[20] G. Shimura, *On the theory of automorphic functions*, Ann. of Math. 70 (1959), pp. 101–144.
[21] O. Zariski, *An introduction to the theory of algebraic surfaces*, Springer Lecture Notes, No. 68, p. 67.

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Cyclic overlattices, II
(Diophantine approximation and sums of roots of unity)

by

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In memory of Professor H. Davenport

1. Introduction. In this paper I shall examine some consequences of three theorems on the Geometry of Numbers which were proved in an earlier paper ([2]). The notation and terminology of that paper will be assumed, often without special comment.

In 1967 Davenport and Schinzel ([1]) considered the following question. Given integers $a_1, \ldots, a_k, q$ with

$$(a_1, \ldots, a_k, q) = 1, \tag{1}$$

can we find an integer n with

$$(n, q) = 1 \tag{2}$$

for which

$$\max_{1\leqslant i\leqslant k} \|na_i/q\| < \delta. \tag{3}$$

Here it is to be understood that δ is a fixed positive number $(0 < \delta < 1/2)$ and that q is large. If the condition (2) were replaced by $n \not\equiv 0 \pmod{q}$ the answer would be affirmative, and by Dirichlet's theorem on Diophantine approximation it would suffice if $q \geqslant \delta^{-k}$. As they observed the answer to the above question cannot, however, be unconditionally affirmative. For suppose there is a linear relation

$$h_1a_1 + \ldots + h_ka_k = hq, \tag{4}$$

in which $h \neq 0$ and

$$(h_1, \ldots, h_k, h) = 1, \tag{5}$$

$$(h_1, \ldots, h_k) > 1, \tag{6}$$

and

$$\sum_{i=1}^{k} |h_i| < \delta^{-1}. \tag{7}$$

Then there is no solution of (2) and (3). For if

$$d = (h_1, \ldots, h_k), \quad h_i = dh_i' \quad (1 \leq i \leq k),$$

then $(d, h) = 1$ by (5) and $d \mid q$ by (4). Now

$$\|n(h_1'a_1 + \ldots + h_k'a_k)/q\| = \|nh/d\| > d^{-1},$$

since $(n, d) = 1$ by (2). Hence

$$|h_1'| \, \|na_1/q\| + \ldots + |h_k'| \, \|na_k/q\| > d^{-1}$$

which by virtue of (7) contradicts (3). (This whole statement is in fact a somewhat weaker version of [2] Theorem 1.)

In their paper Davenport and Schinzel proved that in principle (that is, apart from the particular function of δ and, if we wish, k on the right of (7)), the non-existance of a linear relation of the above kind is in fact sufficient for the solubility of (2) and (3). To be explicit they proved

THEOREM. *Let $a_1, \ldots, a_k, q$ be integers satisfying* (1). *Suppose that for every integer n with $(n, q) = 1$ we have*

(8) $$\max_{1 \leq i \leq k} \|na_i/q\| > \delta \quad (0 < \delta < \tfrac{1}{2}).$$

Then for all sufficiently large q (i.e. $q > q_0(k, \delta)$) there exist integers $h_1, \ldots, h_k, h$ with $h \neq 0$ satisfying (4), (5) *and* (6) *such that*

(9) $$\sum_{i=1}^{k} |h_i| \leq c(k)\, \delta^{-(k+1)} \left(\log (2\delta)^{-k}\right)^{k+2}.$$

Returning to the notation of [2] we shall prove the following

THEOREM 1. *Suppose F^* satisfies condition* C *and that for every point $x \in \Lambda$ which generates Λ over M we have*

(10) $$F(x) > \delta.$$

Let k be the dimension of the space of Λ and M. Then, if $k \geq 3$, given any (small) $\varepsilon > 0$ there exists an integer r, $1 \leq r \leq k$, and r linearly independent points $z_1^*, \ldots, z_r^*$ of Λ^* such that

(11) $$F^*(z_i^*) \leq c(k, \varepsilon)\, \delta^{-(1+\varepsilon)} \quad (1 \leq i \leq r).$$

Also there exists a point $z^* \in \Lambda^*$, primitive in Λ^* but not primitive in M^*, linearly dependent on $z_1^*, \ldots, z_r^*$ and such that

(12) $$F^*(z^*) \leq c(k, \varepsilon)\, \delta^{-(1+\varepsilon)r}.$$

Furthermore if

(13) $$[\Lambda : M] = q > c(k, \varepsilon)\, \delta^{-(1+\varepsilon)k}$$

then (11) *and* (12) *hold for some r with $1 \leq r \leq k-1$.*

Finally if $k = 2$ then (11), (12) *and* (13) *hold with $\varepsilon = 0$ and constants which depend only on k.*

It is perhaps worth stressing that the conclusions (11) and (12) hold for *all* $q > 1$ and not simply for all sufficiently large q.

To interpret this in terms of the Davenport–Schinzel theorem we take M to be the integer lattice in k dimensions and put

(14) $$F(x) = \max_{1 \leq i \leq k} |x_i|$$

where $x = (x_1, \ldots, x_k)$ with respect to the standard basis. Then

(15) $$F^*(x^*) = \sum_{i=1}^{k} |x_i^*|,$$

where $x^* = (x_1^*, \ldots, x_k^*)$, and hence clearly satisfies condition C. Given integers $a_1, \ldots, a_k, q$ which satisfy (1) we consider the lattice Λ generated by the point $a = (a_1/q, \ldots, a_k/q)$ and M. The hypothesis (10) is then precisely the assertion of (8). Moreover Λ^* is now the set of integer points $(h_1, \ldots, h_k)$ which satisfy

$$h_1 a_1 + \ldots + h_k a_k = hq$$

for some integer h. Such a point is primitive in Λ^* if (5) holds and imprimitive in M^* if (6) holds (observe that (5) and (6) together imply $h \neq 0$, provided the point in question is not the origin).

We now see that Theorem 1 implies the Davenport–Schinzel theorem. If $r = 1$ ($k \geq 3$) the right-hand side of (9) is replaced by $c(k, \varepsilon)\, \delta^{-(1+\varepsilon)}$ and if (as is the worst case, assuming (13) holds) $r = k-1$ we obtain the bound $c(k, \varepsilon)\, \delta^{-(k-1)(1+\varepsilon)}$, which is still an improvement over (9). However the theorem implies somewhat more, for although as r increases (12) gives a progressively worse bound for $F^*(z^*)$, this loss is compensated by an increasing number of well bounded, linearly independent, points of Λ^* given by (11).

In a second application of the theorems of [2] we shall consider what information can be extracted concerning sums of roots of unity (see § 3, Theorem 2). In particular we shall show that if

$$a = 1 + e\left(\frac{a_1}{q}\right) + \ldots + e\left(\frac{a_k}{q}\right) \quad (e(\theta) = e^{2\pi i\theta})$$

where $(a_1, \ldots, a_k, q) = 1$, is a sum of $k+1$ roots of unity, then by making all the prime factors of q sufficiently large we can make $|\overline{a}|$(¹) as near to $k+1$ as we please (see § 3, Theorem 2, Corollary or § 7, Theorem 3).

(¹) We denote by $|\overline{a}|$ the maximum absolute value of any algebraic conjugate of a, including a itself.

2. Proof of Theorem 1. We first deal with the case $k \geqslant 3$.

In [2] Theorem 2, take $\varepsilon = \varepsilon_1 < 1/k$, and let $c_1(k, \varepsilon_1)$ be the corresponding constant in [2] (27). We saw in [2] § 4 that condition C on F^* implies $V_F^{-1} \leqslant c(k)$, hence the inequalities [2] (27) and [2] (28) certainly imply bounds on the right of the type $c_2(k, \varepsilon_1)D^{-1}$ and $c_3(k)D^{-1}$ respectively. We can (by increasing $c_2(k, \varepsilon_1)$ and $c_1(k, \varepsilon_1)$ if necessary) arrange that $c_1(k, \varepsilon_1) > 1$ and

(16) $$c_3(k)D^{-1} \leqslant c_2(k, \varepsilon_1)D^{-1} \leqslant c_1(k, \varepsilon_1)D^{-1+\varepsilon_1}$$

for all $D > 1$.

Now choose $D > 1$ so that

(17) $$\delta = c_1(k, \varepsilon_1)D^{-1+\varepsilon_1}.$$

For this D, (10) and (16) provide a contradiction to the conclusion of [2] Theorem 2. Hence

(18) $$\Lambda_D^* \neq M_D^*,$$

which certainly implies that $1 \leqslant r \leqslant k$, where r is the dimension of W_D^*.

Choose ε_1 $(< 1/k)$ so that

$$\varepsilon = \varepsilon_1/(1-\varepsilon_1),$$

where ε is as in the enunciation. From (17) we have

(19) $$D = c(k, \varepsilon)\,\delta^{-(1+\varepsilon)}.$$

The existence of linearly independent points $z_1^*, \ldots, z_r^*$ satisfying (11) now follows from (19) and the definition of W_D^*.

To obtain the point z^* of (12) we observe that (18) permits us to apply [2] Theorem 3, with the D given by (19). Hence [2] (37) implies

$$F^*(z^*) \leqslant c(k)[c(k, \varepsilon)\,\delta^{-(1+\varepsilon)}]^r \leqslant c(k, \varepsilon)\,\delta^{-(1+\varepsilon)r}$$

as required. Similarly to obtain (13) we use the final statement of [2] Theorem 3.

If $k = 2$ then in [2] Theorem 2, only the inequalities [2] (28) and [2] (29) apply. This means that in (16) we may ignore the last inequality and putting $\varepsilon_1 = 1/(k+1)$ we may choose $c_2(k, \varepsilon_1) = c_2(k)$ so that $c_2(k) > 1$ and

$$c_1(k)D^{-1} \leqslant c_2(k)D^{-1}$$

for all $D > 1$. We now choose D so that

$$\delta = c_2(k)D^{-1}$$

and proceed as before. This concludes the proof of Theorem 1.

3. Sums of roots of unity. Let

(20) $$\alpha = 1 + e\left(\frac{a_1}{q}\right) + \ldots + e\left(\frac{a_k}{q}\right)$$

where $(a_1, \ldots, a_k, q) = 1$. To apply the results of [2] we shall again take M to be the integer lattice in k dimensions and Λ to be the lattice generated by the point $\boldsymbol{a} = (a_1/q, \ldots, a_k/q)$, where $a_1, \ldots, a_k$ and q are the integers of (20). The case $k = 1$ is trivial, from now on we always suppose $k \geqslant 2$. Our result is then

Theorem 2. *Let α be a sum of $k+1$ roots of unity given by (20) with associated lattice Λ. Let $D > 1$ be any given real number. Then* **either** (i) *$\Lambda_D^* \neq M_D^*$ and the following conditions are satisfied:*

(A) *There are r $(1 \leqslant r \leqslant k)$ linearly independent relations between the a_i which satisfy*

(21) $$h_1a_1 + \ldots + h_ka_k = hq,$$

(22) $$(h_1, \ldots, h_k, h) = 1$$

and

(23) $$\sum_{i=1}^{k} |h_i| < D.$$

(B) *There is a relation between the a_i which is linearly dependent on those of* (A), *satisfies* (21), (22) *and*

(24) $$(h_1, \ldots, h_k) > 1,$$

(25) $$\sum_{i=1}^{k} |h_i| \leqslant c(k)D^r.$$

(C) *If $D > 2$*

(26) $$\overline{|\alpha|}^2 \leqslant (k+1)^2 - 8(k+1)D^{-2};$$

or (ii) *$\Lambda_D^* = M_D^*$, and*

(27) $$\overline{|\alpha|}^2 > (k+1)^2 - c(k, \varepsilon)D^{-2(1-\varepsilon)}.$$

Furthermore if $q > c(k)D^k$ then in (i) *we may take $1 \leqslant r \leqslant k-1$.*

We have the immediate

Corollary. *If $q > c(k)D^k$ and every prime factor p of q satisfies $p > c(k)D^{k-1}$ then* (27) *holds.*

4. Two lemmas.

LEMMA 1. *The region $\mathscr{R}$ in k dimensional space defined by*

(28) $$|1+e(x_1)+\ldots+e(x_k)|^2 > (k+1)^2-K,$$

(29) $$|x_i| \leqslant \tfrac{1}{2} \quad (1 \leqslant i \leqslant k),$$

where $0 < K < (k+1)^2$, *contains the ellipsoid defined by*

(30) $$\sum_{i=1}^{k} x_i^2 + \sum_{i<j} (x_i - x_j)^2 < K/4\pi^2,$$

provided $0 < K \leqslant \frac{1}{2}\pi^2(k+1)$.

Proof. We may rewrite (28) as

$$k+1+\sum_{i=1}^{k} 2\cos 2\pi x_i + \sum_{i<j} 2\cos 2\pi(x_i-x_j) > (k+1)^2 - K.$$

Since $\cos\theta \geqslant 1-\frac{1}{2}\theta^2$ for all real θ, this inequality will be satisfied if

$$\sum_{i=1}^{k} 2(1-2\pi^2 x_i^2) + \sum_{i<j} 2\big(1-2\pi^2(x_i-x_j)^2\big) > (k+1)^2-(k+1)-K,$$

which easily reduces to (30). Furthermore (30) implies $|x_\nu| \leqslant \frac{1}{2}$ $(1 \leqslant \nu \leqslant k)$ since $K \leqslant \frac{1}{2}\pi^2(k+1)$. To see this consider the following chain of inequalities, where to avoid notational complications we take x_1 to be a typical x_ν.

$$\begin{aligned}\sum_{i=1}^{k} x_i^2 + \sum_{i<j} (x_i-x_j)^2 &= x_1^2 + \sum_{i=2}^{k} x_i^2 + \sum_{j=2}^{k} (x_1-x_j)^2 + \sum_{\substack{i<j\\ i\neq 1}} (x_i-x_j)^2 \\ &\geqslant kx_1^2 - 2x_1 \sum_{j=2}^{k} x_j + 2\sum_{j=2}^{k} x_j^2 \\ &\geqslant \frac{(k+1)}{2} x_1^2 + \sum_{j=2}^{k} \left(\frac{x_1^2}{2} - 2x_1x_j + 2x_j^2\right) \\ &= \frac{(k+1)}{2} x_1^2 + \sum_{j=2}^{k} \left(\frac{x_1}{\sqrt{2}} - \sqrt{2}x_j\right)^2 \geqslant \frac{(k+1)}{2} x_1^2.\end{aligned}$$

Hence by symmetry

(31) $$\sum_{i=1}^{k} x_i^2 + \sum_{i<j} (x_i-x_j)^2 \geqslant \frac{(k+1)}{2} x_\nu^2$$

for any ν $(1 \leqslant \nu \leqslant k)$. Clearly equality occurs in (31) only if

$$x_i = x_\nu/2 \quad (1 \leqslant i \leqslant k,\ i \neq \nu).$$

From (30), (31) and $K \leqslant \frac{1}{2}\pi^2(k+1)$ we have $|x_\nu| \leqslant \frac{1}{2}$ $(1 \leqslant \nu \leqslant k)$ as required.

COROLLARY. *The region $\mathscr{R}$ contains the cube* $|x_i| < H$ $(1 \leqslant i \leqslant k)$ *provided*

(32) $$H^2 \leqslant K/2\pi^2(k+1).$$

Proof. By Lemma 1 and (31).

LEMMA 2. *The region $\mathscr{R}$ is contained in the ellipsoid defined by*

(33) $$\sum_{i=1}^{k} x_i^2 + \sum_{i<j} (x_i-x_j)^2 < K/16$$

provided $0 < K \leqslant 4k$.

Proof. As in Lemma 1 we write the inequality (28) as

(34) $$k+1+\sum_{i=1}^{k} 2\cos 2\pi x_i + \sum_{i<j} 2\cos 2\pi(x_i-x_j) > (k+1)^2 - K.$$

The next step is to show that for K in the range $0 < K \leqslant 4k$ (34) implies $|x_i - x_j| \leqslant \frac{1}{2}$ for $1 \leqslant i < j \leqslant k$. If this is not the case then we may suppose, without loss of generality, that

(35) $$\tfrac{1}{2} < |x_1 - x_2| \leqslant 1.$$

Now

$$\cos 2\pi x_1 + \cos 2\pi x_2 + \cos 2\pi(x_1-x_2) = 4\cos\pi(x_1-x_2)\cos\pi x_1\cos\pi x_2 - 1.$$

By (35) and (29) with $i = 1, 2$ the first term in this last expression is negative. Hence

(36) $$\cos 2\pi x_1 + \cos 2\pi x_2 + \cos 2\pi(x_1-x_2) < -1.$$

Denote the left-hand side of (34) by L. Then using (36) we see that

(37) $$\mathrm{L} < (k+1)-2+\sum_{i=3}^{k} 2\cos 2\pi x_i + \sum_{\substack{i<j\\ (i,j)\neq(1,2)}} 2\cos 2\pi(x_i-x_j).$$

In the final summation of (37) replace each cosine for which $i \geqslant 3$ by its maximum (viz. unity). Then

$$\mathrm{L} < (k+1)-2+2\left[\tfrac{1}{2}k(k-1)-1-2(k-2)\right] + 2\sum_{j=3}^{k}\big(\cos 2\pi x_j + \cos 2\pi(x_1-x_j) + \cos 2\pi(x_2-x_j)\big),$$

that is

(38) $$\mathrm{L} < k^2-4k+5+2\sum_{j=3}^{k}\big(\cos 2\pi x_j + \cos 2\pi(x_1-x_j) + \cos 2\pi(x_2-x_j)\big).$$

Consider the terms in the summation of (38). We have

$$C = \cos 2\pi x_j + \cos 2\pi(x_1 - x_j) + \cos 2\pi(x_2 - x_j)$$
$$= (1 + \cos 2\pi x_1 + \cos 2\pi x_2)\cos 2\pi x_j + (\sin 2\pi x_1 + \sin 2\pi x_2)\sin 2\pi x_j.$$

Whence by Cauchy's inequality

$$C \leqslant \big((\cos 2\pi x_j)^2 + (\sin 2\pi x_j)^2\big)^{1/2}\big((\ldots)^2 + (\ldots)^2\big)^{1/2}$$
$$\leqslant \big(3 + 2\,(\cos 2\pi x_1 + \cos 2\pi x_2 + \cos 2\pi(x_1 - x_2))\big)^{1/2} < 1,$$

by (36). We now obtain from (38)

$$\mathrm{L} < k^2 - 4k + 5 + 2(k-2) = k^2 - 2k + 1.$$

On the other hand from (34) we have

$$\mathrm{L} > (k+1)^2 - K.$$

These two inequalities taken together imply $K > 4k$, which is contrary to hypothesis. We have shown that (34) implies $|x_i - x_j| \leqslant \frac{1}{2}$ for $1 \leqslant i < j \leqslant k$.

Now $\cos\theta \leqslant 1 - \left(\dfrac{2}{\pi^2}\right)\theta^2$ for $-\pi \leqslant \theta \leqslant \pi$, so (34) implies

$$2\sum_{i=1}^{k}(1 - 8x_i^2) + 2\sum_{i<j}\big(1 - 8(x_i - x_j)^2\big) > (k+1)^2 - (k+1) - K,$$

which reduces to (33).

COROLLARY. *The region $\mathscr{R}$ where $0 < K \leqslant 4k$, is contained in the cube $|x_i| < H$ $(1 \leqslant i \leqslant k)$ provided*

(39) $$H^2 \geqslant K/8(k+1).$$

Proof. By Lemma 2, (31) and the remark following (31).

5. The geometric interpretation. From the two lemmas of § 4 it is clear that we ought to take our distance function F to be the one corresponding to the quadratic form which appears in (30) and (33). On the other hand by retaining the distance function

$$F(\boldsymbol{x}) = \operatorname*{Max}_{1 \leqslant i \leqslant k} |x_i|$$

of (14) we shall only lose a little on the constants in the final result and we are not bothering about the constants anyway. Thus we shall not use the full strength of Lemmas 1 and 2 but merely the corollaries.

The conjugates of α are

$$1 + e\left(\frac{na_1}{q}\right) + \ldots + e\left(\frac{na_k}{q}\right)$$

where $(n, q) = 1$. If

$$na_i \equiv n_i \pmod q \quad (1 \leqslant i \leqslant k),$$

where $|n_i| \leqslant \frac{1}{2}q$, we shall call $(n_1/q, \ldots, n_k/q)$ the *point corresponding to the n-th conjugate of α*. Each point corresponding to a conjugate of α is a generating point of Λ over M, and conversely.

We now state as two lemmas, for convenience of reference, the deductions from the corollaries of § 4 which are relevant to the problem.

LEMMA 3. *If $F(\boldsymbol{x}) < H$ for some generating point of Λ over M then*

$$\overline{|\alpha|}^2 > (k+1)^2 - 2\pi^2(k+1)H^2.$$

Proof. Take $K = 2\pi^2(k+1)H^2$ in (28). Then by hypothesis and Lemma 1 Corollary there is a point corresponding to a conjugate of α which lies in the region $\mathscr{R}$.

LEMMA 4. *If $F(\boldsymbol{x}) > H$ $(0 < H < \frac{1}{2})$ for all generating points of Λ over M then*

$$\overline{|\alpha|}^2 \leqslant (k+1)^2 - 8(k+1)H^2.$$

Proof. Take $K = 8(k+1)H^2$ in (28). Then by hypothesis and Lemma 2 Corollary every point which corresponds to a conjugate of α lies outside the region $\mathscr{R}$.

6. Proof of Theorem 2. If $\Lambda_D^* \neq \mathsf{M}_D^*$ then $1 \leqslant r \leqslant k$ where $r = \dim W_D^*$ and conclusion (A) follows from the definition of Λ_D^*. Conclusion (B) and the final statement of the enunciation follow immediately from [2] Theorem 3. Also by [2] Theorem 1, we have

$$F(\boldsymbol{x}) > 1/D$$

for every point $\boldsymbol{x}$ which generates Λ over M. Hence by Lemma 4 with $H = 1/D$ (so we require $D > 2$) we have

$$\overline{|\alpha|}^2 \leqslant (k+1)^2 - 8(k+1)D^{-2}$$

which is (26).

The other possibility is $\Lambda_D^* = \mathsf{M}_D^*$. In this case by [2] Theorem 2, we have

(40) $$F(\boldsymbol{x}) < c(k, \varepsilon)D^{-1+\varepsilon}$$

for some generating point $\boldsymbol{x}$ of Λ over M. By Lemma 3 with

$$H = c(k, \varepsilon)D^{-1+\varepsilon}$$

this means that

$$\overline{|\alpha|}^2 > (k+1)^2 - c(k, \varepsilon) D^{-2(1-\varepsilon)}$$

which is (27).

7. A remark on the Corollary to Theorem 2. One should perhaps remark that the hypothesis that every prime p which divides q satisfies $p > c(k) D^{k-1}$ enables one to eliminate the ε dependence of (27) and replace it by

$$\overline{|\alpha|}^2 > (k+1)^2 - c(k) D^{-2}. \tag{41}$$

To prove this we must consider the proof of [2] Theorem 2. We used there the fact that

$$g(n) < \frac{n 2^{\nu(n)}}{\varphi(n)} \leqslant c(\varepsilon) n^{\varepsilon} \tag{42}$$

applied to divisors of q. On our new hypothesis we can give an improved estimate for $g(n)$.

Lemma 5. *If every prime p which divides n satisfies $p > T > 1$ then*

$$g(n) < n^{(\log 2 + 2/T)/\log T}$$

so that

$$g(n) < n^{2/\log T} \qquad (T \geqslant 2). \tag{43}$$

Proof. We use the first estimate for $g(n)$ in (42). Our hypothesis clearly implies

$$\nu(n) < \log n / \log T. \tag{44}$$

For each prime $p|n$ we have

$$\frac{2}{(1-1/p)} \leqslant 2\left(1+\frac{2}{p}\right) < 2\left(1+\frac{2}{T}\right) = 2e^{\log(1+2/T)} < 2e^{2/T}.$$

Whence from (42)

$$g(n) < (2e^{2/T})^{\nu(n)} < n^{(\log 2 + 2/T)/\log T}$$

by (44). This proves the lemma.

If we now return to the proof of [2] Theorem 2, and use (43) with $T = c(k) D^{k-1}$ to obtain the upper bounds for the $|u_i|$, we see that the ε dependence of the first constant of [2] (36) vanishes and we can put

$$q^{\varepsilon} = q^{2/\log T}$$

and

$$\tau_i^{\varepsilon} = \tau_i^{2/\log T}.$$

To estimate the terms $\tau_i^{\varepsilon} \lambda_{i+1}$ we write

$$\tau_i^{\varepsilon} \lambda_{i+1} \leqslant \lambda_{i+1} \big(c(k) \mu_1 \ldots \mu_{k-i}\big)^{2/\log T} \leqslant c(k) \lambda_{i+1} (\mu_1 \ldots \mu_{k-i})^{2/\log T}$$

and then carry on as before. The final result is now

$$F(\boldsymbol{x}) \leqslant c(k) D^{-1+(k-1)2/\log T} \tag{45}$$

where $T = c(k) D^{k-1}$. The error term in (45) is

$$D^{(k-1)2/\log T} = \exp\left[(\log D)\left(\frac{2(k-1)}{(k-1)\log D + c(k)}\right)\right] \leqslant e^2.$$

Hence (45) asserts

$$F(\boldsymbol{x}) \leqslant c(k) D^{-1}. \tag{46}$$

We now use (46) instead of (40) in the proof of Theorem 2. To sum up we have

Theorem 3. *Let α be a sum of $k+1$ roots of unity given by (20). Let $D > 1$ be any given real number. If $q > c(k) D^k$ and every prime p which divides q satisfies $p > c(k) D^{k-1}$ then*

$$\overline{|\alpha|}^2 > (k+1)^2 - c(k) D^{-2}.$$

References

[1] H. Davenport and A. Schinzel, *Diophantine approximation and sums of roots of unity*, Math. Ann. 169 (1967), pp. 118–135.

[2] A. J. Jones, *Cyclic overlattices, I*, Acta Arith. 17 (1970), pp. 303–314.

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