeta(\tfrac{1}{2}+it, \chi_q, \chi_{\mathfrak{R}})|^2$$
$$\ll D^4 M^2 (1+|t|) \log^{13} M(1+|t|).$$

Proof of the theorem

5. Let $\mu(\mathfrak{a})$ be the Möbius function of ideals and let $\chi_q(N\mathfrak{a})\chi_{\mathfrak{R}}(\mathfrak{a}) = \chi(\mathfrak{a})$,

(34) $$Q(s, \chi) = Q(s, \chi_q, \chi_{\mathfrak{R}}) = \sum_{\substack{\mathfrak{a} \\ N\mathfrak{a} \leqslant M^2}} \mu(\mathfrak{a}) \chi(\mathfrak{a}) N\mathfrak{a}^{-s},$$

(35) $$f(s, \chi) = f(s, \chi_q, \chi_{\mathfrak{R}}) = \zeta(s, \chi_q, \chi_{\mathfrak{R}}) Q(s, \chi_q, \chi_{\mathfrak{R}}) - 1.$$

By (34), (18) and (21)

(36) $$\sum_{\chi_{\mathfrak{R}}} \sum_{\substack{q \leqslant M \\ d(q) \leqslant D}} \frac{1}{\varphi(q)} \sum_{\chi_q} |\tau(\chi_q)|^2 |Q(\tfrac{1}{2}+it, \chi_q, \chi_{\mathfrak{R}})|^2$$
$$\ll hM^2 (\log M) \sum_{n \leqslant M^2} \frac{d(n)^2}{n} \ll M^2 \log^5 M.$$

Since, by (35),

$$|f(s, \chi)| \leqslant 1 + |\zeta \cdot Q| \leqslant 1 + \tfrac{1}{2}|\zeta|^2 + \tfrac{1}{2}|Q|^2,$$

from (36) and (33) we deduce

(37) $$\sum_{\chi_{\mathfrak{R}}} \sum_{\substack{q \leqslant M \\ d(q) \leqslant D}} \frac{1}{\varphi(q)} \sum_{\chi_q}{}' |\tau(\chi_q)|^2 |f(\tfrac{1}{2}+it, \chi_q, \chi_{\mathfrak{R}})|$$
$$\ll D^4 M^2 (1+|t|) \log^{13} M(1+|t|).$$

Let

$$\sigma_0 = 1 + \frac{1}{\log MT}.$$

By (35) and (34)

$$f(\sigma_0+it, \chi) = \sum_{n > M^2} \frac{\chi_q(n) a_n}{n^{\sigma_0+it}}$$

where $a_n = a_n(\chi_{\mathfrak{R}})$ is in modulus $\leqslant d(n)^3$. Writing

$$M_1 = M^{(\log MT)^2}$$

and using (21) we get

$$f(\sigma_0+it, \chi) = \sum_{M^2 < n \leqslant M_1} \frac{\chi_q(n) a_n}{n^{\sigma_0+it}} + O(M^{-4}).$$

Splitting the last sum into $\ll \log M_1$ parts $U \leqslant n < 2U$ and using (18) together with Cauchy's inequality we prove that

$$\sum_{q \leqslant M} \frac{1}{\varphi(q)} \sum_{\chi_q} |\tau(\chi_q)|^2 |f(\sigma_0+it, \chi)|^2$$
$$\ll (\log M_1)^2 \max_{M^2 \leqslant U \leqslant M_1} (M^2+U) \sum_{U \leqslant n < 2U} \frac{d(n)^6}{n^{2+2/\log MT}} + O(M^{-6}).$$

Since by (21)

$$\sum_{n\leqslant x} d(n)^6 \ll x(\log x)^{63} \quad (x\geqslant 3),$$

it follows that

$$(38)\qquad \sum_{\chi_{\mathfrak{K}}} \sum_{q\leqslant M} \frac{1}{\varphi(q)} \sum_{\chi_q}{}' |\tau(\chi_q)|^2 \, |f(\sigma_0+it, \chi_q, \chi_{\mathfrak{K}})|^2 \ll (\log M_1)^{65} < (\log MT)^{195}.$$

Now we introduce the integral function

$$F(s) = \prod_{\chi_{\mathfrak{K}}} \prod_{\substack{q\leqslant M \\ d(q)\leqslant D}} \prod_{\substack{\chi_q \in \Gamma_1 \\ \chi_q\neq\chi_q^0}} \big(1-f^2(s, \chi_q, \chi_{\mathfrak{K}})\big)^{e(\chi_q)}$$

where

$$e(\chi_q) = (M!/q)|\tau(\chi_q)|^2$$

and consider that any zero of $\zeta(s, \chi_q, \chi_{\mathfrak{K}})$ is also a zero (of at least the same order) of the function $1-f^2(s, \chi_q, \chi_{\mathfrak{K}})$. Using (37) and (38) (which are the analogues of [1], Lemmas 8 and 9) and arguing as in the proof of [1], Theorem 5, we get (3).

References

[1] E. Bombieri, *On the large sieve*, Mathematika 12 (1965), pp. 201–225.
[2] H. Davenport and H. Halberstam, *The values of trigonometric polynomial at well spaced points*, Mathematika 13 (1966), pp. 91–96.
[3] — — *Corrigendum and addendum*, Mathematika 14 (1967), pp. 229–232.
[4] E. Fogels, *On the distribution of prime ideals*, Acta Arith. 7 (1962), pp. 255–269.
[5] — *Approximate functional equation for Hecke's L-functions of quadratic field*, Acta Arith. 16 (1969), pp. 161–178.
[6] — *Большое решето*, Latvijas PSR Zin. akad. Vēstis, Fiz. u. tehn. zin sēr. 4 (1969), pp. 1–14.
[7] E. Hecke, *Theorie der algebraischen Zahlen*, Leipzig 1923.
[8] E. Landau, *Über Ideale und Primideale in Idealklassen*, Math. Zeitschr. 2 (1918), pp. 52–154.
[9] — *Vorlesungen über Zahlentheorie III*, Leipzig 1927.
[10] C. Mardjanichvili, *Estimation d'une somme arithmétique*, Comptes Rendus (Doklady) de l'Académie des Sciences de l'URSS, 22 (1939), pp. 387–389.
[11] K. Prachar, *Primzahlverteilung*, Berlin 1957.

Received on 15. 2. 1970

ACTA ARITHMETICA
XVIII (1971)

Diophantine approximation and certain sequences of lattices

by

WOLFGANG M. SCHMIDT (Boulder, Colo.)

In memory of H. Davenport

1. Introduction. The present paper is a continuation of the joint work [2], [3] by Davenport and the author, but most of it can be read independently.

Let $\alpha_1, \ldots, \alpha_n$ be real numbers. There are two forms of Dirichlet's theorem on simultaneous approximation.

(a) *For any positive integer N there exist integers $x_1, \ldots, x_n, y$, not all zero, and satisfying*

$$(1)\qquad |\alpha_1 x_1+\ldots+\alpha_n x_n+y| < N^{-n}, \quad \max(|x_1|, \ldots, |x_n|) \leqslant N;$$

(b) *for any positive integer N there are integers $x_1, \ldots, x_n, y$, not all zero, with*

$$(2)\qquad \max(|\alpha_1 y-x_1|, \ldots, |\alpha_n y-x_n|) < N^{-1}, \quad |y|\leqslant N^n.$$

Now let $\Lambda(\alpha_1, \ldots, \alpha_n; N)$ be the lattice in the space of dimension

$$(3)\qquad l = n+1$$

with basis vectors

$$(4)\qquad \begin{aligned} \boldsymbol{g}_1 &= (N^{-1}, 0, \ldots, 0, \alpha_1 N^n), \\ \boldsymbol{g}_2 &= (0, N^{-1}, \ldots, 0, \alpha_2 N^n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \boldsymbol{g}_n &= (0, 0, \ldots, N^{-1}, \alpha_n N^n), \\ \boldsymbol{g}_l &= (0, 0, \ldots, 0, N^n). \end{aligned}$$

Form (a) of Dirichlet's theorem says precisely that $\Lambda(\alpha_1, \ldots, \alpha_n; N)$ has a nonzero point (namely $x_1\boldsymbol{g}_1+\ldots+x_n\boldsymbol{g}_n+y\boldsymbol{g}_l$) in the cube defined by $|\xi_1|\leqslant 1, \ldots, |\xi_n|\leqslant 1, |\xi_l| < 1$. Dirichlet's theorem in form (a) can be improved for particular $\alpha_1, \ldots, \alpha_n; N$ if the lattice $\Lambda(\alpha_1, \ldots, \alpha_n; N)$