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The quickest proof of the prime number theorem

by

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In memoryof H. Davenport

Introduction. It is natural to raise the question raised by the title. The proof I give has never been published before: there have, however, been near misses. (i) Hardy and Littlewood [1] give the formula (8) below, without giving the details of its proof, but do not draw the "Tauberian" conclusion of the P. N. T. (and go on to assume the Riemann Hypothesis for some different purposes). (ii) Ingham ([2], p. 38, § 10) says that Tauberian theorems for positive coefficients can be used to prove the P. N. T., but goes into no further detail, and it does not appear that he has the present proof in mind. (iii) Ingham (l.c., Theorem 28) gives an explicit formula for

$$\psi_1(x) = \int_0^x \psi(u)\,du, \tag{1}$$

and deduces by a Tauberian argument that $\psi_1(x) \sim \frac{1}{2}x^2$. The proof, however, of the explicit formula is a good deal longer and more delicate than the fairly crude one of the explicit formula (8) below (§ 4). Moreover there remains the deduction of $\psi(x) \sim x$ from $\psi_1(x) \sim \frac{1}{2}x^2$, which is not at all trivial.

§ 1. The theorems and formulae on which my proof depends are as follows:

(I) *If* $a_n > 0$ *and* $y\sum s_n e^{-ny} \to s$ *as* $y \to +0$, *then* $s_n \to s$.

Karamata's proof [3] of this is highly sophisticated, but quite short and easy to follow. (Other Tauberian theorems can be deduced from (I) with a certain amount of trouble, but it happens that (I) is precisely the form we need to use.)

(II) *The functional equations for* $\zeta(s)$ *and* $\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\times$ $\times\Gamma(s)\zeta(s)$; $\xi(1-s) = \xi(s)$ *for the latter.* $\xi(s)$ *is an integral function of order* < 2 (*actually of course* 1).

(III) *If the zeros of* $\xi(s)$ *are* $\varrho = \beta+i\gamma$,

$$\frac{\zeta'}{\zeta}(s) = O(\log t)+s\sum_{\varrho}\frac{1}{\varrho(s-\varrho)}.$$

(IV) *For* $\sigma=-1$, $\frac{\zeta'}{\zeta}(s) = O(t^A)$, *where* A *is a positive absolute constant.*

(We use A's to mean this in the sequel; they are not generally the same from one occurrence to the next. Constants of O's will be of type A.)

Much stronger inequalities for ζ'/ζ are known, but this one would be easier to prove.

(V) $$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)y^{-s}ds = e^{-y} \quad (y>0).$$

This is the well-known Cohen–Mellin integral.

(VI) $-\frac{\zeta'}{\zeta}(s) = \sum\Lambda(n)n^{-s}$ $(\sigma>1)$, *where*

$$\Lambda(n) = \sum_{p^m\leqslant n}\log p.$$

(VII) *Let* $N(T)$ *be the number of zeros* $\varrho=\beta+i\gamma$ *with* $0\leqslant\gamma\leqslant T$. *Then* $N(T) = O(T^A)$.

(VIII) *For the* $\varrho=\beta+i\gamma$ *we have* $0<\beta<1$.

§ 2. LEMMA 1. *Given a large positive* T_0, *there is a* T, *with* $AT_0<T<AT_0$, *such that*

$$\frac{\zeta'}{\zeta}(s) = O(T^A) \quad \textit{for} \quad s=\sigma+iT,\ -1\leqslant\sigma\leqslant 2.$$

The corresponding result for $s=\sigma-iT$ *follows by the symmetry.*

We suppose always that $-1\leqslant\sigma\leqslant 2$.

The number of $\varrho=\beta+i\gamma$ with γ in the range $\Re_1$, $\frac{3}{4}T_0\leqslant\gamma\leqslant\frac{5}{4}T_0$, is $O(T_0^A)$, by (VII). Hence there is a T in $\Re_1$ such that $|s-\varrho|\geqslant|\gamma-T| > T_0^{-A}$ for $\frac{1}{2}T_0\leqslant\gamma\leqslant 2T_0$, or $\Re_2$.

So

(2) $$\sum_{\gamma\in\Re_2}\frac{s}{(s-\varrho)\varrho} = O\left(T_0T_0^A\sum_{\Re_2}\frac{1}{|\varrho|}\right) = O(T_0^A).$$

In $\gamma\geqslant 2T_0$, $|s-\varrho|\geqslant A\gamma > A|\varrho|$,

(3) $$\sum_{\gamma\geqslant 2T_0}\frac{s}{(s-\varrho)\varrho} = O(T_0|\varrho|^{-2}) = O(T_0),$$

since ξ has order <2.

In $0\leqslant\gamma\leqslant\frac{1}{2}T_0$, or $\Re_3$, $|s-\varrho|>AT_0$ and $\min|\varrho|>A$,

(4) $$\sum_{\gamma\in\Re_3}\frac{s}{(s-\varrho)\varrho} < A\sum_{\gamma\in\Re_3}\frac{1}{|\varrho|} < AN(\tfrac{1}{2}T_0) = O(T_0^A).$$

Finally, for $\gamma<0$ we have $|s-\varrho|>A|\gamma|>A|\varrho|$

$$\sum_{\gamma<0}\frac{s}{(s-\varrho)\varrho} = O\left(T_0\sum\frac{1}{|\varrho^2|}\right) = O(T_0).$$

From this and (2), (3), (4) we have the result of Lemma 1.

§ 3. LEMMA 2. *For* $y>0$,

$$\sum\Lambda(n)e^{-ny} = -\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)\frac{\zeta'}{\zeta}(s)y^{-s}ds.$$

We have

$$e^{-ny} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\Gamma(s)(ny)^{-s}ds$$

by (V). On $s=r+it$

$$-\frac{\zeta'}{\zeta}(s) = \sum\Lambda(n)n^{-s}, \text{ absolutely convergent},$$

and Lemma 2 follows.

§ 4. Choose T (large) as in Lemma 1, and consider the s-contour $C = C_1+C_2+C_3+C_4$,

C_1: $2-Ti$ to $2+Ti$; C_2: $2+Ti$ to $-1+Ti$;

C_3: $-1+Ti$ to $-1-Ti$; C_4: $-1-Ti$ to $2-Ti$.

We have

$$-\frac{1}{2\pi i}\int_C\frac{\zeta'}{\zeta}(s)\Gamma(s)y^{-s}ds = \text{sum of residues of } -\frac{\zeta'}{\zeta}\Gamma(s)y^{-s}$$

inside C. The poles are: $s=1$, residue y^{-1}; $s=0$, residue $O(1)$ (for $y\to 0+$); $s=\varrho$, residue $-\Gamma(\varrho)$. Thus

(5) $$-\frac{1}{2\pi i}\int_C\Gamma(s)\frac{\zeta'}{\zeta}(s)y^{-s}ds = y^{-1}+O(1)-\sum\nolimits_1\Gamma(\varrho)y^{-\varrho},$$

where $\sum_1$ is taken over the ϱ in C.

We now make $T\to\infty$ through its special values. Now by Lemma 1

(6) $$\int_{C_2+C_4} = O(1)\int_{2+iT}^{-2+iT}T^s\Gamma(s)ds = o(1)$$

as $T\to\infty$, since $\Gamma(s) = O(e^{-A|T|})$ is the range of integration.

On C_3 $(-1-Ti$ to $-1+Ti)$

$$\Gamma(s)\frac{\zeta'}{\zeta}(s)y^{-s} = O(e^{-A|t|}|t|^{A}y),$$

and so

(7) $$\int_{C_3} = O(y).$$

Collecting from (5), (6), (7) and multiplying by y, we have, as $y\to 0+$,

$$-\frac{y}{2\pi i}\int_{2-Ti}^{2+Ti}\Gamma(s)\frac{\zeta'}{\zeta}(s)y^{-s}ds = 1-\sum\nolimits_1 y^{1-\varrho}\Gamma(\varrho)+o(1)$$

and making $T\to\infty$

$$-\frac{y}{2\pi i}\int_{2-\infty i}^{2+\infty i}\Gamma(s)\frac{\zeta'}{\zeta}(s)y^{-s}ds = 1-\sum_{(\varrho)} y^{1-\varrho}\Gamma(\varrho)+o(1),$$

or, by Lemma 2, as $y\to 0+$

(8) $$y\sum\Lambda(n)e^{-ny} = 1-\sum_{(\varrho)} y^{1-\varrho}\Gamma(\varrho)+o(1).$$

Now, $\beta<1$, $|y^{1-\varrho}\Gamma(\varrho)|<|\Gamma(\varrho)|$, and $\sum|\Gamma(\varrho)|$ converges absolutely. Thus $\sum y^{1-\varrho}\Gamma(\varrho)$ is uniformly convergent in $y\geqslant 0$. Also each $|y^{1-\varrho}\Gamma(\varrho)|\to 0$ as $y\to 0$. By the uniform convergence theorem

$$\lim_{y\to+0}\sum_{(\varrho)} y^{1-\varrho}\Gamma(\varrho) = 0,$$

and from (8)

$$\lim y\sum\Lambda(n)e^{-ny} = 1.$$

It now follows from (I) that

$$\sum^{x}\Lambda(n)\sim x$$

which is equivalent to the P. N. T.

References

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[3] J. Karamata, *Über die Hardy–Littlewoodschen Umkehrung des Abelschen Stetigkeitssatzen*, Math. Zeitschr. 32 (1930), pp. 319–320.

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On the graded rings of modular forms

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The results. It needs some words of justification why the present article fits in the frame of a volume dedicated to the memory of H. Davenport. One may expect contributions which are more or less related to his work or which at least deal with algebraic or analytic number theory. The nature of this note may be, on the contrary, described as number theoretical analysis or algebra, but number theory nevertheless.

It is known since works by A. Selberg [16] and A. Weil [19] that the transformation groups underlying automorphic functions and forms in several variables have algebraic coefficients (or are equivalent to such). So number theory plays an important part in the definition of these groups. But the contribution of number theory is not exhausted by this. I need not mention the vast connections between modular and automorphic functions and class field theory; already Siegel's theory of Eisenstein series [18] and a recent generalization by Baily [1] show the contrary. Moreover, investigations by Igusa [13], Hammond [12], Freitag [9], and Gundlach [10], [11] derive the algebraic structure of the rings of special modular forms from number theoretical properties. The present note reports on some consequences following from this knowledge. The details of this investigation appear independently in a series of lecture notes [8]. Here we only describe the results and possible implications.

For a large class of automorphic forms Baily and Borel [2] — see also [7] — showed that they form finitely generated graded rings $\mathfrak{J}$. The degree is the weight of an automorphic form. Here we only deal with Hilbert and Siegel modular forms in the most narrow sense, the underlying groups being

$$\Gamma = \mathbf{SL}(2,\mathfrak{O}),\quad \text{resp } \mathbf{Sp}(m,\mathbf{Z})$$

where $\mathfrak{O}$ is the principal order of a totally real algebraic number field of degree n. The number of independent variables is n in the first case and $n=\frac{1}{2}m(m+1)$ in the second. By normalization of a system of genera-