Soient A, B, D, E et p définis comme au Théorème 3, et soit $A < B$. Si m désigne le nombre d'équations résolubles en nombres entiers x et y parmi les equations (25) on a ou $m = 0$ ou $m = 1$.

Dans le cas exceptionnel

$$x^2+y^2 = Ep,$$

où $p \equiv 1 (\mathrm{mod}\, 4)$ on a évidemment $m = 2$.

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Representations of real numbers by series of reciprocals of odd integers

by

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Harold Davenport in memoriam

1. It is well-known that a real number x between 0 and 1 can be expanded into a series of reciprocals of integers (a "sorites" of Sylvester) originally found by Lambert (see Perron [2]) as follows:

$$(1.1)\qquad x = x_1 = \frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots$$

where the positive integers a_i are given in succession uniquely by the algorithm

$$(1.2)\qquad a_i = 1+[1/x_i], \quad x_{i+1} = x_i - \frac{1}{a_i}, \quad 0 < x_{i+1} < x_i, \quad \ldots$$

The process is unending: the integers a_i satisfy the inequalities

$$(1.3)\qquad a_i \geqslant 2, \quad a_{i+1} \geqslant a_i^2 - a_i + 1 \quad (i \geqslant 1).$$

A convergent series (1.1) in which the integers a_i satisfy (1.3) is necessarily the Sylvester expansion of its sum. For rational x equality must occur eventually in (1.3), i.e. for all $i > i_0$, $a_{i+1} = a_i^2 - a_i + 1$. The converse is trivially true.

I have taken the algorithm so that the process is non-ending. If we take $1/a_i \leqslant x_i < 1/(a_i - 1)$, the process ends for rational x; for irrational numbers the two processes naturally yield the same series.

Variations of (1.1) exist in which signs can be attached to the terms in accordance with prescribed rules (and appropriate changes in (1.3)).

2. Engel (anticipated by Lambert: see Perron [2]) obtained another kind of series for x in $(0, 1)$:

$$(2.1)\qquad x = \frac{1}{c_1}+\frac{1}{c_1c_2}+\frac{1}{c_1c_2c_3}+\ldots,$$

by the algorithm

(2.2) $c_i = 1+[1/x_i] \geq 2, \quad x_{i+1} = c_i x_i - 1, \quad 0 < x_{i+1} \leq x_i.$

The process is unending; the integers c_i satisfy the inequalities

(2.3) $c_i \geq 2, \quad c_{i+1} \geq c_i \quad (i \geq 1).$

A convergent series (2.1) in which the integers c_i satisfy (2.3) is necessarily the Engel series of its sum. For rational x, equality occurs from some point on in (2.3), i.e. $c_{i+1} = c_i$ for all $i > i_0$. The converse is trivial.

Variations exist wherein signs are attached to the successive terms. Modifications are needed in the inequalities (2.3).

3. Some years ago I obtained (but have not published) a remarkable extension of these series which include not merely the series of Lüroth (Perron [2]) but also the well-known infinite product of Cantor (Perron [2]) and its generalizations by myself (Oppenheim [1]). In this note I consider series of the types (1.1) and (2.1) where in place of the integers a_i, c_i we use numbers with residue $\frac{1}{2}$ modulo 1 or alternatively numbers which are odd integers.

The following series arise:

(3.1) $$\sum 1/(2d_i+1),$$

(3.2) $$\sum \varepsilon_1 \ldots \varepsilon_{i-1}/(2d_i+1),$$

(3.3) $$\sum 2^i/(2d_1+1)(2d_2+1)\ldots(2d_i+1),$$

(3.4) $$\sum \varepsilon_1 \ldots \varepsilon_{i-1} 2^i/(2d_1+1)(2d_2+1)\ldots(2d_i+1),$$

(3.5) $$\sum 1/(2d_1+1)(2d_2+1)\ldots(2d_i+1),$$

(3.6) $$\sum \varepsilon_1 \ldots \varepsilon_{i-1}/(2d_1+1)(2d_2+1)\ldots(2d_i+1).$$

In these expansions of an arbitrary positive x (which need not be restricted to the interval $0 < x < 1$), the d_i are integers to be determined by algorithms shortly to be described; the ε_i take the values $+1$ or -1 according to certain rules. In each case a unique expansion exists. Various questions arise: (i) to determine the inequalities necessarily satisfied by the integers d_i; (ii) given a series (3.1) say in which the d_i satisfy appropriate inequalities, to determine whether the series is derived from its sum by the algorithm in question; (iii) to find the kind of expansion which necessarily obtains when x is rational.

It is curious to note that for the series (3.2), (3.4), (3.6) complete answers can be given to these questions. For the seemingly simpler series (3.1), (3.5) I cannot answer questions (ii) and (iii). For (3.3) I cannot answer question (iii). But some interesting conjectures arise which I illustrate with (3.3). This series is derived by the algorithm

$$d_i = \left[\tfrac{1}{2} + \frac{1}{x_i}\right], \quad x_{i+1} = (d_i + \tfrac{1}{2})x_i - 1 \quad (i = 1, 2, \ldots; x_1 > 0);$$

the integers d_i satisfy the inequalities

$$d_{i+1} \geq d_i \geq 0 \quad (\text{and at least one } d_i \geq 1).$$

The expansion is unique. A convergent series (3.3) in which the d_i satisfy the conditions $d_{i+1} \geq d_i \geq 0$ is necessarily the expansion of its sum by algorithm.

Conjecture 1. *The series (3.3) in which the integers d_i satisfy the conditions $d_{i+1} \geq d_i \geq 0$ and at least one $d_i \geq 1$ is rational if and only if from some point on either all the d_i are equal or each d_i is twice its predecessor.*

This conjecture can be put thus: take positive coprime integers p_1, q_1. Define positive coprime integers p_i, q_i and an integer $d_i \geq 0$ by the relations $(i = 1, 2, \ldots)$

$$0 < \lambda_i p_{i+1} = (2d_i+1)p_i - 2q_i \leq 2p_i,$$
$$\lambda_i q_{i+1} = 2q_i$$

so that $d_i = [\frac{1}{2} + q_i/p_i]$ and λ_i is a positive integer.

Conjecture 2. *Either $p_i = 1$ (all large i) or $p_i = 2$ (all large i). The first case leads to $d_{i+1} = 2d_i$; the second case to $d_{i+1} = d_i$.*

Theorems relating to these series are stated below but not all proofs are given. Some further conjectures are also made.

In the expansions so far described the numbers a_i, c_i, d_i have been selected from a single set. Plainly we may consider a sequence of sets $\{S_i\}$,

$$S_i\colon\ 1 \leq a_i(1) < a_i(2) < a_i(3) \ldots \to \infty$$

and associate with given $x_1 > 0$ a unique number of S_1 and a real number $x_2 > 0$ by a rule R_1; with x_2 a unique number of S_2 and a real number $x_3 > 0$, and so on. The numbers $a_i(n)$ need not be integers. Thus arise series such as

$$\sum \frac{1}{a_i(n_i)}, \quad \sum \frac{1}{a_i(n_i)\ldots a_i(n_i)},$$

generalisations of Sylvester-series and Engel-series. The first series includes also Cantor-series (Perron [2]). I have obtained a number of theorems relating to these expansions.

4. The series (3.2). For $x = x_1 > 0$ define a sequence of positive integers d_i, a sequence of real numbers x_i, θ_i and signs $\varepsilon_i = \operatorname{sgn}\theta_i$ by the algorithm (for $i = 1, 2, \ldots$)

$$(4.1)\qquad \begin{cases} d_i = [1/x_i],\ 1/x_i = d_i + \tfrac{1}{2} - \theta_i,\ -\tfrac{1}{2} < \theta_i \leqslant \tfrac{1}{2}, \\ \varepsilon_i = \operatorname{sgn}\theta_i\ (1 \text{ if } \theta_1 > 0,\ 0 \text{ if } \theta_i = 0,\ -1 \text{ if } \theta_1 < 0), \\ x_i = 1/(d_i + \tfrac{1}{2}) + \varepsilon_i x_{i+1}\ (\text{so that } x_{i+1} > 0 \text{ if } \varepsilon_i \neq 0). \end{cases}$$

The process terminates if a zero θ_i is reached (and in this case x_i must be rational).

THEOREM 1. *The algorithm* (4.1) *applied to $x = x_1 > 0$ yields either a finite series with sum x or an infinite series*

$$(4.2)\qquad \sum_{1}^{\infty} \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{i-1}/(d_i + \tfrac{1}{2}) \qquad (\varepsilon_0 = 1)$$

with sum x. The series is in any case uniquely determined. The non-negative integers d_i satisfy the conditions

$$(4.3)\qquad d_{i+1} \geqslant \begin{cases} 2d_i^2 + d_i & (\varepsilon_i = 1); \\ 2d_i^2 + 3d_i + 1 & (\varepsilon_i = -1). \end{cases}$$

Proof. It is plain that if $\theta_1 \neq 0$

$$\frac{\varepsilon_1}{d_2 + \frac{1}{2} - \theta_2} = x_2 = x_1 - \frac{1}{d_1 + \frac{1}{2}} = \frac{\theta_1}{(d_1 + \frac{1}{2})(d_i + \frac{1}{2} - \theta_1)}$$

so that

$$d_2 + \tfrac{1}{2} - \theta_2 = (d_1^2 + d_1 + \tfrac{1}{4})/|\theta_1| - \varepsilon_1(d_1 + \tfrac{1}{2}),$$

$$d_2 \geqslant 2d_1^2 + 2d_1 - \varepsilon_1(d_1 + \tfrac{1}{2}) + \theta_2.$$

Thus $d_2 \geqslant 2d_1^2 + d_1$ if $\varepsilon_1 = 1$; $d_2 \geqslant 2d_1^2 + 3d_1 + 1$ if $\varepsilon_1 = -1$ since

$$-\tfrac{1}{2} < \theta_2 \leqslant \tfrac{1}{2}.$$

But this is (4.3) for $i = 1$. So for $i \geqslant 2$.

Suppose now that $2n < x_1 \leqslant 2n + 2$ for some integer $n \leqslant 1$. Then $d_1 = \ldots = d_{n-1} = 0$; $x_2 = x_1 - 2, \ldots, x_n = x_1 - 2n$ so that $0 < x_n \leqslant 2$. But, for $0 < x_n \leqslant 1$, $d_n \geqslant 1$ so that d_i now increases (and indeed rapidly) for $i > n$. If however $1 < x_n \leqslant 2$, then $d_n = 0$; either $\theta_n = 0$, $x_n = 2$ or $\varepsilon_n = -1$ and so $d_{n+1} \geqslant 1$; d_i increases for $i > n + 1$.

In any case the series terminates or $d_i \to \infty$ and $x_{i+1} \to 0$; the series (3.2) has sum x.

Conversely we have

THEOREM 2. *Suppose that the positive integers d_i satisfy the inequalities*

$$d_{i+1} \geqslant 2d_i^2 + d_i \qquad (\varepsilon_i = 1),$$
$$d_{i+1} \geqslant 2d_i^2 + 3d_i + 1 \qquad (\varepsilon_i = -1),$$

for all $i \geqslant 1$. Then with one exception the convergent infinite series

$$\sum \varepsilon_1 \varepsilon_2 \ldots \varepsilon_{i-1}/(d_i + \tfrac{1}{2}) \qquad (\varepsilon_0 = 1)$$

is the expansion of its sum by the algorithm (4.1).

The exception occurs when for some i

$$\varepsilon_i = -1, \quad d_{i+1} = 2d_i^2 + 3d_i + 1,$$

and

$$\varepsilon_j = 1, \quad d_{j+1} = 2d_j^2 + d_j \quad (\textit{all } j \geqslant i + 1).$$

Remarks. 1. I have assumed for simplicity that $d_i \geqslant 1$ and that the series is non-terminating. It is easy to adapt the proof for the other cases.

2. In the excepted case $x_i = 1/(d_i + 1)$ for which there is the expansion $\sum_{j \geqslant i} 1/(u_j + \frac{1}{2})$, $u_i = d_i + 1$, $u_{j+1} = 2u_j^2 + u_j$ $(j \geqslant i)$.

To prove Theorem 2 it is enough to prove that for $i \geqslant 1$,

$$\frac{1}{d_i + \frac{1}{2}} < x_i \leqslant \frac{1}{d_i} \qquad (\varepsilon_i = 1),$$

$$\frac{1}{d_i + 1} \leqslant x_i < \frac{1}{d_i + \frac{1}{2}} \qquad (\varepsilon_i = -1).$$

Note that

$$\sum_{j=i}^{\infty} 1/(\delta_j + \tfrac{1}{2}) = 1/\delta_i \quad \text{when} \quad \delta_i \geqslant 1$$

and

$$\delta_{j+1} = 2\delta_j^2 + \delta_j \quad (j = i, i+1, \ldots).$$

It follows at once that for any i $x_i \leqslant 1/d_i$ (with equality if and only if $d_{j+1} = 2d_j^2 + d_j$ (all $j \geqslant i$)). Since $x_{i+1} \leqslant 1/d_{i+1}$, it follows that for $\varepsilon_i = -1$

$$x_i = \frac{1}{d_i + \frac{1}{2}} - x_{i+1} \geqslant \frac{2}{2d_i + 1} - \frac{1}{(2d_i + 1)(d_i + 1)} = \frac{1}{d_i + 1}.$$

There is equality iff $\varepsilon_i = -1$, $d_{i+1} = 2d_i^2 + 3d_i + 1$ and $\varepsilon_j = 1$, $d_{j+1} = 2d_j^2 + d_j$ for all $j \geqslant i + 1$. The argument shows also that $x_i > 0$ (all i) and so, for $\varepsilon_i = 1$, $x_i > 1/(d_i + \frac{1}{2})$, for $\varepsilon_i = -1$, $x_i < 1/(d_i + \frac{1}{2})$ as stated. And Theorem 2 is proved.

THEOREM 3. *If x is a positive rational number, then either the series terminates or from some stage on*

$$d_{i+1} = 2d_i^2 + d_i, \quad \varepsilon_i = 1.$$

And conversely.

The converse is trivial. To prove the first part: if x_1 is rational, each x_i is rational. Let $x_i = p_i/q_i$ for positive coprime integer p_i and q_i (except that some $p_i = 0$ if the process ends). Then

$$\frac{p_{i+1}}{q_{i+1}} = \left|\frac{p_i}{q_i} - \frac{2}{2d_i+1}\right|,$$

$$\lambda_i p_{i+1} = |p_i(2d_i+1) - 2q_i| \leqslant p_i,$$

$$\lambda_i q_{i+1} = q_i(2d_i+1),$$

for some integer $\lambda_i \geqslant 1$. Hence $p_i \geqslant p_{i+1} \geqslant 0$; either we reach a suffix i for which $p_i = 0$ and process ends or else for some suffix i $p_{i+1} = p_i$ and so $\lambda_i = 1$, $p_i(2d_i+1) - 2q_i = p_i$, $p_i = 1$, $q_i = d_i$, $d_{i+1} = d_i(2d_i+1)$, $p_{i+1} = 1$ and so on for all $j \geqslant i$.

5. The series (3.4). For $x = x_1 > 0$ the algorithm (4.1) is slightly modified:

(5.1) $$\varepsilon_i x_{i+1} = (d_i + \tfrac{1}{2})x_i - 1, \quad d_i = [1/x_i],\ 1/x_i = d_i + \tfrac{1}{2} - \theta_i,$$
$$-\tfrac{1}{2} < \theta_i \leqslant \tfrac{1}{2}, \quad \varepsilon_i = \operatorname{sgn}\theta_i \quad (i \geqslant 1).$$

The process ends if $\theta_i = 0$ for some i (and then of course x is rational).

THEOREM 4. *The algorithm* (5.1) *applied to $x > 0$ leads to a unique series (finite or infinite)*

(5.2) $$\sum_1^{\infty} \varepsilon_1\varepsilon_2 \ldots \varepsilon_{i-1} 2^i/(2d_1+1)\ldots(2d_i+1) \quad (\varepsilon_0 = 1)$$

with sum x. The non-negative integers d_i satisfy the inequalities

(5.3) $$d_{i+1} \geqslant 2d_i \quad (\varepsilon_i = 1),$$
$$d_{i+1} \geqslant 2d_i + 2 \quad (\varepsilon_i = -1)$$

and at least one $d_i \geqslant 1$.

THEOREM 5. *Given a convergent series* (5.2) *in which the integers $d_i \geqslant 0$ satisfy* (5.3), *then with one exception the series* (5.2) *is the expansion of its sum by* (5.1).

The exception occurs when for some i

$$\varepsilon_i = -1, \quad d_{i+1} = 2d_i + 2$$

and

$$\varepsilon_j = 1, \quad d_{j+1} = 2d_j \quad (all\ j \geqslant i+1).$$

THEOREM 6. *For rational $x > 0$, the algorithm* (5.1) *leads either to a finite sum or an infinite series in which eventually $d_{i+1} = 2d_i$.*

I omit the proofs of Theorems 4, 5, 6.

6. The series (3.6) $\sum \varepsilon_1\varepsilon_2 \ldots \varepsilon_{i-1}/(2d_1+1)\ldots(2d_i+1)$ $(\varepsilon_0 = 1)$. Precise results can also be given for (3.6). These are given without proof in

THEOREM 7. *For $x = x_1 > 0$ use the algorithm*

(6.1) $$d_i = [1/2x_i], \quad 1/x_i = 2d_i + 1 - \varphi_i, \quad -1 < \varphi_i \leqslant 1,$$
$$\varepsilon_i = \operatorname{sgn}\varphi_i, \quad \varepsilon_i x_{i+1} = (2d_i+1)x_i - 1 \quad (i = 1, 2, \ldots)$$

(the process ending if any $\varphi = 0$). The series (3.6) *so found is unique; its sum is x. The integers $d_i \geqslant 0$ satisfy the conditions*

(6.2) $$d_{i+1} \geqslant d_i \ (\varepsilon_i = 1), \quad d_{i+1} \geqslant d_i + 1 \ (\varepsilon_i = -1).$$

The necessary conditions are also sufficient. An infinite series (3.6) *in which the integers $d_i \geqslant 0$ satisfy* (6.2) *and at least one $d_i \geqslant 1$ is the expansion of its sum by* (6.1).

For rational x the series terminates or else the d_i become periodic, i.e. $\varepsilon_i = 1$ and $d_{i+1} = d_i$ for all large i.

7. The series (3.5) $\sum 1/(2d_1+1)\ldots(2d_i+1)$. For this series we use the algorithm on $x = x_1 > 0$,

(7.1) $$d_i = [(x_i+1)/2x_i], \quad x_{i+1} = (2d_i+1)x_i - 1 \quad (i \geqslant 1).$$

A unique convergent series with sum x is obtained. Necessary conditions satisfied by the integers $d_i \geqslant 0$ are

(7.2) $$d_{i+1} \geqslant \tfrac{1}{2}d_i \ (d_i \text{ even}), \quad d_{i+1} \geqslant \tfrac{1}{2}(d_i - 1) \ (d_i \text{ odd})$$

but these conditions are not sufficient. A simple set of sufficient conditions to ensure that (3.5) is the expansion of its sum by the algorithm (7.1) is given by

(7.3) $$d_{i+1} \geqslant \tfrac{1}{2}d_i + 1 \ (d_i \text{ even}), \quad d_{i+1} \geqslant \tfrac{1}{2}(d_i + 1) \ (d_i \text{ odd}).$$

This set can be weakened but the result is not the best possible. For rational $x > 0$ a precise theorem can be stated.

THEOREM 8. *When the algorithm* (7.1) *is applied to rational $x > 0$, the resulting sequence of integers $\{d_i\}$ is ultimately periodic.*

Proof. From the algorithm

$$x = x_1 = u_1 + u_2 + \ldots + u_i x_{i+1},$$
$$u_i = 1/(2d_1+1)\ldots(2d_i+1)$$

where

$$0 < x_{j+1} = (2d_j+1)x_j - 1 \leqslant 2x_j.$$

Hence in succession

$$\sum_1^i u_j < x; \quad \sum_1^\infty u_j = u \leqslant x; \quad u_i \to 0 \text{ as } i \to \infty;$$
$$u_i x_{i+1} \to v \geqslant 0; \quad u + v = x_1.$$

The sequence $\{x_i\}$ must contain a bounded subsequence. For if not then $x_i \to \infty$ as $i \to \infty$; $d_i = [\frac{1}{2} + 1/2x_i] = 0$, all $i > i_0$. But then $u_{i+1} = u_i$ (all $i > i_0$) and $\sum u_j$ cannot be convergent.

Thus the sequence $\{x_i\}$ contains a bounded subsequence. For this subsequence $u_i x_{i+1} \to 0$, $v = 0$, $\sum u_j = x_1$. Now suppose that x_1 is rational and so all x_i are rational, $x_i = p_i/q_i$ for positive coprime integers p_i, q_i.

From $x_{i+1} = (2d_i+1)x_i - 1$ we obtain

$$\lambda_i p_{i+1} = (2d_i+1)p_i - q_i \leqslant 2p_i,$$
$$\lambda_i q_{i+1} = q_i \quad (\lambda_i \geqslant 1 \text{ integer}).$$

Hence, for all $i > i_0$, $q_i = Q \geqslant 1$, $\lambda_i = 1$. Now we showed above that there is a bounded subsequence of x_i. Since $q_i = Q$ (all $i > i_0$) there is a bounded subsequence of p_i. Hence there must be one value of $p_i = P$ which occurs at least twice:

$$p_h = P = p_k, \quad q_h = Q = q_k \quad (1 < h < k).$$

But clearly periodicity of the d_i results. Theorem 8 follows.

8. The series (3.3) $\sum 2^i/(2d_1+1)\ldots(2d_i+1)$.

THEOREM 9. *Apply to $x = x_1 > 0$ the algorithm*

(8.1) $$d_i = [\tfrac{1}{2} + 1/x_i], \quad x_{i+1} = (d_i + \tfrac{1}{2})x_i - 1 \quad (i = 1, 2, \ldots).$$

Then

(8.2) $$x = \sum_1^\infty 2^i/(2d_1+1)\ldots(2d_i+1)$$

where the integers $d_i \geqslant 0$ satisfy the conditions

(8.3) $$d_{i+1} \geqslant d_i \quad (\text{and at least one } d_i \geqslant 1).$$

These conditions are both necessary and sufficient for the expansion.

I omit the proof.

The rationality question remains open. It is easy to see that the sum is rational in each of the cases:

(i) $d_{i+1} = d_i$ eventually,

(ii) $d_{i+1} = 2d_i$ eventually.

To repeat what was stated earlier (Section 3) I conjecture that these are the only cases of rationality; in other words the sequence of operations on positive coprime integers p_1, q_1 defined by

$$\frac{p_{i+1}}{q_{i+1}} = \frac{(2d_i+1)p_i - 2q_i}{2q_i}, \quad 0 < (2d_i+1)p_i - 2q_i \leqslant 2p_i \quad (i = 1, 2, \ldots)$$

gives ultimately all $p_i = 2$ or all $p_i = 1$. The first case leads to $d_{i+1} = d_i$ eventually; the second to $d_{i+1} = 2d_i$ eventually.

9. The series (3.1) $\sum 1/(2d_i+1)$. The algorithm for this series is given by

$$d_i = [\tfrac{1}{2} + 1/2x_i], \quad x_{i+1} = x_i - \frac{1}{2d_i+1}, \quad x_1 = x > 0.$$

Necessary conditions for the validity of the expansion are

$$d_{i+1} \geqslant d_i^2 \quad (\text{and at least one } d_i \geqslant 2).$$

But these conditions are not sufficient. A simple set of sufficient conditions is given by

$$d_{i+1} \geqslant d_i^2 + 1.$$

Another and a weaker set of sufficient conditions is provided by

$$d_{i+1} \geqslant d_i^2 \quad \text{or} \quad d_{i+1} = d_i^2 \quad \text{and} \quad d_{i+2} = \tfrac{4}{3}d_{i+1}^2 + \tfrac{1}{3}d_{i+1} + \tfrac{1}{3}.$$

Naturally a complete set of necessary and sufficient conditions is given by

$$\frac{1}{2d_i - 1} - \left\{\frac{1}{2d_i+1} + \frac{1}{2d_{i+1}+1} + \ldots + \frac{1}{2d_j+1}\right\} = x_{j+1} > \frac{1}{2d_{j+1}+1}$$

for $1 \leqslant i \leqslant j$; $i, j = 1, 2, 3, \ldots$ But to obtain in a simple form a set of necessary and sufficient conditions appears to be very difficult.

I am unable to determine the form the expansion takes when x_1 (and so each x_i) is rational. We have

CONJECTURE 3. *Given coprime positive integers p_1, q_1, determine coprime positive integers p_i, q_i $(i > 1)$ as follows:*

$$0 < \lambda_i p_{i+1} = p_i(2d_i+1) - q_i \leqslant 2p_i \quad (d_i \text{ integer}),$$
$$\lambda_i q_{i+1} = q_i(2d_i+1) \quad (\lambda_i \text{ integer}).$$

Then the numerators p_i are ultimately periodic.

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Representation of Markoff's binary quadratic forms by geodesics on a perforated torus

by

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In memory of Harold Davenport

1. Introduction. One of Harold Davenport's most remarkable contributions was a succession of papers (notably [4] and [5]) on the minima of the product of three ternary homogeneous linear forms (compare Mordell [11]). Davenport showed that (with unit determinant) the two largest minima, $1/7$ and $1/9$, are discrete. No further minima have been established since then.

One of the reasons that this problem is so intriguing and challenging is the comparison one naturally makes with the Markoff theory of binary (indefinite) quadratic forms (see [10], [6], [2]). The Markoff theory represents a state of perfection at the fringes of utter chaos! A discrete, convergent sequence of minima exists with a limit point $(1/3)$ below which the spectrum of minima varies locally from continuous to discrete ([9], [12]). The original theory depended heavily on continued fractions, although a revision of Frobenius [7] made the theory depend more on chains of reduced forms. A paper of the author [3] used as a substitute tool some algebraic (matrix) identities which, in principle, are less specialized than continued fractions.

We now return to our earlier approach [3] in the hope that additional insight might be gained in understanding the discrete nature of the minima by an exploration of the geometric aspects of the Markoff forms. We interpret these forms in terms of closed geodesics of preassigned homology type on a perforated torus. It is possible, specifically, to gain a better understanding of some of the "fringe" behavior at the limit point of the discrete set of minima.

2. Rational Markoff forms. We briefly summarize the classical theory. Let [10]

$$(1) \qquad Q(x, y) = ax^2+bxy+cy^2, \quad d = b^2-4ac > 0$$