On C_3 $(-1-Ti$ to $-1+Ti)$

$$\Gamma(s)\frac{\zeta'}{\zeta}(s)y^{-s} = O(e^{-A|t|}|t|^A y),$$

and so

$$\int_{C_3} = O(y). \tag{7}$$

Collecting from (5), (6), (7) and multiplying by y, we have, as $y\to 0+$,

$$-\frac{y}{2\pi i}\int_{2-Ti}^{2+Ti}\Gamma(s)\frac{\zeta'}{\zeta}(s)y^{-s}ds = 1-\sum\nolimits_1 y^{1-\varrho}\Gamma(\varrho)+o(1)$$

and making $T\to\infty$

$$-\frac{y}{2\pi i}\int_{2-\infty i}^{2+\infty i}\Gamma(s)\frac{\zeta'}{\zeta}(s)y^{-s}ds = 1-\sum_{(\varrho)} y^{1-\varrho}\Gamma(\varrho)+o(1),$$

or, by Lemma 2, as $y\to 0+$

$$y\sum\Lambda(n)e^{-ny} = 1-\sum_{(\varrho)}y^{1-\varrho}\Gamma(\varrho)+o(1). \tag{8}$$

Now, $\beta<1$, $|y^{1-\varrho}\Gamma(\varrho)|<|\Gamma(\varrho)|$, and $\sum|\Gamma(\varrho)|$ converges absolutely. Thus $\sum y^{1-\varrho}\Gamma(\varrho)$ is uniformly convergent in $y\geqslant 0$. Also each $|y^{1-\varrho}\Gamma(\varrho)|\to 0$ as $y\to 0$. By the uniform convergence theorem

$$\lim_{y\to+0}\sum_{(\varrho)}y^{1-\varrho}\Gamma(\varrho) = 0,$$

and from (8)

$$\lim y\sum\Lambda(n)e^{-ny} = 1.$$

It now follows from (I) that

$$\sum^{x}\Lambda(n)\sim x$$

which is equivalent to the P. N. T.

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On the graded rings of modular forms

by

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Dedicated to the memory of H. Davenport

The results. It needs some words of justification why the present article fits in the frame of a volume dedicated to the memory of H. Davenport. One may expect contributions which are more or less related to his work or which at least deal with algebraic or analytic number theory. The nature of this note may be, on the contrary, described as number theoretical analysis or algebra, but number theory nevertheless.

It is known since works by A. Selberg [16] and A. Weil [19] that the transformation groups underlying automorphic functions and forms in several variables have algebraic coefficients (or are equivalent to such). So number theory plays an important part in the definition of these groups. But the contribution of number theory is not exhausted by this. I need not mention the vast connections between modular and automorphic functions and class field theory; already Siegel's theory of Eisenstein series [18] and a recent generalization by Baily [1] show the contrary. Moreover, investigations by Igusa [13], Hammond [12], Freitag [9], and Gundlach [10], [11] derive the algebraic structure of the rings of special modular forms from number theoretical properties. The present note reports on some consequences following from this knowledge. The details of this investigation appear independently in a series of lecture notes [8]. Here we only describe the results and possible implications.

For a large class of automorphic forms Baily and Borel [2] — see also [7] — showed that they form finitely generated graded rings $\mathfrak{J}$. The degree is the weight of an automorphic form. Here we only deal with Hilbert and Siegel modular forms in the most narrow sense, the underlying groups being

$$\Gamma = \mathbf{SL}(2,\mathfrak{O}), \quad \text{resp } \mathbf{Sp}(m,\mathbf{Z})$$

where $\mathfrak{O}$ is the principal order of a totally real algebraic number field of degree n. The number of independent variables is n in the first case and $n=\frac{1}{2}m(m+1)$ in the second. By normalization of a system of generators of $\mathfrak{J}$ one obtains $n+1$ algebraically independent modular forms $y_0, \ldots, y_n$ of equal weight (which will be called h_0) such that every modular form depends integrally on the subring

$$\mathfrak{h} = C[y_0, \ldots, y_n].$$

The y_i form a system of projective coordinates of the modular variety M which covers the projective space P^n in a finite number of sheets.

In many cases (see [9], [10], [11], [12], [13]) $\mathfrak{J}$ is a free $\mathfrak{h}$-module for a suitable system of y_ν, and there are reasons why the following may be true under rather general conditions:

HYPOTHESIS. *There exists a system y_ν of modular forms such that $\mathfrak{J}$ is a free $\mathfrak{h}$-module.*

We give some arguments in favour of this hypothesis although they only concern Hilbert modular forms.

A necessary and sufficient criterion for a free graded $\mathfrak{h}$-module $\mathfrak{A}$ is the vanishing of all functions $\mathrm{Ext}^i_{\mathfrak{h}}(\mathfrak{A}, \mathfrak{h})$ $(i = 1, \ldots, n-1)$, together with the reflexivity of $\mathfrak{A}$. The Ext^i are also graded $\mathfrak{h}$-modules. Of particular interest are the submodules of homogeneous elements of degree $-n-1$. They are spaces of finite dimensions over C, and Serre [17] has shown under quite general conditions that, for $\mathfrak{A} = \mathfrak{J}$, the dimensions are equal to the ranks of the $(n-i)$th cohomology groups of the underlying projective variety M. We may abbreviate this fact as

$$H\big(-n-1, \mathrm{Ext}^i_{\mathfrak{h}}(\mathfrak{J}, \mathfrak{h})\big) = H^{n-i}(M).$$

If the variety were regular a theorem of Dolbeault [4] could be applied by which these ranks are equal to the ranks of de Rham cohomology groups of our variety M of degrees $n-i$. If this were so we could conclude from our Hypothesis that there do not exist non-vanishing holomorphic differential forms of degrees $1, \ldots, n-1$. Unfortunately the modular variety has always singular points, but one may ask whether Dolbeault's theorem still holds for singular varieties.

In the case of Hilbert modular forms one can easily prove the vanishing of all these differential forms by a method which is due to Matsushima and Shimura, although they only develop it in a slightly easier case [15]. One can do this even for groups which are commensurable to the Hilbert modular group.

Under the assumption that Dolbeault's theorem is still true, this would state that the ranks $H\big(-n-1, \mathrm{Ext}^i_{\mathfrak{h}}(\mathfrak{J}, \mathfrak{h})\big)$ vanish. Now it may be reasonably conjectured that a system of y_ν exists such that $\mathrm{Ext}^i_{\mathfrak{h}}(\mathfrak{J}, \mathfrak{h}) = 0$.

Before we can enunciate our chief theorem, we have to explain two concepts.

An $\mathfrak{J}$-ideal $\mathfrak{A}$ is called *quasi-invertible* if there exists another $\mathfrak{J}$-ideal $\mathfrak{A}^{-1}$ such that the product $\mathfrak{A}\mathfrak{A}^{-1}$ coincides with $\mathfrak{J}$ in all homogeneous elements of sufficiently large degree. A special quasi-invertible ideal $\mathfrak{A} = \mathfrak{P}$ is the ideal of multiples of a prime divisor which does not pass through a singular point.

Furthermore we need a concept introduced simultaneously by Bourbaki [3] and the author [5]. We consider all valuations of $\mathfrak{h}$ which are attached to homogeneous prime polynomials $p \in \mathfrak{h}$. It is clear how a p-adic extension $\mathfrak{A}_p$ of a $\mathfrak{h}$-module $\mathfrak{A}$ can be defined. A $\mathfrak{h}$-module $\mathfrak{A}$ is called *reflexive* if

$$\mathfrak{A} = \bigcap_p \mathfrak{A}_p.$$

(The author first used the term "arithmetisch abgeschlossen" instead of "reflexive", which has been suggested by Bourbaki.) A free module is always reflexive.

In the following we will give a criterion for principal ideals. It holds in the following situation: $\mathfrak{J}$ is either the Siegel modular group or the Hilbert modular group for the field $Q(\sqrt{d})$ in two variables z_1, z_2. In this case we further assume that the Eisenstein series $G_3(z_1, z_2)$, is not mapped on 0 by the specialization $z_1 = z_2$. It then maps the ring of Hilbert modular forms of even weights on the full ring of elliptic modular forms of weights divisible by 4. This ring is the ring of polynomials in the classical Eisenstein series g_2 and g_3^2. A similar statement can be made in the case of Siegel modular forms, the specialization is this time Siegel's operator Φ (see [18], No. 32), applied $m-1$ times.

Under these conditions and the above hypothesis, a quasiinvertible $\mathfrak{J}$-ideal which is a reflexive $\mathfrak{h}$-module is a principal ideal.

Some applications. Restricting ourselves to the Hilbert modular forms under the conditions just mentioned, we study the following curves:

$$\mathfrak{P}_0\colon z_1 = z_2, \qquad \mathfrak{P}_q\colon z_1 z_2 = -q$$

with a positive integer q. The curves $\mathfrak{P}_q$ have been discovered by Poincaré in a different context and studied extensively by Shimura [20]. While $\mathfrak{P}_0$ maps Hilbert modular forms on elliptic modular forms, $\mathfrak{P}_q$ maps them on automorphic forms with respect to the following group

$$\Gamma_q = \left\{\begin{pmatrix}\alpha & -q\beta \\ \beta' & \alpha'\end{pmatrix} \in \mathbf{SL}(2, \mathfrak{O})\right\}$$

where $\alpha \to \alpha'$ means the non-identical automorphism of $\mathfrak{O}$.

We will calculate the number of intersections $d(\mathfrak{P}, \mathfrak{Q})$ for two of these curves. The first way to do this applies number theoretical considerations concerning the matrices and groups involved. They are similar to those used in the theory of complex multiplication. It is easy to give the following estimate:

$$d(\mathfrak{P}_0, \mathfrak{P}_q) \leqslant \sum h\big(4(du^2-q)f^{-2}\big)$$

where u, f run over all integers (f must be positive) such that the argument of h is the discriminant of a ring of integers in an imaginary quadratic number field, and h means the ideal class number of this ring. One can show that the coordinates of the intersection points lie in class fields of $\boldsymbol{Q}(\sqrt{du^2-q})$. The exact number of intersections in this connection is not yet known.

In a similar way one gets

$$d(\mathfrak{P}_{q_1}, \mathfrak{Q}_{q_2}) \leqslant \sum w(d, q_1, q_2; u, f)\, h\big((u^2-q_1q_2)df^{-2}\big)$$

with certain elementary factors $w(\ldots)$ which are either 0 or powers of 2.

On the other hand, the number of intersections can be given by a formula which was first proved by Zariski [21] and which we use in a slightly different notation:

$$d(\mathfrak{P}, \mathfrak{Q}) = \gamma(\mathfrak{J}) - \gamma(\mathfrak{P}) - \gamma(\mathfrak{Q}) + \gamma(\mathfrak{P}\mathfrak{Q})$$

where γ is the "genus coefficient", namely the constant term in the rank polynomial. Since our variety has always singular points, we need the assumption that the ideals $\mathfrak{P}, \mathfrak{Q}$ are quasiinvertible ([8], § 14). The $\mathfrak{P}_q$ are indeed quasiinvertible if they avoid all singular points of the variety ([8], § 15). The conditions for that are cumbersome ([8], § 19, prop. 1). $\mathfrak{P}_0$ is only quasiinvertible if $\mathfrak{O} = \boldsymbol{Z}(\sqrt{3})$ and Γ is the larger group of substitutions of determinants ε, units in $\mathfrak{O}$.

Inserting $\mathfrak{P}_0, \mathfrak{P}_q$ for $\mathfrak{P}, \mathfrak{Q}$ such that $\mathfrak{P} = \mathfrak{J}P$, $\mathfrak{Q} = \mathfrak{J}Q$ with modular forms P, Q of weights v, w, we obtain

$$d(\mathfrak{P}, \mathfrak{Q}) = \frac{D(\sqrt{d})^{3/2}\zeta(2, \sqrt{d})}{4\pi^4 d(\sqrt{d})}\, vw,$$

where $D(\sqrt{d})$, $\zeta(s, \sqrt{d})$, $h(\sqrt{d})$ mean the discriminant, the zeta function, and the ideal class number of $\boldsymbol{Q}(\sqrt{d})$. (Here we used Shimizu's formula for the number of linearly independent modular forms of given weights.)

There is another application of the intersection number ([8], § 19) which allows to express the weight of the modular form P_q generating $\mathfrak{P}_q = \mathfrak{J}P_q$ by the genus of the field of automorphic functions attached to the group Γ_q.

Without doubt, the cooperation of number theory, function theory, and algebraic geometry leads once more to quite a few new results and problems which are not yet exhausted.

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Cyclic overlattices, II

(Diophantine approximation and sums of roots of unity)

by

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In memory of Professor H. Davenport

1. Introduction. In this paper I shall examine some consequences of three theorems on the Geometry of Numbers which were proved in an earlier paper ([2]). The notation and terminology of that paper will be assumed, often without special comment.

In 1967 Davenport and Schinzel ([1]) considered the following question. Given integers $a_1, \ldots, a_k, q$ with

$$(a_1, \ldots, a_k, q) = 1, \tag{1}$$

can we find an integer n with

$$(n, q) = 1 \tag{2}$$

for which

$$\max_{1 \leqslant i \leqslant k} \|na_i/q\| < \delta. \tag{3}$$

Here it is to be understood that δ is a fixed positive number ($0 < \delta < 1/2$) and that q is large. If the condition (2) were replaced by $n \not\equiv 0 \pmod{q}$ the answer would be affirmative, and by Dirichlet's theorem on Diophantine approximation it would suffice if $q \geqslant \delta^{-k}$. As they observed the answer to the above question cannot, however, be unconditionally affirmative. For suppose there is a linear relation

$$h_1 a_1 + \ldots + h_k a_k = hq, \tag{4}$$

in which $h \neq 0$ and

$$(h_1, \ldots, h_k, h) = 1, \tag{5}$$

$$(h_1, \ldots, h_k) > 1, \tag{6}$$

and

$$\sum_{i=1}^{k} |h_i| < \delta^{-1}. \tag{7}$$