Hence, for all i, j with $1 \leqslant i < j$,

$$\binom{i}{j} A_j = 0. \tag{5.25}$$

It follows that $A_j = 0$ unless $j = p^t$, where p is the characteristic of $GF(q)$. Since $p^t \equiv 1 (\operatorname{mod} q-1)$, we must have $p^t = q^k$, where $k \geqslant 0$.

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On the changes of sign of a certain class of error functions

by

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§ 1. Introduction. Since its introduction by Euler in the eighteenth century, $\varphi(n)$ and its behavior have been of great interest in number theory [1]. During the next century G. L. Dirichlet [2] proved that $\sum_1^N \varphi(n) \sim 3N^2/\pi^2$, and F. Mertens ([6]; [4], p. 268) showed the error to be $O(N \log N)$; this has only recently been improved, to $O\big(N \log^{2/3} N (\log\log N)^{4/3}\big)$, by A. Walfisz [11]. The average order of $\varphi(n)$ is thus $6n/\pi^2$, and it is well known ([4], p. 267) that $\limsup \varphi(n)/n = 1$ and that $n^{\delta-1}\varphi(n) \to \infty$ for all positive δ; there is also the theorem due to Landau [5] that $\sum_1^N \big(1/\varphi(n)\big) \sim \big(315\zeta(3)/2\pi^4\big) \log N$.

These results all support the assertion that $\varphi(n)$ behaves asymptotically very much like n. It is then reasonable to look at $\sum_1^N n - \frac{1}{2}N^2$ and $\sum_1^N 1 - N$ (which are $\frac{1}{2}N$ and 0) for qualitative information about the errors $E(N) = \sum_1^N \varphi(n) - 3N^2/\pi^2$ and $H(N) = \sum_1^N \varphi(n)/n - 6N/\pi^2$, on which basis one would expect $E(x) \nearrow +\infty$ and $H(x)$ very small. Sylvester ([9], [10]) conjectured that $E(x) > 0$ for all x. Between 1930 and 1950 it was shown that in each of these respects $\varphi(n)$ differs radically from n. Pillai and Chowla [7] proved that the average order of $H(n)$ is $3/\pi^2$ and that of $E(n)$ is $3n/2\pi^2$, which comes up to expectation; but they also proved that $E(x) = \Omega(x \log\log\log x)$. It follows that $H(x) = \Omega(\log\log\log x)$ refuting the conjecture that $H(x)$ is small. Subsequently M. L. N. Sarma [8] showed that $H(820)$ is negative; in 1950 P. Erdös and H. N. Shapiro [3] proved that $H(x) = \Omega_{\pm}(\log\log\log\log x)$.

The purpose of this paper is to show that this behavior is not peculiar to $\varphi(n)$, but is shared by a large class of functions $f(n) = n \sum_{c|n} \mu(c) p(c)/c$, where $p(n)$ satisfies certain admissibility conditions given below. The method is based on an extension of that used by Erdös and Shapiro, and is motivated by the following: if $H(x)$ were linear we would have $H(An-B) = H(An)-H(B)$, and to contradict $H(x) > 0$ it would suffice to find A, B, and n such that $An > B$ and $H(An) < H(B)$. $H(x)$ is not linear, but can be smoothed out by averaging; it is possible to find an averaging operator Ψ which gives

$$0 \leqslant \Psi H(An-B) < 1-H(B).$$

The operator Ψ is constructed in two stages. The first has the nature of a convolution; it is the average

$$\left(\frac{1}{x\sum\limits_{m\leqslant Ax} p(m)/m}\right)\Big(\sum_{\substack{mn\leqslant Ax\\ n\equiv -B(A)}} p(m)H(n)\Big).$$

The second is a limiting procedure; letting $A_\varkappa$ be the product of all primes less than or equal to $\varkappa$ then Ψ is given by:

$$\Psi[H, B] = \lim_{\varkappa\to\infty}\lim_{x\to\infty}\Big(x\sum_{x\leqslant A_\varkappa x}\frac{p(m)}{m}\Big)^{-1}\Big(\sum_{\substack{mn\leqslant A_\varkappa x\\ n\equiv -B(A_\varkappa)}} p(m)H(n)\Big).$$

Under suitable hypotheses it follows that

$$\Psi[H, B] = -H(B)+f(B)/B. \tag{1.1}$$

If we have $f(B) < B$ and can prove that $H(B)$ is large for infinitely many B, equation (1.1) immediately implies that $H(x)$ changes sign infinitely often. The same argument is refined to provide $\Omega_\pm$-estimates for $H(x)$.

The results will concern functions $f(n)$ generated by $p(n)$ satisfying:

(i) $p(n)$ is a completely multiplicative, integer-valued, arithmetic function.

(ii) $p(1) = 1$.

(iii) $0 < p(n) < n$ for $n > 1$.

(iv) $\sum\limits_{n\leqslant x} p(n) = o\big(x\sum\limits_{n\leqslant x} p(n)/n\big)$ as $x\to\infty$.

For such functions we prove:

(i) $H(N) = \sum\limits_{1}^{N} f(n)/n - aN = O\big(\sum\limits_{1}^{N} p(n)/n\big) = o(N)$.

(ii) $H(N) = \Omega(\log\log\log N)$.

(iii) $\limsup H(N) = +\infty$; $\liminf H(N) = -\infty$.

Under somewhat stronger hypotheses, (iii) is strengthened to

(iv) $H(N) = \Omega_\pm(\log\log\log\log N)$.

Finally, an attempt is made to estimate the number of changes of sign of $H(x)$; however unresolved difficulties remain, related to transforming the information obtained by these methods into explicit form. The nature of this difficulty is exhibited in the special case $f(n) = \varphi(n)$; for this case the method produces $IL(x)$ as a lower bound for the number of changes of sign of $H(n)$ in the interval $(1, x)$, where $IL(x)$ is the smallest integer K such that the $4K$-fold iterated logarithm of x (to a sufficiently large base) is less than 2.

§ 2. Notations and notions. We will use the following notations and conventions:

(i) $\sum\limits_{n\leqslant x} g(n)$ will always mean $\sum\limits_{1\leqslant n\leqslant [x]} g(n)$; if the lower limit is not 1, it will be explicitly displayed.

(ii) Wherever q is used as index in a sum or product it will be understood to take only prime values.

(iii) $F(x) = \Omega\big(G(x)\big)$ means $F(x) \neq o\big(G(x)\big)$.

(iv) $F(x) = \Omega_+\big(G(x)\big)$ means there is a positive constant C such that $F(x) > C\cdot G(x)$ for infinitely many x. $F(x) = \Omega_-\big(G(x)\big)$ means there is a constant $D > 0$ such that $F(x) < -D\cdot G(x)$ for infinitely many x. $F(x) = \Omega_\pm\big(G(x)\big)$ means that $F(x) = \Omega_+\big(G(x)\big)$ and $F(x) = \Omega_-\big(G(x)\big)$.

The following functions will be needed, and are listed here for reference:

$$\alpha = \sum_{1}^{\infty}\mu(n)p(n)/n^2, \qquad \beta = \sum_{1}^{\infty}p(n)/n^2,$$

$$B(N) = \sum_{n\leqslant N}p(n)/n^2, \qquad C(N) = \prod_{q\nmid N}\Big(1-\frac{p(q)}{q^2}\Big),$$

$$D(N) = \prod_{q\mid N}\Big(1-\frac{p(q)}{q^2}\Big), \qquad J(N) = \sum_{n\leqslant N}p(n)L\Big(\frac{N}{n}\Big),$$

$$K(N) = \sum_{n\leqslant N}p(n), \qquad L(N) = \sum_{n\leqslant N}p(n)/n,$$

$$T(N) = \sum_{n>N}p(n)/n^2, \qquad S(N) = \sum_{n>N}\mu(n)p(n)/n^2,$$

$$M(A, B) = \begin{cases}\alpha B-\frac{1}{2}f(A)C(A)/A - C(A)\sum\limits_{c<B}\Big(\dfrac{f\big((A,c)\big)}{(A,c)}\Big) & \text{if } B>1,\\[2ex] \alpha B-\frac{1}{2}f(A)C(A)/A & \text{if } B=0,1.\end{cases}$$

A function $p(n)$ will be called *admissible* if:

(1) $p(n)$ is a completely multiplicative, integer-valued, arithmetic function.

(2) $p(1) = 1$.

(3) $1\leqslant p(n)\leqslant n-1$, for $n>1$.

(4) $K(x) = o\big(xL(x)\big)$ as $x\to\infty$.

The objects of our investigations will be the functions

$$f(n) = n\prod_{q|n}\left(1-\frac{p(q)}{q}\right) = \sum_{d|n}\mu(d)p(d)n/d,$$

$$H(N) = \sum_{n\leqslant N} f(n)/n - aN$$

and

$$E(N) = \sum_{n\leqslant N} f(n) - \tfrac{1}{2}aN^2.$$

§ 3. Preparatory lemmas. In this section the basic properties of admissible functions are established.

LEMMA 1. *If $P(n)$ is completely multiplicative, then*

(a) $G(x) = \sum_{n\leqslant x} P(n)F(x/n)$, $1\leqslant x\leqslant w$, *if and only if*

$$F(x) = \sum_{n\leqslant x}\mu(n)P(n)G(x/n), \quad 1\leqslant x\leqslant w.$$

(b) $g(n) = \sum_{d|n} P(d)f(n/d)$ *if and only if*

$$f(n) = \sum_{d|n}\mu(d)P(d)g(n/d).$$

This lemma is a generalization of the Möbius inversion formulae, and the proof is well known ([4], pp. 236–237).

LEMMA 2. *If $p(n)\geqslant 1$, $p(n)$ is completely multiplicative, and*

$$\beta = \sum_{1}^{\infty} p(n)/n^2$$

converges, then

$$a = \sum_{1}^{\infty}\mu(n)p(n)/n^2$$

converges absolutely and $a\beta = 1$.

Proof. Absolute convergence follows immediately from $\mu^2(n) < 1$; then, as both sums converge absolutely, they can be multiplied and rearranged to give

$$a\beta = \sum_{1}^{\infty}\sum_{1}^{\infty}\mu(m)p(m)p(n)/m^2n^2 = \sum_{c=1}^{\infty}\sum_{m|c}\mu(m)p(c)/c^2 = p(1).$$

LEMMA 3. *If $p(m)\geqslant 0$, $K(x) = o\big(xL(x)\big)$, and $B(x) = O(1)$, then* $L(x) = o(x)$.

Proof. Using Schwartz's inequality, we have

$$\big(L(x)\big)^2 = \left(\sum_{m\leqslant x}\sqrt{p(m)}\,\frac{\sqrt{p(m)}}{m}\right)^2 \leqslant \left(\sum_{m\leqslant x}p(m)\right)\left(\sum_{m\leqslant x}\frac{p(m)}{m^2}\right)$$

or

$$\big(L(x)\big)^2 = o\big(xL(x)\big).$$

LEMMA 4. *If $p(n)\geqslant 0$, $B(x)$ converges, and $L(x)$ diverges, then the following three conditions are equivalent:*

(i) $K(x) = o\big(xL(x)\big)$,

(ii) $T(x) = o\big(L(x)/x\big)$,

(iii) $J(x) = o\big(xL(x)\big)$.

Proof. That (iii) implies (i) is clear, since $L(x)\geqslant 1$ for $x\geqslant 1$. We show next that (i) implies (iii).

The hypotheses of this lemma together with (i) provide, via Lemma 3, that $L(x) = o(x)$. Hence:

$$\sum_{n\leqslant x}p(n)L\left(\frac{x}{n}\right) = \sum_{n\leqslant x/x_0}p(n)L\left(\frac{x}{n}\right) + \sum_{x/x_0<n\leqslant x}p(n)L\left(\frac{x}{n}\right)$$

$$\leqslant \varepsilon x\sum_{n\leqslant x/x_0}\frac{p(n)}{n} + \sum_{x/x_0<n\leqslant x}p(n)L(x_0)$$

$$= \varepsilon xL(x) + L(x_0)K(x)$$

which implies $J(x) = o\big(xL(x)\big)$.

We now prove that (i) is equivalent to (ii). Assume (i); then, summing by parts,

$$\sum_{x+1\leqslant n\leqslant y}\frac{p(n)}{n^2} = \sum_{x\leqslant n\leqslant y-1}K(n)\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right) + \frac{K(y)}{y^2} - \frac{K(x)}{x^2}. \tag{3.1}$$

For n sufficiently large, $K(n) < \varepsilon nL(n)$; thus for x sufficiently large we have

$$\sum_{x+1\leqslant n\leqslant y}\frac{p(n)}{n^2} < \varepsilon\frac{L(y)}{y} + 2\varepsilon\sum_{x\leqslant n\leqslant y-1}L(n)\left(\frac{1}{n}-\frac{1}{n+1}\right).$$

Summing again by parts,

$$\sum_{x+1\leqslant n\leqslant y}\frac{p(n)}{n^2} < \varepsilon\frac{L(y)}{y} + 2\varepsilon\left(\sum_{x+1\leqslant n\leqslant y}\frac{p(n)}{n^2} + \frac{L(x)}{x} - \frac{L(y)}{y}\right)$$

or

$$\sum_{x+1\leqslant n\leqslant y}\frac{p(n)}{n^2}<\left(\frac{2\varepsilon}{1-2\varepsilon}\right)\frac{L(x)}{x} \tag{3.2}$$

which yields (ii).

Now assume that (ii) holds; summing by parts

$$K(x)=\sum_{m\leqslant x}m\frac{p(m)}{m}=xL(x)-\sum_{n\leqslant x-1}L(n),$$

$$L(x)=\sum_{m\leqslant x}m\frac{p(m)}{m^2}=\sum_{0\leqslant n\leqslant x-1}(n+1)\big(T(n)-T(n+1)\big)$$

$$=\beta-xT(x)+\sum_{n\leqslant x-1}T(n)$$

thus

$$K(x)=\sum_{n\leqslant x-1}nT(n)-\sum_{c\leqslant x-1}\sum_{n\leqslant c-1}T(n)+x\sum_{n\leqslant x-1}T(n)+$$
$$+\beta-x^2T(x)+O(1)$$

which, summing by parts, is

$$K(x)=2\sum_{n\leqslant x}nT(n)-(x^2+2x)T(x)+O(x)=o\big(xL(x)\big).$$

LEMMA 5. *Under the hypotheses of Lemmas* 3 *and* 4,

$$\prod_{q\leqslant x}\left(1-\frac{p(q)}{q^2}\right)=\alpha+o\left(\frac{L(x)}{x}\right).$$

Proof.

$$\prod_{q\leqslant N}\left(1-\frac{p(q)}{q^2}\right)=\prod_{q|N!}\left(1-\frac{p(q)}{q^2}\right)=\sum_{n|N!}\mu(n)\frac{p(n)}{n^2}=\alpha+S(N),$$

$$|S(N)|\leqslant\sum_{n>N}\mu^2(n)\frac{p(n)}{n^2}\leqslant\sum_{n>N}\frac{p(n)}{n^2}=T(N)=o\left(\frac{L(N)}{N}\right)$$

and the lemma follows.

LEMMA 6. *Under the hypotheses of Lemma* 3

$$L(kx)\sim L(x).$$

Proof.

$$L(kx)-L(x)=\sum_{x<m\leqslant kx}\frac{p(m)}{m}$$
$$=\sum_{x-1<m\leqslant kx-1}K(m)\left(\frac{1}{m}-\frac{1}{m+1}\right)+\frac{K(kx)}{kx}-\frac{K(x)}{x}.$$

For all sufficiently large n, $K(n)<\varepsilon nL(n)$, and thus

$$L(kx)-L(x)\leqslant\varepsilon\sum_{x-1<m\leqslant kx-1}\frac{L(m)}{m}+\varepsilon L(kx).$$

The sum may be estimated as follows:

$$\sum_{x-1<m\leqslant kx-1}\frac{L(m)}{m}\leqslant L(kx)\sum_{x-1<m\leqslant kx-1}\frac{1}{m}$$

and

$$\sum_{x-1<m\leqslant kx-1}\frac{1}{m}=\sum_{[x]<m\leqslant k[x]}\frac{1}{m}+O(1)=\log k+O(1).$$

Thus:

$$L(kx)-L(x)\leqslant 4(\log k)L(kx)$$

and the conclusion follows.

The preceding lemmas may now be combined to give:

LEMMA 7. *If* $p(n)$ *is admissible, then*:

(i) $\alpha\beta=1$,
(ii) $L(x)=o(x)$,
(iii) $J(x)=o\big(xL(x)\big)$,
(iv) $T(x)=o\big(L(x)/x\big)$,
(v) $S(x)=o\big(L(x)/x\big)$,
(vi) $L(kx)\sim L(x)$.

Proof. If we can show that $B(x)$ converges and $L(x)$ diverges, the conclusions follow from Lemmas 2, 3, 4, 5, and 6.

As $p(n)\geqslant 1$, we have that $L(x)\geqslant\log x$ and therefore diverges. As $p(n)\leqslant n$, we have that $L(x)\leqslant x$, and equation (3.2) yields $T(x)=o(1)$; thus $B(x)$ converges.

§ 4. Order of magnitude of $f(n)$ and $H(n)$. We first show that $H(n)$ is indeed the error function associated with $f(n)$, and estimate its order of magnitude.

THEOREM 1. *Let* $p(n)$ *be admissible; then*

(i) $|H(x)|<xT(x)+L(x)=O\big(L(x)\big)$,
(ii) $|E(x)|<\frac{1}{2}x^2T(x)+\frac{3}{2}xL(x)+K(x)=O\big(xL(x)\big)$.

Proof.

(i) $$\sum_{n\leq x}\frac{f(n)}{n} = \sum_{cd\leq x}\frac{\mu(d)p(d)}{d} = \sum_{d\leq x}\frac{\mu(d)p(d)}{d}\left[\frac{x}{d}\right]$$

$$= x\sum_{n\leq x}\frac{\mu(n)p(n)}{n^2} - \sum_{n\leq x}\frac{\mu(n)p(n)}{n}\left\{\frac{x}{n}\right\}$$

$$= ax - xT(x) - \sum_{n\leq x}\frac{\mu(n)p(n)}{n}\left\{\frac{x}{n}\right\},$$

thus

$$|H(x)| \leq xT(x)+L(x) = L(x)+o\big(L(x)\big).$$

(ii) $$\sum_{n\leq x} f(n) = \sum_{n\leq x}\sum_{d|n}\mu(d)p(d)\frac{n}{d} = \frac{1}{2}\sum_{n\leq x}\mu(n)p(n)\left(\left[\frac{x}{n}\right]^2+\left[\frac{x}{n}\right]\right)$$

$$= \frac{1}{2}x^2\sum_{n\leq x}\frac{\mu(n)p(n)}{n^2} - x\sum_{n\leq x}\frac{\mu(n)p(n)}{n}\left\{\frac{x}{n}\right\} +$$

$$+\frac{1}{2}\sum_{n\leq x}\mu(n)p(n)\left\{\frac{x}{n}\right\}^2 + \frac{1}{2}x\sum_{n\leq x}\frac{\mu(n)p(n)}{n} - \frac{1}{2}\sum_{n\leq x}\mu(n)p(n)\left\{\frac{x}{n}\right\}$$

and thus

$$|E(x)| \leq \tfrac{1}{2}x^2T(x)+\tfrac{3}{2}xL(x)+K(x) = O\big(xL(x)\big).$$

THEOREM 2. *If R is a positive integer, then*

$$\sum_{n\leq R}H(n) = (R+1)H(R)-E(R)+\tfrac{1}{2}aR.$$

Proof.

(4.1)
$$\sum_{n\leq R}H(n) = \sum_{n\leq R}\left(\sum_{d\leq n}\frac{f(d)}{d}-an\right) = \sum_{n\leq R}(R-n+1)\frac{f(n)}{n} - \tfrac{1}{2}a(R^2+R)$$

$$= (R+1)\sum_{n\leq R}\frac{f(n)}{n} - \tfrac{1}{2}aR - \tfrac{1}{2}aR^2 - \sum_{n\leq R}f(n)$$

$$= (R+1)\big(H(R)+aR\big) - \big(E(R)+\tfrac{1}{2}aR^2\big) - \tfrac{1}{2}aR^2 - \tfrac{1}{2}aR$$

$$= (R+1)H(R)-E(R)+\tfrac{1}{2}aR.$$

This identity will be used in § 7 to prove that $\sum_{n\leq x}H(n)\sim\frac{1}{2}ax$, as in the special case $f(n)=\varphi(n)$. The proof requires estimates involving $p(n)$, which in the special case depend on the Prime Number Theorem.

It is possible to do without these estimates by using a weighted hyperbolic summation.

THEOREM 3. *If $p(n)$ is admissible, then*

$$\sum_{mn\leq x}p(m)H(n) = \tfrac{1}{2}axL(x)+R_0 = \tfrac{1}{2}axL(x)+o\big(xL(x)\big)$$

where

(4.2) $$R_0 = \tfrac{1}{2}ax^2T(x)-\tfrac{1}{2}[x]+\tfrac{1}{2}\{x\}^2+O\big(J(x)\big).$$

Proof. Replacing x by x/m in the identity of Theorem 2, multiplying by $p(m)$, and summing yields:

$$\sum_{mn\leq x}p(m)H(n) = \sum_{m\leq x}\left(\frac{x}{m}+1\right)p(m)H\left(\frac{x}{m}\right)+\tfrac{1}{2}axL(x)-\sum_{m\leq x}p(m)E\left(\frac{x}{m}\right)$$

$$= \tfrac{1}{2}axL(x)+\sum_{m\leq x}\frac{x}{m}p(m)H\left(\frac{x}{m}\right)-\sum_{m\leq x}p(m)E\left(\frac{x}{m}\right)+$$

$$+\sum_{m\leq x}p(m)H\left(\frac{x}{m}\right).$$

Lemma 1(b) gives

$$n = \sum_{d|n}f(d)p\left(\frac{n}{d}\right),$$

and thus

$$[x] = \sum_{n\leq x}\frac{1}{n}\sum_{d|n}f(d)p\left(\frac{n}{d}\right) = \sum_{cd\leq x}\frac{p(c)f(d)}{cd} = \sum_{c\leq x}\frac{p(c)}{c}\sum_{d\leq x/c}\frac{f(d)}{d},$$

$$[x] = axB(x)+\sum_{m\leq x}\frac{p(m)}{m}H\left(\frac{x}{m}\right)$$

or:

$$\sum_{m\leq x}\frac{p(m)}{m}H\left(\frac{x}{m}\right) = -\{x\}+axT(x);$$

similarly,

$$\tfrac{1}{2}[x]^2+\tfrac{1}{2}[x] = \sum_{m\leq x}p(m)E\left(\frac{x}{m}\right)+\tfrac{1}{2}ax^2\sum_{m\leq x}\frac{p(m)}{m^2}$$

or:

$$\sum_{m\leq x}p(m)E\left(\frac{x}{m}\right) = \tfrac{1}{2}[x]^2-\tfrac{1}{2}x^2+\tfrac{1}{2}[x]+\tfrac{1}{2}ax^2T(x);$$

combining these and substituting back in,

$$\sum_{mn\leqslant x} p(m)H(n) = \tfrac{1}{2}axL(x)+\tfrac{1}{2}x^2T(x)-\tfrac{1}{2}[x]+\tfrac{1}{2}\{x\}^2+\sum_{m\leqslant x} p(m)H\left(\frac{x}{m}\right).$$

As $p(m)\geqslant 1$, $[x]\leqslant K(x) = o\big(xL(x)\big)$;

$$H(x) = O\big(L(x)\big) \quad \text{implies} \quad \sum_{m\leqslant x} p(m)H\left(\frac{x}{m}\right) = O\big(J(x)\big) = o\big(xL(x)\big)$$

and

$$\sum_{mn\leqslant x} p(m)H(n)-\tfrac{1}{2}axL(x) = R_0 = o\big(xL(x)\big).$$

Our method involves summing $H(n)$ on an arithmetic progression; for this we will need the corresponding result for $f(n)/n$.

THEOREM 4. *If $p(d)$ is admissible, then*

$$\sum_{\substack{m\leqslant z\\ m\equiv\beta(A)}} \frac{f(m)}{m} = \frac{zC(A)}{A}\sum_{\tau|(\beta,A)}\frac{\mu(\tau)p(\tau)}{\tau}+R_3$$

where

$$R_3 = O\Big(\sum_{\tau|(\beta,A)} L(z/\tau)\mu^2(\tau)p(\tau)/\tau\Big)$$

and the O is uniform in A and β.

Proof.

$$\sum_{\substack{m\leqslant z\\ m\equiv\beta(A)}} \frac{f(m)}{m} = \sum_{\substack{cd\leqslant z\\ cd\equiv\beta(A)}}\frac{\mu(d)p(d)}{d} = \sum_{d\leqslant z}\sum_{\substack{c\leqslant z/d\\ cd\equiv\beta(A)}}\frac{\mu(d)p(d)}{d}$$

$$= \sum_{\substack{d\leqslant z\\ (d,A)|\beta}}\frac{\mu(d)p(d)}{d}\left(\frac{z(d,A)}{dA}+r_1\right)$$

where $0\leqslant r_1\leqslant 1$.

Let

$$R_1 = \sum_{\substack{d\leqslant z\\ (d,A)|\beta}}\frac{r_1\mu(d)p(d)}{d} \quad\text{and}\quad R_2 = -\sum_{\substack{d>z\\ (d,A)|\beta}}\frac{z\mu(d)p(d)(d,A)}{d^2A}.$$

Then

$$\sum_{\substack{m\leqslant z\\ m\equiv\beta(A)}} \frac{f(m)}{m} = \frac{z}{A}\sum_{(d,A)|\beta}\frac{\mu(d)p(d)(d,A)}{d^2}+R_1+R_2$$

$$= \frac{z}{A}\sum_{(d,A)|(\beta,A)}\frac{\mu(d)p(d)(d,A)}{d^2}+R_1+R_2$$

$$= \frac{z}{A}\sum_{\tau|(\beta,A)}\tau\sum_{(d,A)=\tau}\frac{\mu(d)p(d)}{d^2}+R_1+R_2$$

$$= \frac{z}{A}\sum_{\tau|(\beta,A)}\tau\sum_{(c,A)=1}\frac{\mu(c\tau)p(c\tau)}{c^2\tau^2}+R_1+R_2$$

$$= \frac{z}{A}\sum_{\tau|(\beta,A)}\frac{\mu(\tau)p(\tau)}{\tau}\sum_{(c,A)=1}\frac{\mu(c)p(c)}{c^2}+R_1+R_2$$

$$= \frac{zC(A)}{A}\sum_{\tau|(\beta,A)}\frac{\mu(\tau)p(\tau)}{\tau}+R_1+R_2.$$

The remainders are:

$$|R_1|\leqslant \sum_{\substack{d\leqslant z\\ (d,A)|\beta}}\frac{\mu^2(d)p(d)}{d}\leqslant\sum_{\tau|(\beta,A)}\sum_{\substack{c\leqslant z/\tau\\ (c,A)=1}}\frac{\mu^2(\tau)p(\tau)\mu^2(c)p(c)}{c\tau}$$

thus

$$|R_1|\leqslant\sum_{\tau|(\beta,A)}\frac{\mu^2(\tau)p(\tau)}{\tau}L\left(\frac{z}{\tau}\right);$$

similarly,

$$|R_2|\leqslant\sum_{\tau|(\beta,A)}\frac{z\mu^2(\tau)p(\tau)}{\tau A}\sum_{\substack{(c,A)=1\\ c>z/\tau}}\frac{\mu^2(c)p(c)}{c^2}$$

$$\leqslant\frac{1}{A}\sum_{\tau|(\beta,A)}\mu^2(\tau)p(\tau)\frac{z}{\tau}T\left(\frac{z}{\tau}\right).$$

As $xT(x)<\varkappa L(x)$ for all x, $\varkappa$ a constant,

$$|R_2|\leqslant\varkappa\sum_{\tau|(\beta,A)}\frac{\mu^2(\tau)p(\tau)}{\tau}L\left(\frac{z}{\tau}\right)$$

and

$$|R_3| = |R_1+R_2| = O\left(\sum_{\tau|(\beta,A)}\frac{\mu^2(\tau)p(\tau)}{\tau}L\left(\frac{z}{\tau}\right)\right)$$

uniformly in A and β.

§ 5. Ω-estimates for $H(n)$ and $E(n)$.

THEOREM 5. *If $p(n)$ is admissible, then*

$$H(x) = \Omega(\log\log\log x).$$

Proof.

$$1 \leqslant p(n) < n \quad \text{and} \quad f(n) = n\prod_{q|n}\left(1-\frac{p(q)}{q}\right)$$

imply that

$$f(n) \leqslant \varphi(n) < n.$$

Merten's theorem ([4], p. 351) says that:

$$\prod_{q\leqslant z}\left(1-\frac{1}{q}\right) \sim \frac{e^{-\gamma}}{\log z}$$

which implies: for $z \neq 1$, θ a constant,

$$\prod_{z<q\leqslant z^{\theta}}\left(1-\frac{1}{q}\right) = \frac{1}{\theta}+o(1).$$

Define

$$P(a, b) = \prod_{a<q<b} q.$$

Then if $a \neq 1$ and $P(a, a^{\theta})\,|\,n$ we have

$$\left|\frac{f(n)}{n}\right| \leqslant \left|\frac{\varphi(n)}{n}\right| \leqslant \prod_{a<q<a^{\theta}}\left(1-\frac{1}{q}\right) = \frac{1}{\theta}+o(1).$$

We can then find constants ξ, θ, and δ, all positive, such that:

$$\prod_{x<q<x^{\theta}}\left(1-\frac{1}{q}\right) < \delta < a \quad \text{for} \quad x > \xi.$$

This provides the key to the proof; for,

$$(5.1)\qquad H(x_0+k)-H(x_0) = \sum_{x_0<n\leqslant x_0+k}\frac{f(n)}{n} - ak$$

$$= \sum_{x_0<n\leqslant \tau+x_0}\frac{f(n)}{n} + \sum_{x_0+\tau<n\leqslant x_0+k}\frac{f(n)}{n} - ak.$$

We now choose τ, k, and x_0 so that the second sum in (5.1) is less than δk, as follows:

Let x_0 be the least positive solution of the system

$$(5.2)\qquad \begin{aligned} x &\equiv 0 \pmod 2, \\ x+\lambda &\equiv 0 \left(\operatorname{mod} P(2^{\theta^{\lambda-1}}, 2^{\theta^{\lambda}})\right), \quad 1 \leqslant \lambda \leqslant k. \end{aligned}$$

It follows that there are positive constants χ, ψ such that

$$\chi\log\log\log x_0 < k < \psi\log\log x_0.$$

Define $\tau = \min\{n\,|\,2^{\theta^n} > \xi\}$; it is independent of k, x_0. Choosing $k > \tau$ yields:

$$\sum_{x_0<n\leqslant x_0+k}\frac{f(n)}{n} \leqslant \sum_{x_0<n\leqslant x_0+\tau} 1 + \sum_{x_0+\tau<n\leqslant x_0+k}\frac{f(n)}{n}$$

$$\leqslant \sum_{n\leqslant\tau} 1 + \sum_{\tau\leqslant t\leqslant k}\frac{f(x_0+t)}{(x_0+t)} \leqslant \tau+\delta(k-\tau) \leqslant (1-\delta)\tau+\delta k.$$

Thus (5.1) implies

$$H(x_0+k)-H(x_0) \leqslant (\delta-a)k+O(1)$$

or

$$(5.3)\qquad |H(x_0+k)-H(x_0)| \geqslant (a-\delta)k+O(1) \geqslant \chi(a-\delta)\log\log\log x_0+O(1).$$

As $\chi(a-\delta) > 0$, and x_0 can be made arbitrarily large by making k large, (5.3) implies $H(x) = \Omega(\log\log\log x)$.

Observe that in inequality (5.3) we have proved more than is needed for this theorem. (5.3) implies not merely that $|H(x)| > c\log\log\log x$ for suitable arbitrarily large x; but also that there are arbitrarily large x for which $H(x)$ changes by more than $c\log\log\log x$ within the interval $[x, x+\psi\log\log x]$. This stronger statement will be used in § 8.

THEOREM 6. *If $p(n)$ is admissible, then*

$$E(x) = \Omega(x\log\log\log x).$$

The proof is almost identical to that of Theorem 5, and will be omitted. As with Theorem 5, the proof yields the stronger statement that there are arbitrarily large x for which $E(x)$ changes by more than $cx\log\log\log x$ in the interval $[x, x+\psi\log\log x]$.

§ 6. On the changes of sign of $H(x)$.

6.1. Averages over arithmetic progressions.

THEOREM 7. *If $p(n)$ is admissible, then*

$$A\sum_{\substack{n\leqslant z\\ n\equiv -B(A)}} H(n) = \sum_{n\leqslant z} H(n) - \tfrac{1}{2}az + M(A, B)z + O\big(L(z)\big)$$

where

$$M(A, B) = \begin{cases} aB - \frac{1}{2}f(A)C(A)/A - C(A)\displaystyle\sum_{c<B}\left(\frac{f\big((A, c)\big)}{(A, c)}\right) & \text{if} \quad B > 1, \\ aB - \frac{1}{2}f(A)C(A)/A & \text{if} \quad B = 0, 1. \end{cases}$$

Proof. Let $z = Ax-B$. Then for integral x we have:

$$A\sum_{\substack{n\leqslant z\\ n\equiv -B(A)}} H(n) = A\sum_{n\leqslant x} H(An-B)$$

$$= A\sum_{n\leqslant x}\sum_{m\leqslant An-B}\frac{f(m)}{m} - aA\sum_{n\leqslant x}(An-B)$$

$$\doteq A\sum_{m\leqslant Ax-B}\Bigg(\sum_{(m+B)/A<n\leqslant x}\frac{f(m)}{m}\Bigg) - \tfrac12 a(A^2x^2+A^2x-2ABx),$$

$$A\sum_{\substack{n\leqslant Ax-B\\ n\equiv -B(A)}} H(n) = Ax\sum_{m\leqslant Ax-B}\frac{f(m)}{m} - A\sum_{m\leqslant Ax-B}\left[\frac{m+B}{A}\right]\frac{f(m)}{m} +$$

$$+A\sum_{\substack{m\leqslant Ax-B\\ m\equiv -B(A)}}\frac{f(m)}{m} - \tfrac12 a(A^2x^2+A^2x-2ABx),$$

$$A\sum_{\substack{n\leqslant z\\ n\equiv -B(A)}} H(n) = (z+B)\sum_{m\leqslant z}\frac{f(m)}{m} - A\sum_{m\leqslant z}\left[\frac{m+B}{A}\right]\frac{f(m)}{m} +$$

$$+A\sum_{\substack{m\leqslant z\\ m\equiv -B(A)}}\frac{f(m)}{m} - \tfrac12 a(Az+BA+z^2-B^2).$$

Taking this a piece at a time we have:

$$-A\sum_{m\leqslant z}\left[\frac{m+B}{A}\right]\frac{f(m)}{m} = -\sum_{\sigma=0}^{A-1}\sum_{\substack{m\leqslant z\\ m+B\equiv\sigma(A)}}\frac{f(m)}{m}(m+B-\sigma)$$

$$= -\sum_{m\leqslant z} f(m) + \sum_{\sigma=0}^{A-1}\sum_{\substack{m\leqslant z\\ m+B\equiv\sigma(A)}}(\sigma-B)\frac{f(m)}{m}.$$

Therefore

$$A\sum_{\substack{n\leqslant z\\ n\equiv -B(A)}} H(n) = (z+B)\sum_{m\leqslant z}\frac{f(m)}{m} - \sum_{m\leqslant z} f(m) + \tfrac12 a(B^2-z^2-AB-Az) +$$

$$+\sum_{\sigma=0}^{A-1}(\sigma-B)\sum_{\substack{m\leqslant z\\ m+B\equiv\sigma(A)}}\frac{f(m)}{m} + A\sum_{\substack{m\leqslant z\\ m\equiv -B(A)}}\frac{f(m)}{m}.$$

Using the identity of Theorem 2, and combining the last two sums, we have:

$$A\sum_{\substack{n\leqslant z\\ n\equiv -B(A)}} H(n) = \sum_{n\leqslant z} H(n) + \sum_{\sigma\leqslant A}(\sigma-B)\sum_{\substack{m\leqslant z\\ m+B\equiv\sigma(A)}}\frac{f(m)}{m} + (B-1)H(z) +$$

$$+aBz - \tfrac12 aAz - \tfrac12 az + \tfrac12 a(B^2-AB).$$

If $B>1$, then, from Theorem 4,

$$\sum_{\sigma\leqslant A}(\sigma-B)\sum_{\substack{m\leqslant z\\ m+B\equiv\sigma(A)}}\frac{f(m)}{m} = \sum_{\sigma\leqslant A}(\sigma-B)\frac{zC(A)}{A}\sum_{\tau|(\sigma-B,A)}\frac{\mu(\tau)p(\tau)}{\tau} + R_4$$

where

$$|R_4| \leqslant \theta\sum_{\sigma\leqslant A}(\sigma-B)\sum_{\tau|(\sigma-B,A)}\frac{\mu^2(\tau)p(\tau)}{\tau}L\left(\frac{z}{\tau}\right),$$

θ a positive constant independent of A and B.

$$(6.1)\qquad \sum_{\sigma\leqslant A}(\sigma-B)\sum_{\tau|(\sigma-B,A)}\frac{\mu(\tau)p(\tau)}{\tau}$$

$$= \sum_{c=1-B}^{A-B} c\sum_{\tau|(c,A)}\frac{\mu(\tau)p(\tau)}{\tau} = \sum_{c=0}^{A-B} c\sum_{\tau|(c,A)}\frac{\mu(\tau)p(\tau)}{\tau} + \sum_{c=-B+1}^{-1} c\sum_{\tau|(c,A)}\frac{\mu(\tau)p(\tau)}{\tau}$$

$$= \sum_{c=0}^{A-B} c\sum_{\tau|(c,A)}\frac{\mu(\tau)p(\tau)}{\tau} + \sum_{c=A-B+1}^{A-1}(c-A)\sum_{\tau|(c,A)}\frac{\mu(\tau)p(\tau)}{\tau}$$

$$= \sum_{c<A} c\sum_{\tau|(c,A)}\frac{\mu(\tau)p(\tau)}{\tau} - A\sum_{c<B}\sum_{\tau|(c,A)}\frac{\mu(\tau)p(\tau)}{\tau}.$$

The first sum is:

$$\sum_{c<A} c\sum_{\tau|(c,A)}\frac{\mu(\tau)p(\tau)}{\tau} = \sum_{d|A}\mu(d)p(d)\sum_{\substack{c<A\\ c\equiv 0(d)}}\frac{c}{d} = \frac12\sum_{d|A}\mu(d)p(d)\left(\frac{A^2}{d^2}-\frac{A}{d}\right).$$

Then

$$\frac{zC(A)}{A}\sum_{\sigma\leqslant A}(\sigma-B)\sum_{\tau|(\sigma-B,A)}\frac{\mu(\tau)p(\tau)}{\tau}$$

$$= \tfrac12 aAz - \tfrac12 z\frac{C(A)f(A)}{A} - zC(A)\sum_{c<B}\frac{f\big((A,c)\big)}{(A,c)}.$$

Substituting back in,

$$(6.2)\quad A\sum_{\substack{n\leqslant z\\ n\equiv -B(A)}} H(n) = \sum_{n\leqslant z} H(n)+M(A,B)z-\tfrac{1}{2}az+(B-1)H(z)+ +\tfrac{1}{2}a(B^2-AB)+R_4.$$

Proceeding as with (6.1),

$$(6.3)\quad |R_4|\leqslant \theta L(z)\sum_{\sigma\leqslant A}(\sigma-B)\sum_{\tau|(\sigma-B,A)}\frac{\mu^2(\tau)p(\tau)}{\tau} \leqslant \theta L(z)\sum_{\tau|A}\mu^2(\tau)p(\tau)\frac{A^2}{\tau^2}\leqslant aA^2\theta L(z).$$

The error introduced by not requiring z to be an integer is $O(1)$.

Corollary 7a. *If $p(n)$ is admissible, then*

$$(6.4)\quad A\sum_{\substack{mn\leqslant z\\ n\equiv -B(A)}} H(n)p(m) = M(A,B)zL(z)+o\big(zL(z)\big).$$

Proof. This follows immediately from Theorems 7 and 3, and Lemma 7.

6.2. Infinitude of the number of changes of sign.

Theorem 8. *If $p(n)$ is admissible, then $H(n)$ changes sign infinitely often.*

Proof. In Corollary 7a, replace z by Ay to get:

$$(6.5)\quad \sum_{\substack{mn\leqslant Ay\\ n\equiv -B(A)}} p(m)H(n) = M(A,B)yL(Ay)+o\big(yL(Ay)\big).$$

Then, for $A_\varkappa = \prod_{q\leqslant\varkappa} q$ and $\varkappa\geqslant B$

$$\sum_{c<B}\frac{f\big((A_\varkappa,c)\big)}{(A_\varkappa,c)} = \sum_{c<B}\frac{f(c)}{c} = H(B-1)+a(B-1)$$

and

$$M(A_\varkappa,B) = aC(A_\varkappa)+aB\big(1-C(A_\varkappa)\big)-\frac{f(A_\varkappa)C(A_\varkappa)}{2A_\varkappa}-H(B-1).$$

As $\varkappa\to\infty$, $C(A_\varkappa)\to 1$ and $f(A_k)/A_\varkappa\to 0$; thus

$$\lim_{\varkappa\to\infty} M(A_\varkappa,B) = a-H(B-1) = \frac{f(B)}{B}-H(B).$$

Define the operator $\Psi[H;B]$ as

$$\Psi[H;B] = \lim_{\varkappa\to\infty}\lim_{y\to\infty}\frac{1}{yL(A_\varkappa y)}\sum_{\substack{mn\leqslant A_\varkappa y\\ n\equiv -B(A_\varkappa)}} p(m)H(n).$$

Then

$$\Psi[H;B] = \frac{f(B)}{B}-H(B)$$

which becomes

$$(6.6)\quad -H(B)\leqslant \Psi[H;B]\leqslant 1-H(B).$$

If $H(n)>0$ for all sufficiently large n, then for all sufficiently large B we have that

$$\Psi[H;B]\geqslant 0.$$

As $H(n) = \Omega(\log\log\log n)$ we may choose B so that $H(B)>2$, and thus

$$0\leqslant \Psi[H;B] < -1$$

which is impossible.

If we assume $H(n)<0$ for all sufficiently large n, then we can find an arbitrarily large B for which

$$1<\Psi[H;B]\leqslant 0$$

which is also impossible.

6.3. Large positive and negative values of $H(n)$. Theorem 8 does not make full use of Theorem 5. If we do then we can get correspondingly stronger statements, namely that $\limsup H(n) = +\infty$ and $\liminf H(n) = -\infty$.

Theorem 5 implies that for any integer N, there are arbitrarily large B for which either $H(B)>N$ or $H(B)<-N$. Suppose that $H(B)>N$; inequality (6.6) implies that, for $\varkappa>\varkappa_0$,

$$\lim_{y\to\infty}\sum_{\substack{mn\leqslant A_\varkappa y\\ n\equiv -B(A_\varkappa)}} p(m)H(n)/yL(A_\varkappa) < 2-H(B)$$

and thus for $y>y_0(\varkappa_0)$

$$(6.7)\quad \sum_{\substack{mn\leqslant A_\varkappa y\\ n\equiv -B(A_\varkappa)}} p(m)H(n) < \big(3-H(B)\big)yL(A_\varkappa y).$$

Inequality (6.7) implies that for some n^* larger than B:

$$H(n^*) < 3-N < -\tfrac{1}{2}N.$$

Similarly, if $H(B)<-N$ we derive from inequality (6.6) that for some n' larger than B we have $H(n')>\tfrac{1}{2}N$. Combining both parts we have:

Theorem 9. *If $p(n)$ is admissible, then as $n\to\infty$*

$$\operatorname{Lim\,sup} H(n) = +\infty,\qquad \operatorname{Lim\,inf} H(n) = -\infty.$$

§ 7. More precise estimates. It is reasonable to expect that stronger estimates for sums involving $p(n)$ will lead to correspondingly stronger results for $H(n)$. In this section we will consider only functions $f(n)$ generated by $p(n)$ satisfying:

(i) $\sum_{d\leqslant x} \mu(d)p(d) = o(x)$,

(ii) $\sum_{d\leqslant x} \mu(d)p(d)[x/d] = o(x)$,

(iii) $\sum_{d\leqslant x} \mu(d)p(d)\{x/d\}^2 = o(x)$,

(iv) $p(n)$ is admissible.

These conditions are assumed to hold for each theorem of this section, without being restated each time. (Note that even stronger estimates hold for the special case $f(n) = \varphi(n)$.)

THEOREM 10.

$$\sum_{n\leqslant x} H(n) \sim \tfrac{1}{2}ax.$$

Proof.

$$\sum_{n\leqslant x} f(n) = \sum_{n\leqslant x}\sum_{d|n} n\mu(d)p(d)/d = \sum_{de\leqslant x} e\mu(d)p(d)$$

$$= \sum_{n\leqslant x} \mu(d)p(d)\left(\frac{1}{2}\left[\frac{x}{d}\right]^2 + \frac{1}{2}\left[\frac{x}{d}\right]\right),$$

$$\sum_{n\leqslant x} \frac{f(n)}{n} = \sum_{n\leqslant x}\sum_{d|n} \frac{\mu(d)p(d)}{d} = \sum_{d\leqslant x} \frac{\mu(d)p(d)}{d}\left[\frac{x}{d}\right].$$

Substituting into the identity of equation (4.1),

$$\sum_{n\leqslant R} H(n) = R\sum_{n\leqslant R}\frac{f(n)}{n} - \sum_{n\leqslant R} f(n) + H(R) + \tfrac{1}{2}aR - \tfrac{1}{2}aR^2 \tag{7.1}$$

$$= \tfrac{1}{2}aR + H(R) - \tfrac{1}{2}R^2 S(R) - \tfrac{1}{2}\sum_{d\leqslant R}\mu(d)p(d)\left[\frac{R}{d}\right] -$$

$$- \tfrac{1}{2}\sum_{d\leqslant R}\mu(d)p(d)\left\{\frac{x}{d}\right\}^2$$

where R is an integer and $S(R) = \sum_{n>R} \mu(n)p(n)/n^2$. That $S(R) = o(1/R)$ follows immediately from condition (i) above, using an argument similar to that of Lemma 4. (Note: if $p(n) < M < \infty$, then (iii) follows from (i), using a theorem of Pillai and Chowla ([7], pp. 95–97) that $\sum_{n\leqslant x} a_n = o(x)$ and $|a_n| < M < \infty$ implies $\sum_{n\leqslant x} a_n\left\{\frac{x}{n}\right\}^2 = o(x)$.) Thus

$$\sum_{n\leqslant R} H(n) = \tfrac{1}{2}aR + o(R). \tag{7.2}$$

COROLLARY 10a.

$$A\sum_{\substack{n\leqslant z\\ n\equiv -B(A)}} H(n) = M(A,B)z + (B-1)H(z) + \tfrac{1}{2}a(B^2-AB) + \tag{7.3}$$

$$+ O\big(A^2L(z)\big) + O\big(AM(A,B)\big) + o(z)$$

where the O's are uniform in both A and B.

This follows from equations (6.2) and (6.3), and from Theorem 10.

THEOREM 11. $H(n) = \Omega_{\pm}(\log\log\log\log n)$.

Proof. In equation (7.3) set $z = Ax-B$, divide by Ax, and set $x = AL(A^2)$ to get

$$\frac{1}{AL(A^2)}\sum_{n\leqslant AL(A^2)} H(An-B) = M(A,B)\left(1-\frac{B}{A^2L(A^2)}\right) + \tag{7.4}$$

$$+\frac{a}{2}\left(\frac{B^2-AB}{A^2L(A^2)}\right) + O\left(\frac{M(A,B)}{AL(A^2)}\right) + \frac{(B-1)H\big(A^2L(A^2)-B\big)}{A^2L(A^2)} +$$

$$+ O\left(\frac{L\big(A^2L(A^2)\big)}{L(A^2)}\right) + o\left(\frac{1}{L(A^2)}\right).$$

As $L(x) = o(x)$, for A sufficiently large $A^2L(A^2) \leqslant 2A^2$ and, by Lemma 6, for A sufficiently large, $L(2A^2) \leqslant 2L(A^2)$.

If we set $A = \prod_{q\leqslant B} q$, then

$$M(A,B) = \frac{f(B)}{B} - H(B) + o(1)$$

and (7.4) becomes

$$\left|\frac{1}{AL(A^2)}\sum_{n\leqslant AL(A^2)} H(An-B) + H(B)\right| = O(1). \tag{7.5}$$

Thus for B sufficiently large, there is a constant $c_1 > 0$ such that

$$\left|\frac{1}{A^2L(A^2)}\sum_{n\leqslant AL(A^2)} H(An-B) + H(B)\right| < c_1. \tag{7.6}$$

As before, we may choose B so that

$$|H(B)| > c_2\log\log\log B.$$

Either $H(B) > 0$ or $H(B) < 0$. If $H(B) > 0$, equation (7.6) implies the existence of an n^*, $n^* \leqslant AL(A^2)$, for which

$$H(An^*-B) \leqslant c_1 - c_2\log\log\log B \leqslant -c_3\log\log\log B$$

or

$$H(An^*-B) \leqslant -c^*\log\log\log\log(An^*-B).$$

Similarly if $H(B)$ is negative there is an $n' \leqslant AL(A^2)$ for which

$$H(An'-B) \geqslant c_2 \log\log\log B - c_1 \geqslant c_4 \log\log\log B$$

or

$$H(An'-B) \geqslant c' \log\log\log\log(An'-B).$$

§ 8. On the number of changes of sign of $H(n)$.

8.1. The general case. For these estimates it is convenient to set $M(A, B) = C(A)\, M^*(A, B)$ and thus, for $B > 1$,

$$M^*(A, B) = BD(A) - \tfrac{1}{2} f(A)/A - \sum_{c<B} f\big((A, c)\big)/(A, c),$$

$$D(A) = \sum_{n|A} \mu(n) p(n)/n^2.$$

We need a more precise version of Corollary 7a; it is obtained by applying our summation operation to equation (6.2).

Corollary 7b. *If $p(n)$ is admissible,*

$$(8.1)\quad A \sum_{\substack{mn \leqslant z\\ n \equiv -B(A)}} p(m) H(n) = C(A)\, M^*(A, B) z L(z) + W_1(A, B, z) + W_2(z),$$

where we have

$$(8.2)\qquad W_1(A, B, z) = O\Big(\big(B + AB + AM(A, B)\big) K(z)\Big) + O\big(A^2 J(z)\big),$$

$$(8.3)\qquad W_2(z) = \tfrac{1}{2} a z^2 T(z) - \tfrac{1}{2}[z] + \tfrac{1}{2}\{z\}^2 + O\big(J(z)\big),$$

where the O's are uniform in A and B.

Proof. In equation (6.2) replace z by z/m, multiply by $p(m)$, and sum. Equations (8.2) and (8.3) are the errors from (6.2) and (4.2), respectively.

Define the operator $\Xi(H; A, B, z)$ as:

$$\Xi(H; A, B, z) = \frac{A}{C(A) z L(z)} \sum_{\substack{mn \leqslant z\\ n \equiv -B(A)}} p(m) H(n).$$

As usual, let $A_\lambda = \prod_{q \leqslant \lambda} q$, for $\lambda > B$; then

$$(8.4)\quad \Xi(H; A_\lambda, B, z) = a - H(B-1) - \tfrac{1}{2}\frac{f(A_\lambda)}{A_\lambda} + BS(\lambda) + \frac{W_1(A_\lambda, B, z) + W_2(z)}{C(A_\lambda) z L(z)}.$$

Theorem 12. *If $p(n)$ is admissible, there is a function $\zeta(N)$ such that*
(i) $H(N) > 0$ *implies* $H(n^*) < 0$ *for some* $n^* \in [N, \zeta(N)]$,
(ii) $H(N) < 0$ *implies* $H(n^*) > 0$ *for some* $n^* \in [N, \zeta(N)]$.

Proof. Assume $H(N) < 0$; replacing B by $N+1$ in equation (8.4) yields:

$$(8.5)\quad \Xi(H; A_\lambda, N+1, z) + H(N) = a - \tfrac{1}{2}\frac{f(A_\lambda)}{A_\lambda} + (N+1)S(\lambda) + \frac{W_1(A_\lambda, N+1, z) + W_2(z)}{C(A_\lambda) z L(z)}.$$

It does not suffice merely to observe that by taking first λ and then z large we would have

$$(8.6)\qquad \Xi(H; A_\lambda, N+1, z) + H(N) > H(N).$$

While this implies that $H(n) > 0$ somewhere in $[N, z]$, the choice of z depends on knowing the value of $H(N)$.

We can avoid this by picking first λ and then z large enough that

$$(8.7)\qquad \Xi(H; A_\lambda, N+1, z) + H(N) > -a.$$

This choice of z depends only on N and, of course, on the estimates for $K(z)$ etc.; but it does not guarantee a change of sign. For that we would have to know that $H(N) < -a$. While we do not have this, we can get that $H(N^*) < -a$ for an N^* slightly larger than N, and an estimate for how much larger; this will suffice.

The penultimate step of the proof that $H(x) = \Omega(\log\log\log x)$ was inequality (5.3):

$$|H(x_0 + k) - H(x_0)| > c \log\log\log x_0$$

where

$$k < \psi \log\log x_0$$

and x_0 is a solution of the system of congruences (5.2). As we noted there, this inequality implies that $H(n)$ changes by more than $c\log\log\log n$ in less than $\psi\log\log n$ for suitably chosen, arbitrarily large n.

We have now arrived at a procedure that works:

(i) Pick x^* such that $x^* > N$ and x^* is a solution of the system (5.2) for some k (note that x^* and k are integers), and that $c\log\log\log x^* > 2a$.

(ii) Set $B = x^* + \psi \log\log x^*$.

(iii) Take first λ, and then z, large enough that $\lambda > B$ and

$$(8.8)\quad -2a < a - \tfrac{1}{2}\frac{f(A_\lambda)}{A_\lambda} + BS(\lambda) + \frac{W_1(A_\lambda, B, z)}{C(A_\lambda) z L(z)} + \frac{W_2(z)}{C(A_\lambda) z L(z)} < 2a.$$

(iv) Define $\zeta(n)$ to be the z so determined.

To illustrate how this works, assume that $H(N) > 0$. Then either $H(x^*) > 0$ or $H(x^*) < 0$. If less than, we have a change of sign; otherwise, either $H(x^*) > 2\alpha$ or $H(x^*) < 2\alpha$. If greater, then $\zeta(N)$ is certainly large enough so that

$$\Xi\big(H;\, A_\lambda,\, x^*,\, \zeta(N)\big) + H(x^*) < 2\alpha$$

or

$$\Xi\big(H;\, A_\lambda,\, x^*,\, \zeta(N)\big) < 0$$

and $H(n)$ is negative somewhere in $[N, \zeta(N)]$. If less, then $H(x^*+k)$ differs from $H(x^*)$ by at least $c\log\log\log x^*$. But then either $H(x^*+k) < 0$ or $H(x^*+k) > 2\alpha$; in either case $H(n)$ must be negative somewhere on $[N, \zeta(N)]$. The corresponding argument applies if $H(N) < 0$.

To get a lower bound for the number of changes of sign in a given interval $[N, M]$, simply iterate $\zeta(n)$. We know that there is at least 1 change of sign between N and $\zeta(N)$, and therefore δ changes of sign between N and $\zeta^{(\delta)}(N)$. (The superscripts here denote iteration and not exponentiation.) The lower bound is given by the largest δ for which $\zeta^{(\delta)}(N) \leqslant M$.

A difficulty arises from the possibility that $H(n) = 0$ somewhere during the iteration; this can occur only if α is rational. Even then, it is not a real difficulty; the same procedure used in steps (i) and (ii) can be used to produce an $\tilde{x}$ for which $H(\tilde{x}) \neq 0$, and then steps (i) through (iv) carried out as above.

A more serious difficulty is that $\zeta(n)$ depends on how large z must be so that $K(z) < \varepsilon z L(z)$, for a given ε, and the corresponding estimates for $J(z)$ and $K(z)$. To have $\zeta(N)$ as an explicit function of N (and, of course, $p(n)$) we need an explicit function $g(\varepsilon)$ such that

$$\big(K(z)/zL(z)\big) < \varepsilon \quad \text{if} \quad z > g(\varepsilon) \tag{8.9}$$

and the corresponding functions for $J(z)$ and $T(z)$. We have only that such functions exist.

8.2. The special case $f(n) = \varphi(n)$. If we apply the above procedure to the special case $f(n) = \varphi(n)$, corresponding to $p(n) = 1$, we obtain explicit estimates for the number of changes of sign of $H(n)$ for $n < x$. We have the following estimates:

$$\begin{cases} \alpha = 6/\pi^2, \\ f(A_\lambda)/A_\lambda < 1/\log\lambda, \\ |S(\lambda)| \leqslant T(\lambda) < 1/\lambda, \\ K(x) = [x] \leqslant x, \\ J(x) < x+1+1/(x-1), \\ \log x < L(x) < \log x + \gamma + 1/(x-1) \end{cases} \tag{8.10}$$

and therefore:

$$\begin{cases} z^2 T(z) < 2/\log z, \\ K(z) < 2/\log z, \\ J(z) < 2/\log z. \end{cases} \tag{8.11}$$

Proceeding step by step: Given an integer N.

(i) In the system of congruences (5.2) set $\theta = 3$, $\psi = 2$, $c = 3/2\pi^2$, and take

$$k = \max(2\log\log N,\, e^4). \tag{8.12}$$

(ii) $x^* + 2\log\log x^* < 2\prod q \leqslant 2^{(2^{3^k}+1)}$, where the product is taken over all primes not exceeding 2^{3^k}. (The estimate ([4], p. 341) used here holds for all k; for large k better estimates are available.) Therefore set

$$B = 2^{(2^{3^k}+1)}. \tag{8.13}$$

(iii) Set

$$\lambda = \exp\big(2(B+1)/\alpha\big), \tag{8.14}$$

$$\zeta(N) = \exp(8A_\lambda^2/\alpha^2). \tag{8.15}$$

This guarantees that

$$|BS(\lambda) - \tfrac{1}{2}f(A_\lambda)/A_\lambda| < (B+1)/\log\lambda < \tfrac{1}{2}\alpha \tag{8.16}$$

and

$$\Big|\big(W_1(A_\lambda, B, \zeta) + W_2(\zeta)\big)/C(A_\lambda)\zeta L(\zeta)\Big| < \tfrac{1}{2}\alpha. \tag{8.17}$$

Collecting everything, given N we have:

$$\begin{cases} \zeta(N) = \exp(8A_\lambda^2/\alpha^2), \\ A_\lambda = \prod_{q<\lambda} q, \\ \lambda = \exp\big(2(B+1)/\alpha\big), \\ B = 2^{(2^{3^k}+1)}, \\ k = \max(2\log\log N,\, e^4). \end{cases} \tag{8.18}$$

To make this qualitatively simpler, we may increase the bases so that all of the exponentials in equations (8.18) are to the same base. This gives a function $z(N)$ which is a 4-fold exponential to some sufficiently large base. To get a lower bound for the number of changes of sign in $[1, N]$ reduces to finding the smallest integer ν for which the 4ν-fold iterated logarithm (to the same base) of N is less than 2. This is precisely the function $IL(N)$ mentioned in the introduction.

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Errata to the paper "A general arithmetic construction of transcendental non-Liouville normal numbers from rational fractions"

Acta Arithmetica 16 (1970), pp. 240–253

by

R. G. Stoneham (New York, N. Y.)

Page 246, (2.25)

for

$$\ldots = \ldots + (Z_2/m^2 - Z_1/m)\, g^{a_1 \omega(m)} \ldots$$

read

$$\ldots = \ldots + (Z_2/m^2 - Z_1/m)/g^{a_1 \omega(m)} \ldots$$

Page 248, (2.36)

for

$$\ldots = \log m^{s+2} g^{S(s,m)} / \log m^{s+1} g^{S(s,m)},$$

read

$$\ldots = \log m^{s+2} g^{S(s+1,m)} / \log m^{s+1} g^{S(s,m)}.$$

Page 251_6

for

$$x(g, m) = \ldots E_1(a_1 - 1) E_1 E_2(a_2) E_2 \ldots$$

read

$$x(g, m) = \ldots E_1(a_1) E_1 E_2(a_2) E_2 \ldots$$